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On a method to construct magic rectangles of odd order

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Abstract

Magic rectangles are well-known for their very interesting and entertaining combinatorics. Such magic rectangles have been used in designing experiments. For example, Phillips (1964, 1968a, 1968b) illustrated the use of these magic figures for the elimination of trend effects in certain classes of one-way, factorial, latin-square, and graeco-latin-square designs. In a magic rectangle, the integers 1 to mn are arranged in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N. In the present paper we provide a simple and systematic method for constructing any m by $m+2$ magic rectangle for m odd.

Key words: Magic rectangles; Magic constants.

1 Introduction

Magic rectangles are well-known for their very interesting and entertaining combinatorics. A magic rectangle is an arrangement of the integers 1 to mn in an array of m rows and n columns so that each row adds to the same total M and each column to the same total N. The totals M and N are termed the magic constants. Since the average value of the integers is $A = (mn+1)/2$, we must have $M = nA$ and $N = mA$. The total of all the integers in the array is $mnA = mM = nN$. If mn is even $mn+1$ is odd and so for $M = n(mn+1)/2$ and $N = m(mn+1)/2$ to be integers n and m must both be even. On the other hand, since either m or n being even would result in the product mn to be even, therefore if mn is odd then m and n must both be odd. In this case also M and N are integers since $mn + 1$ is even. Therefore, an odd by even magic rectangle is impossible.

For an update on available literature on magic rectangles we refer to Hagedorn (1999) and Bier and Kleinschmidt (1997). Such magic rectangles have been used in designing experiments. For example, Phillips (1964, 1968a, 1968b) illustrated the use of these magic figures for the elimination of trend effects in certain classes of one-way, factorial, latin-square, and graeco-latin-square designs. As highly balanced structures, magic rectangles can be potential tools for use in situations yet unexplored.

A simple construction method for any even by even magic rectangle was recently provided by Reyes, Das, Midha and Vellaisamy (2008). The construction of odd by odd magic rectangles is more challenging. In the present paper we provide a method for constructing any m by $m+2$ magic rectangle for m odd. The construction involves some simple steps. The method has been shaped in form of an algorithm that is very convenient for writing a computer program for constructing such rectangles.

In Section 2 we construct m by $m + 2$ magic rectangle. The proofs related to the construction are given in the Appendix. In Section 3 we illustrate our construction method through some examples of magic rectangles.

2 The construction

We construct magic rectangle with m rows and $m+2$ columns for given positive odd integers m. Let J_m denote a square matrix of order m having all its elements equal to 1.

Step 1. Construct m by m matrix $A = (a_{ij})$ where $a_{ij} = \{j + i + (m - 3)/2\} \mod m$.

Step 2. Construct m by m matrix $B = (b_{ij})$ where $b_{ij} = \{2(j - 1) + i\} \mod m$.

Step 3. Construct m by m matrix C where $C = mA + B + J_m$.

Step 4. Construct m by 2 matrix D where the 1st column of D is $\{m^2+1, m^2+2, m^2+m, m^2+3, m^2+4, m^2+5, \ldots, m^2+m-1\}$ and 2nd column is $\{m^2+2m, m^2+2m-1, m^2+m+1, m^2+2m-2, m^2+2m-3, m^2+2m-4, \ldots, m^2+m+2\}$. We would call this a modified serpentine format for writing the $2m$ consecutive integers from m^2+1 through m^2+2m in the 2 columns of D. The un-modified serpentine format is achieved by shifting the 3rd element in each column of D as the last elements of the respective columns.

Step 5. For $i = 1, 2, \ldots, (m-1)/2$, interchange the unique element $2i-1$ in C with unique element m^2+2i-1 in D. Let the resultant matrices be C_1 and D_1.

Step 6. For $i = 1, 2, \ldots, (m+1)/2$, interchange the unique element $m+2i-1$ in C_1 with unique element $m^2+m+2i-1$ in D_1. Let the resultant matrices be C_2 and D_2.

Step 7. Let $R = (C_2|D_2)$.

R is the required magic rectangle. The proofs related to the construction are given in Appendix.

3 Some illustrative examples

In this section we provide two examples of magic rectangle of sides 7 by 9 and sides 9 by 11.

