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Algorithmic construction of catalogs of non-isomorphic two-level orthogonal designs for economic run sizes

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Abstract

We propose an algorithm for sequentially constructing non-isomorphic orthogonal designs (including both regular and non-regular orthogonal designs). An essential element of the algorithm is using minimal column base to reduce the computations for determining isomorphism between two designs. The algorithm also makes use of the extended word length pattern criterion to reduce the number of designs for isomorphism check. By using this algorithm, we obtain the complete catalogs of $n \times p$ ($p = 2, \ldots, n-1$) two-level orthogonal designs for $n = 12$, 16, and 20. We then study the statistical properties of the designs in terms of the extended word length pattern criterion. The minimum-aberration designs according to this criterion are obtained and provided for practical use.

Key words : Column base; Design isomorphism; Extended word length pattern; Minimum aberration.

1 Introduction

Two-level orthogonal designs are commonly used in practice. Two designs are called *isomorphic* if one can be obtained from the other by row exchanges, column exchanges, and level exchanges within the column. It is desirable to know, for a given size, how many non-isomorphic orthogonal designs exist and what they are. In addition to the obvious mathematical interests of these two questions, practically, knowing the answers will allow one to quickly and easily find the global optimal orthogonal arrays according to some design criteria.

Unfortunately, construction of all non-isomorphic designs is extremely computationally intensive. Many useful orthogonal designs were given in Hedayat, Slone and Stufken (1999). For regular fractional factorial designs, Chen, Sun, and Wu (1993) proposed a comprehensive algorithm for detecting non-isomorphic designs beyond the comparisons of word length patterns and letter patterns. For non-regular designs, identifying non-isomorphic designs is generally more challenging. For instance, repeated-run patterns (two identical rows in a design matrix are called repeated runs) and mirror-image patterns (a row that is identical to another row after a level exchange of the entire row is called a mirror-image of the other row) have been proposed for non-regular designs (see, for example, Draper (1985), Draper and Lin (1990), Wang and Wu (1995)), but neither approach guaranteed the identification of the true isomorphism between two designs. It has been shown by Chen and Lin (1990) that these patterns can be the same even for non-isomorphic designs.

In this article, we propose a sequential method for constructing non-isomorphic orthogonal designs (including both regular and non-regular designs) and an algorithm for detecting isomorphism between any two designs. By using this approach, we obtain complete catalogs of $12 \times p$ ($p \leq 11$), $16 \times p$ ($p \leq 15$), and $20 \times p$ ($p \leq 19$) two-level orthogonal designs. In particular, the complete catalog of non-isomorphic 20-run orthogonal designs is obtained for the first time in the literature. The three resulting 20×19 designs are indeed equivalent to the three well known 20-run Hadamard matrices discovered by Hall (1965).

Once the complete set of all $n \times p$ non-isomorphic designs is found, the next step is to choose optimal designs according to a given criterion. For regular fractional factorial designs, a commonly used criterion is minimum aberration. However, it is not defined for non-regular designs, and all 12-run, 20-run designs and most of the 16-run designs are non-regular designs. In this article, we use the generalized resolution and aberration criteria proposed by Deng and Tang (1999) for evaluating non-regular designs. The original definition of these two criteria are based on the so called J-characteristics and confounding frequency vectors. Li, Lin and Ye (2003) presented an equivalent definition by using *words of fractional length*, which can be briefly explained as follows: In a regular design, the word length equals the number of letters of a word, and each word implies full aliasing among associated factors. In a non-regular design, partial aliasing can exist. Then the word length is defined to be:

$$\text{number of letters} + (1 - \text{correlation among the associated factors}).$$

For instance, suppose a word is represented by 123. If x_1 and x_2x_3 are fully aliased (i.e., the correlation is 1), then word length of 123 is: $3 + (1 - 1) = 3$. For the same word, if partial correlation between x_1 and x_2x_3 exists, and the correlation is 1/3, then the word length of 123 is: $3 + (1 - 1/3) = 3\frac{2}{3}$.

Let $f_{i+j/n}$ be the number of length-$(i + j/n)$ words. The *extended word length pattern* (EWLP) of a design D is defined to be

$$(f_1, f_{1+1/n}, \cdots, f_{1+(n-1)/n}, \cdots, f_k, f_{k+1/n}, \cdots, f_{k+(n-1)/n}).$$

This criterion was proposed as the G-aberration in Deng and Tang (1999). Also, following Deng and Tang (1999), the *generalized resolution* of D can be defined as the length of the shortest word. Consequently, design D_1 is said to have less aberration than design D_2 if $f_t(D_1) < f_t(D_2)$ and $f_s(D_1) = f_s(D_2)$ for all $s < t$.

Deng, Li, and Tang(2000) took an exhaustive search over all projections of Hadamard matrices and presented the optimal designs according to the minimum aberration criterion. One interesting question is whether or not there exist designs with less aberration than what they found. A related but more fundamental question is whether or not all non-isomorphic orthogonal designs are projections of some Hadamard matrices. The following two examples show that the answers to both questions are negative.

Consider the two 20-run designs in Tables 1 and 2. Both designs are found to be not isomorphic to any projection of the 20-run Hadamard matrices. Moreover, they have less aberration than the minimum aberration designs found by Deng *et al.* (2000) among all projections. Extended word length pattern of the design in Tables 1 is $\{(0, 0, 20)_3, (0, 0, 15)_4, (0, 5, 0)_5\}$. Here, $(0, 0, 20)_3$ means that the numbers of words of length-3.4, 3.6 and 3.8 are 0, 0 and 20, respectively. Similarly, the number of words of length 4.4, 4.6 and 4.8 is shown by $(0, 0, 15)_4$. Note that the only possible word lengths of a 20-run orthogonal design are $k.4$, $k.6$ and $k.8$ for $3 \le k \le 5$. In comparison, the best designs of the same size found by Deng *et al.* (2000) have extended word length pattern $\{(0, 0, 20)_3, (1, 0, 14)_4, (0, 2, 0)_5\}$.

Table 1: The minimum aberration $OA(20, 2^6)$

1	1	1	0	1	1
1	1	1	0	0	1
1	1	0	1	1	0
1	1	0	1	0	1
1	1	0	0	0	0
1	0	1	1	1	0
1	0	1	1	0	1
1	0	1	0	0	0
1	0	0	1	1	1
1	0	0	0	1	0
0	1	1	1	1	0
0	1	1	1	0	1
0	1	1	0	1	0
0	1	0	1	0	0
0	1	0	0	1	1
0	0	1	1	0	0
0	0	1	0	1	1
0	0	0	1	1	1
0	0	0	0	0	1
0	0	0	0	0	0

Table 2: The minimum aberration $OA(20, 2^7)$

1	1	1	1	0	1	1
1	1	1	0	1	1	0
1	1	0	1	1	0	0
1	1	0	0	1	1	1
1	1	0	0	0	0	0
1	0	1	1	0	1	0
1	0	1	0	1	0	0
1	0	1	0	0	0	1
1	0	0	1	1	1	1
1	0	0	1	0	0	1
0	1	1	1	0	0	0
0	1	1	0	1	0	1
0	1	1	0	0	1	1
0	1	0	1	1	0	1
0	1	0	1	0	1	0
0	0	1	1	1	1	0
0	0	1	1	1	0	1
0	0	0	0	1	1	0
0	0	0	0	0	1	1
0	0	0	0	0	0	0

The design in Table 2 has extended word length pattern of $\{(0,0,35)_3, (2,0,33)_4, (0,11,0)_5\}$ as compared to $\{(0,0,35)_3, (3,0,32)_4, (0,7,0)_5\}$ found by Deng *et al.* (2000).

The remainder of this article is organized as follows. Section 1 describes the algorithm in details. Section 2 summarizes the orthogonal designs produced by the algorithm. The minimum-aberration designs are tabulated for practical use. Section 3 investigates the relationship between the resulting n-run designs and the projections of n-run Hadamard matrices.