Magic rectangle of order 7 by 9

Step 1 gives

4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	0	1	2	3	4	5
0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2

Step 2 gives

1	3	5	0	2	4	6
2	4	6	1	3	5	0
3	5	0	2	4	6	1
4	6	1	3	5	0	2
5	0	2	4	6	1	3
6	1	3	5	0	2	4
0	2	4	6	1	3	5

Step 3 gives

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

Step 4 gives

50	63
51	62
56	57
52	61
53	60
54	59
55	58

Following Steps 5 and 6, we get the following 7 by 9 magic rectangle in Step 7

30	39	48	**50**	**59**	19	28	**1**	**14**
38	47	7	9	18	27	29	51	62
46	6	**57**	17	26	35	37	56	**8**
54	**63**	16	25	34	36	45	**3**	**12**
13	15	24	33	42	44	4	53	60
21	23	32	41	43	**52**	**61**	**5**	**10**
22	31	40	49	2	11	20	55	58

Magic rectangle of order 9 by 11 Step 1 gives

5	6	7	8	0	1	2	3	4
6	7	8	0	1	2	3	4	5
7	8	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7
0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
4	5	6	7	8	0	1	2	3

Step 2 gives

1	3	5	7	0	2	4	6	8
2	4	6	8	1	3	5	7	0
3	5	7	0	2	4	6	8	1
4	6	8	1	3	5	7	0	2
5	7	0	2	4	6	8	1	3
6	8	1	3	5	7	0	2	4
7	0	2	4	6	8	1	3	5
8	1	3	5	7	0	2	4	6
0	2	4	6	8	1	3	5	7

Step 3 gives

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

Step 4 gives

82	99
83	98
90	91
84	97
85	96
86	95
87	94
88	93
89	92

Following Steps 5 and 6, we get the following 9 by 11 magic rectangle in Step 7

47	58	69	80	**82**	**93**	23	34	45	**1**	**18**
57	68	79	9	11	22	33	44	46	83	98
67	78	8	**91**	21	32	43	54	56	90	**10**
77	**88**	**99**	20	31	42	53	55	66	**3**	**16**
6	17	19	30	41	52	63	65	76	85	96
97	27	29	40	51	62	64	75	**86**	**5**	**14**
26	28	39	50	61	72	74	4	15	87	94
36	38	49	60	71	73	**84**	**95**	25	**7**	**12**
37	48	59	70	81	2	13	24	35	89	92

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Appendix

Steps 1 and 2 construct two orthogonal Latin squares A and B with elements 0 through $m-1$. Step 3 uses the orthogonal property of A and B to construct the matrix C having elements 1 through m^2 such that each of the row and column sums of C are $m(m2+1)/2$.

Step 4 constructs the matrix D with elements m^2+1 through m^2+2m such that every row sum of D is $2m^2+2m+1$. The 1st and 2nd column sums of D are $m\{m^2+(m+1)/2\}$ and $m\{m^2+m+(m+1)/2\}$ respectively.

The magic constants for m by $m+2$ magic rectangle are $M = (m+2)(m+1)2/2$ and $N = m(m+1)2/2$. Thus, in order to convert $(C|D)$ into a magic rectangle, we first observe that $(C|D)$ has the required magic constant $M = (m+2)(m+1)2/2$ but not the required magic constant $N = m(m+1)2/2$. The column sums of C are less by m^2 than the required sum N. Also the column sums of the 1st and 2nd columns of D exceeds N by $m^2(m-1)/2$ and $m^2(m+1)/2$ respectively.

Step 5 interchanges $(m-1)/2$ elements of the 1st column of D with elements in $(m-1)/2$ distinct columns of C. Similarly, Step 6 interchanges $(m+1)/2$ elements of the 2nd column of D with elements in $(m+1)/2$ distinct columns of C. Moreover, the distinct columns of C in step 6 are different from the columns involved in step 5.

Thus, Steps 5 and 6 increase each of the column sums of C by m^2 and also decrease each of the column sums of D so as to achieve the magic constant $N = m(m+1)2/2$.

Step 7 augments C and D after interchanges are carried out as per Steps 5 and 6. This provides the desired magic rectangle.