2 Sequential construction of non-isomorphic two-level orthogonal designs

To construct a complete catalog of non-isomorphic orthogonal designs with all possible number of factors, we propose the following sequential construction approach:

- *Step 1: Start with the* $n \times 1$ *design, which is simply a vector with half 0's and half 1's.*
- *Step 2: For each* $n \times (p-1)$ *design* ($p \geq 2$), *add one column in all possible ways such that the additional column is orthogonal to the existing columns.*
- *Step 3: Classify all* $n \times p$ *augmented designs into groups in terms of the extended word length pattern criterion, such that the designs within one group have the same extended word length pattern.*
- *Step 4: For each group, perform the design isomorphism check for each pair of designs, using the minimum column based method described below. For designs that are isomorphic to each other, only one is retained.*

The most time consuming step in constructing catalogs of non-isomorphic designs is the isomorphism check on two designs. We now describe a method for checking design isomorphism based on the concept of *minimal column base*, an idea motivated by the work of Leon (1979) in the computation of automorphism groups of a Hadamard matrix.

A *column base* is a subset of columns of a design, such that no two rows in the column base are identical to or the mirror images of each other. The size of a column base is defined to be the number of columns in that column base. Because one can always obtain a new column base by adding a column to an existing column base, we define the *minimal column base* to be a column base with the smallest possible number of columns for a given design. Note there may be multiple minimal column bases for one design.

There are two important properties of the column base. First, an isomorphism mapping of a column base of design D_1 onto another design D_2 constitutes a column base of D_2. This implies that two isomorphic designs must have equal number of minimum column basis. Second, the isomorphic mapping between two column bases, if existing, is unique. Based on these two properties, the isomorphism check in Step 4 of the proposed sequential algorithm can be performed as follows:

- *Step 4a: To check isomorphism of two designs D_1 and D_2, obtain the minimum column bases of both designs. If the numbers of minimum column bases are different, then the two designs are non-isomorphic. Otherwise, go to Step 4b.*

- *Step 4b: Pick any column base, say $(b_1, \cdots, b_p)$ of D_1, and check its isomorphism with all column bases of D_2. For a given column base of D_2, if an isomorphism mapping is found between this column base and $(b_1, \cdots, b_p)$, then we only need to perform column exchanges and column level changes of the remaining columns (the ones that are not in the column base) in D_2 and compare the resulting design with D_1. (No row permutation is necessary because the uniqueness of isomorphism mapping of two column bases determines the mapping of rows from D_1 to D_2.)*

- *Step 4c: If an isomorphism mapping from D_1 to D_2 is found, then the two designs are isomorphic. Otherwise, proceed to the next column base of D_2 and repeat Step 4b.*

For more details on the algorithm and column bases, see Sun, Li, and Ye (2002).

Table 3: Summary of non-isomorphic $12 \times p$ orthogonal designs

Design	resolution	EWLP
3.1	3.0	$(1,0,0)_3$
3.2	3.7	$(0,0,1)_3$
4	3.7	$(0,0,4)_3$ $(0,0,1)_4$
5.1	3.7	$(0,0,10)_3$ $(0,0,5)_4$ $(0,0,0)_5$
5.2	3.7	$(0,0,10)_3$ $(0,0,5)_4$ $(0,1,0)_5$
6.1	3.7	$(0,0,20)_3$ $(0,0,15)_4$ $(0,1,0)_5$ $(0,0,0)_6$
6.2	3.7	$(0,0,20)_3$ $(0,0,15)_4$ $(0,0,0)_5$ $(0,1,0)_6$
7	3.7	$(0,0,35)_3$ $(0,0,35)_4$ $(0,3,0)_5$ $(0,1,0)_6$
8	3.7	$(0,0,56)_3$ $(0,0,70)_4$ $(0,8,0)_5$ $(0,4,0)_6$
9	3.7	$(0,0,84)_3$ $(0,0,126)_4$ $(0,18,0)_5$ $(0,12,0)_6$
10	3.7	$(0,0,120)_3$ $(0,0,210)_4$ $(0,36,0)_5$ $(0,30,0)_6$

Note: EWLP lists the numbers of words of length k, $k+1/3$, and $k+2/3$ for $3 \leq k \leq 6$. For example, $(0,0,4)_3$ means that there is no length-3 word, no length-3.3 word, and 4 length-3.7 word.

3 Results

We now use the proposed sequential algorithm to construct complete catalogs of orthogonal designs of 12, 16, and 20 runs. The results are summarized in this section.

3.1 12-run designs

For 12-run designs, there is only one unique $12 \times p$ design for $p = 4$ and $7 \leq p \leq 11$. For $p = 5$ and 6, there are two non-isomorphic $12 \times p$ designs. These results are consistent with those presented previously by Lin and Draper (1992) and Wang and Wu (1995). The generalized resolution of these designs were discussed in Deng *et al.* (2000). We summarize the results in Table 3. It can be easily seen from the table that Design 5.1 and 6.1 are the minimum-aberration designs for $p = 5$ and 6. For $p = 3$, there are two non-isomorphic designs. Note that Design 3.1 consists of 3 replicates of the regular 2^{3-1} orthogonal design, which is the only resolution-3 design and the only design that is not a projection of the 12-run Plakett-Burman design.

3.2 16-run designs

The proposed algorithm finds five 16×15 orthogonal designs, which are equivalent to the five well known non-isomorphic Hadamard matri-

Table 4: Summary of non-isomorphic $16 \times p$ orthogonal designs.

p	total # of design	# of regular designs	minimum aberration designs	
			resolution	EWLP
5	11	4	5.0	$(0,0)_3$ $(0,0)_4$ $(1,0)_5$
6	27	5	4.0	$(0,0)_3$ $(3,0)_4$ $(0,0)_5$
7	55	6	4.0	$(0,0)_3$ $(7,0)_4$ $(0,0)_5$
8	80	6	4.0	$(0,0)_3$ $(14,0)_4$ $(0,0)_5$
9	87	5	3.5	$(0,16)_3$ $(14,0)_4$ $(0,32)_5$
10	78	4	3.5	$(0,32)_3$ $(10,32)_4$ $(0,64)_5$
11	58	3	3.5	$(0,48)_3$ $(14,48)_4$ $(0,112)_5$
12	36	2	3.5	$(0,64)_3$ $(15,96)_4$ $(0,192)_5$
13	18	1	3.5	$(0,88)_3$ $(15,160)_4$ $(0,288)_5$
14	10	1	3.5	$(0,112)_3$ $(21,224)_4$ $(0,448)_5$

Note: EWLP lists the numbers of words of length k, $k+.5$ for $3 \leq k \leq 5$. For example, $(0,16)_3$ means that there is no length-3 word and 16 length-3.5 words. There are no words of other lengths for $3 \leq k \leq 5$.

ces by Hall (1961). Table 4 shows the numbers of p-factor orthogonal designs ($p = 4, \ldots, 14$). It was found that *all 16-run orthogonal designs are projections of the 16-run Hadamard matrices*. Note most of related work in the literature focused on the projections of 16-run Hadamard matrices. For instance, Lin and Draper (1992) studied the projections onto dimensions of 3–5. Chen *et al.* (1993) presented all non-isomorphic projections onto p dimensions ($3 \leq p \leq 15$). Deng *et al.* (2000) classified projections according to the extended word length pattern. However, it was not clear previously whether or not all 16-run non-isomorphic orthogonal designs are the projections of the Hadamard matrices.

Using the complete catalog presented in Sun *et al.* (2002), Li *et al.* (2003) found the $16 \times p$ ($p \leq 15$) minimum-aberration designs. For the convenience of readers, we include the EWLP of 16-run minimum aberration designs in Table 4. For more details, see Li *et al.* (2003).

Because all orthogonal designs of 16 runs are projections of Hadamard matrices, there is no need to give all design matrices. In the Appendix, all 16-run designs are tabulated according to the corresponding projections of Hadamard matrices.

3.3 20-run designs

The results of 20-run designs are summarized in Table 5. There are 11,491 non-isomorphic designs. Among these designs, there is only one design whose resolution is 3.0. It is a 20×3 design that repeats the regular 2^{3-1} design 5 times and it is not a projection of the Hadamard matrices. All other designs have resolutions of 3.4 or 3.8. Recall that when a design has a resolution of 3.8, the maximum absolute correlation between the three effects of all letter-3 words is $4/20 = .2$. Thus, resolution-3.8 designs are generally preferable to resolution-3.4 designs. The third column in Table 5 lists the number of resolution-3.8 designs for each m. It shows that resolution-3.8 designs exist for $p \leq 10$. In addition, the numbers of resolution-3.8 designs are generally small. For example, when $m = 10$, there are 2,389 non-isomorphic designs. But only one of them has a resolution of 3.8.

We investigate all $20 \times p$ designs according to their extended word length patterns. Table 5 reports the extended word length patterns of the minimum-aberration designs for all $3 \leq p \leq 19$. For simplicity, only (W_3, W_4, W_5) is presented. For each W_k $(k = 3, 4, 5)$, the possible fractional lengths are $1-12/20 = .4$, $1-8/20 = .6$, and $1-4/20 = .8$. Thus, the reported EWLP's have three elements for each W_k $(3 \leq k \leq 5)$. For example, when $p = 11$, the minimum-aberration design has $W = \{(5, 0, 160)_3, (30, 0, 300)_4, (0, 142, 0)_5\}$. This design has, among letter-3 words, 5 length-3.4 words, 0 length-3.6 words, and 160 length-3.8 words.

Most of the minimum-aberration designs are projections of the 20-run Hadamard matrices. Thus, they have the same EWLP's as those reported in Deng *et al.* (2000), which did an exhaustive search of all projections. However, as shown at the beginning of this article, the minimum-aberration designs for $p = 6$ and 7 are not projections of the Hadamard matrices.

Non-isomorphic designs may have the same EWLP's. When $p \geq 17$, all p-factor non-isomorphic designs have the same EWLP's. The detailed information of all non-isomorphic designs and their EWLP's is available from the corresponding author upon request.

4 Projections of Hadamard matrices

The proposed algorithm constructs $n \times p$ designs $(2 \leq p \leq n)$ sequentially. When $p = n-1$, the resulting orthogonal designs are equivalent to Hadamard matrices. We find five 16-run designs and three 20-run designs, which are consistent with the results reported by Hall (1961).

Hadamard matrices and their projections are commonly used in

Table 5: Summary of non-isomorphic $20 \times p$ orthogonal designs.

p	total # of design	# of designs for $r = 3.8$	minimum aberration designs: resolution	minimum aberration designs: EWLP
3	3	1	3.8	$(0,0,1)_3$
4	3	2	3.8	$(0,0,4)_3$ $(0,0,1)_4$
5	11	4	3.8	$(0,0,10)_3$ $(0,0,5)_4$ $(0,0,0)_5$
6	75	13	3.8	$(0,0,20)_3$ $(0,0,15)_4$ $(0,5,0)_5$
7	474	21	3.8	$(0,0,35)_3$ $(2,0,33)_4$ $(0,11,0)_5$
8	1,603	6	3.8	$(0,0,56)_3$ $(6,0,64)_4$ $(0,24,0)_5$
9	2,477	2	3.8	$(0,0,84)_3$ $(18,0,108)_4$ $(0,34,0)_5$
10	2,389	1	3.8	$(0,0,120)_3$ $(30,0,180)_4$ $(0,72,0)_5$
11	1,914	0	3.4	$(5,0,160)_3$ $(30,0,300)_4$ $(0,142,0)_5$
12	1,300	0	3.4	$(8,0,212)_3$ $(39,0,456)_4$ $(0,240,0)_5$
13	730	0	3.4	$(14,0,272)_3$ $(47,0,668)_4$ $(0,390,0)_5$
14	328	0	3.4	$(20,0,344)_3$ $(60,0,941)_4$ $(0,601,0)_5$
15	124	0	3.4	$(26,0,429)_3$ $(81,0,1284)_4$ $(0,891,0)_5$
16	40	0	3.4	$(33,0,527)_3$ $(107,0,1713)_4$ $(0,1284,0)_5$
17	11	0	3.4	$(40,0,640)_3$ $(140,0,2240)_4$ $(0,1820,0)_5$
18	6	0	3.4	$(480,0,768)_3$ $(180,0,2880)_4$ $(0,250,0)_5$
19	3	0	3.4	$(57,0,912)_3$ $(228,0,3648)_4$ $(0,3420,0)_5$

Note: EWLP lists the numbers of words of length $k+.4$, $k+.6$, and $k+.8$ for $3 \le k \le 5$. For example, $(0,0,1)_3$ means that there is no length-3.4 word, no length-3.6 word, and 1 length-3.8 word. There are no words of other lengths for $3 \le k \le 5$.

practice. Several criteria, as referenced in the introduction, were proposed to distinguish different designs in the literature. Lin and Draper (1992) discussed the possibly different projections of $H_{16.II} - H_{16.V}$ to $p = 3, 4$, and 5 dimensions by repeated and mirror-image patterns. By applying the isomorphism detecting algorithm based on minimal column bases, we obtain the complete collection of non-isomorphic projections to $p = 2, \cdots, 14$ dimensions. The numbers of non-isomorphic projections for $H_{16.I} - H_{16.V}$ by two methods are shown in Table 6. It can be seen that our results are more complete. Because a sub-matrix of one Hadamard matrix may be isomorphic to a sub-matrix of another Hadamard matrix, the total number of distinct $16 \times p$ sub-matrices from $H_{16.I} - H_{16.V}$ is less than the sum of the numbers in each row. The last column in Table 6 gives the total number of non-isomorphic projections based on the five matrices $H_{16.I} - H_{16.V}$.

One of the major results obtained by the proposed algorithmic approach is that all 16-run orthogonal designs are projections of the 16-run Hadamard matrices. This can be verified by comparing the total number of non-isomorphic projections, which is given in the last column of Table 6, with the total number of 16-run designs obtained through sequential construction. The two numbers match for all $2 \le p \le 15$. Therefore, it is not necessary to list all design matrices, but only their corresponding projections of the Hadamard matrices. Such a list is given in the appendix.

Because there are non-trivial 20-run orthogonal designs (e.g., De-

Table 6: Numbers of non-isomorphic projections of $H_{16.I} - H_{16.V}$

	Lin and Draper (1992)				
p	$H_{16.I}$	$H_{16.II}$	$H_{16.III}$	$H_{16.IV}$	$H_{16.V}$
2	1	1	1	1	1
3	2	2	2	2	3
4	3	4	4	3	5
5	4	6	6	5	8
6	-	-	-	-	-
7	-	-	-	-	-
8	-	-	-	-	-
9	-	-	-	-	-
10	-	-	-	-	-
11	-	-	-	-	-
12	-	-	-	-	-
13	-	-	-	-	-
14	-	-	-	-	-

	new method					
p	$H_{16.I}$	$H_{16.II}$	$H_{16.III}$	$H_{16.IV}$	$H_{16.V}$	total
2	1	1	1	1	1	1
3	2	3	3	3	3	3
4	3	5	5	5	5	5
5	4	10	11	10	10	11
6	5	18	26	18	20	27
7	6	25	48	24	26	55
8	6	29	62	25	26	80
9	5	26	57	20	21	87
10	4	19	43	14	16	78
11	3	13	28	9	11	58
12	2	8	15	5	6	36
13	1	4	7	3	3	18
14	1	2	3	2	2	10

signs in Tables 1 and 2) that are not the projections of the Hadamard matrices, we present 20-run orthogonal designs by giving their design matrices. The complete catalog of non-isomorphic 20-run designs is available from the corresponding author upon request.

In a related work, Beder (1998) reported that there is one 12×20 matrix of orthogonal design that can not be expanded to Hadamard matrix, which is consistent with our findings. Beder (1998) also conjectured that if n is power of 2, every n-run orthogonal design is a projection of Hadamard matrix, and the conjecture is known to be true $n = 2, 4, 8$. Our results confirmed that the conjecture is also true for $n = 16$.

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Appendix: Nonisomorphic Projections of $H_{16.I} - H_{16.V}$

For abbreviation, we use $I, \cdots, V$ to denote $H_{16.I} - H_{16.V}$ respectively, e.g., IV(1, 2, 3, 4, 5) instead of $H_{16.IV}(1, 2, 3, 4, 5)$.

p=3:

1=I(1,2,3); 2=I(1,2,4); 3=II(4,8,12);

p=4:

1=I(1,2,3,4); 2=I(1,2,4,7); 3=I(1,2,4,8); 4=II(1,4,8,12); 5=II(4,5,8,12);

p=5:

1=I(1,2,3,4,5); 2=I(1,2,3,4,8); 3=I(1,2,4,7,8); 4=I(1,2,4,8,15);
5=II(1,2,4,8,12); 6=II(1,4,5,8,12); 7=II(1,4,6,8,12); 8=II(4,5,6,8,12);
9=II(4,5,8,9,12); 10=II(4,5,8,10,12); 11=III(2,4,8,10,12);

Designs for $p = 6, \ldots, 15$ are available on the web page: www.umn.edu/~wli.