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Group Testing Designs: A Combinatorial Marvel

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Abstract

We propose to discuss at length some salient features of what are known as Group Testing Designs [GTDs]. Not too many design theorists/graph theorists have looked into this problem. Surprisingly, it did not escape the attention of Professor M N Das. He ventured into some finer aspects of the design issues in his characteristic style. We pay our homage to Professor Das by contributing this review article. We mainly discuss what are known as (i) non-adaptive or Hypergeometric GTDs, (ii) adaptive or sequential GTDs.

Key words Group tests; hypergeometric group tests, sequential group tests, t-completeness, detecting power of order t, group divisible designs, Petersen graphs.

1 Introduction to GTDs

Group Testing is a technique to test a collection of units in several groups, rather than in isolation (i.e., one-at-a-time), in order to ascertain the 'status' of *each* individual unit in the collection in respect of a well-defined 'feature'. The problem is to plan the testing procedure so as to be able to do so without any ambiguity and with a minimum number of such tests [called Group Tests (GTs)]. The feature to be extracted from each unit is the same and it is 'qualitative' in nature and it is tacitly assumed that 'possession' of the feature by at least one unit within a group [so formed] would render the group 'identifiable' as 'possessed'. When this happens, we need to 'open up' the group and go for further exploration of the status of individual units of the group, possibly by sub-group(s) testing or by other means. Other possibility is that the group would be declared as 'passed', and consequently, it would mean that all constituent units within the group would be declared as 'passed' and 'at one go'! This interpretation is accepted for group testing schemes to work. When this latter phenomenon happens, the merit of group testing prevails over individual unit testing in terms of reduction in the required number of tests. For a given collection of units, we may adopt one-at-a-time testing or group testing with formation of suitable groups, or even a combination of the two strategies. As is mentioned above, the sole purpose is to minimize the number of GTs in such situations.

The above formulation looks deceptively simple! Hidden are probabilistic and combinatorial challenges. In this paper, we will discuss the issues related to combinatorial challenges only.

A Group Testing Design [GTD] is referred to as Non-adaptive or Hypergeometric Group Testing Design if one can specify *all* the group tests in the beginning and execute all of them and then analyze the results together for identification of the status of each unit in the collection. No more group tests are performed at any later stage in this formulation. Otherwise, a GTD is referred to as a Sequential GTD in which the GTs are executed in different stages, each time analyzing the cumulative results of the previous stages and deciding on the nature of GTs in the current stage. There are situations wherein Sequential GTs are prohibitive and one has to take recourse to Hypergeometric GTs only. In either case, the problem is referred to as a combinatorial group testing problem.

There is an impressive literature in the area of combinatorial group testing, even though there are many unresolved issues. We intent to review the existing literature with respect to both Hypergeometric and Sequential GTDs. In both the above formulations, it is assumed that we are dealing with a finite collection of units for testing purpose and these are amenable to group testing with the stipulations about 'possessed' and 'passed' as explained above.

To re-emphasize, the central problem of interest is to decide on the minimum number of group tests for a given number of units in a collection. Non-adaptive or Hypergeometric Group Testing is more challenging and highly non-trivial as a combinatorial problem, even in its simplest version! The rest of this paper mainly addresses the Hypergeometric GT problem. In other words, we are required to spell out all the necessary GTs beforehand and analyze the results and be able to ascertain the nature of each unit in the sample under consideration. The term 'non-adaptive' is coined by Hwang and Sos (1981). We refer to Du and Wang (2000) for an account of theoretical results in this area of research.

There is a remarkable similarity of this problem with what is popularly known as 'Counterfeit Coin Problem' [CCP]. A school-level popular puzzle is stated as : There are 8 coins - 7 are perfect and only one is counterfeit. In how many trials with a chemical balance [a two-pan balance], can one identify the CC ? It is tacitly assumed that there is no actual weighing involved in the whole process. It is interesting to note that one needs 3 trials in the absence of any knowledge about the behavior of the CC. If, however, it is known apriori that the CC is 'heavier' or 'lighter' than the rest, one can identify it in exactly 2 trials. This is possible either following a sequential search or going thru the Hypergeometric GT formulation. And yet a highly non-trivial problem [college-level puzzle ?] deals with 12 coins [including at the most one CC] and without any knowledge about the behavior of the CC. In this case, it turns out that exactly 3 Hypergeometric GTs resolve the problem. We leave it at this stage and only refer to the latest thoughts in this direction as dealt with in a paper [Rao et al (2006b)]. It may be noted that for the more interesting problem i.e., in the absence of any knowledge about the heavier/lighter status of the CC, a single-pan balance [known as Spring Balance] is not useful in a non-trivial sense and hence this is uninteresting.

2 Concepts and Illustrative Examples

We start with an example of $n = 6$ units/items and assume that it is known apriori that at the most one of these units may be 'possessed'. How many GTs are required to identify this unit? This has a solution with 4 GTs, as indicated below; but that may not be the best. [That means, fewer GTs might be available.] Here and all through this paper, the units are labeled as $1, 2, \ldots, n$. In the example below, we take $n = 6$.

$$GT(I) : (1, 2, 3); GT(II) : (1, 4, 5); GT(III) : (2, 4, 6); GT(IV) : (3, 5, 6)$$

A quick check is provided by the following argument, as suggested in Bush et al. (1984) who introduced the concept of $t - completeness$ of a GTD. Meanwhile, Saha et al. (1982) provided further results in this direction. Assuming that unit 1 is 'possessed', the GTs III, IV would turn out to be 'passed' and consequently, all the units captured by these two GTs i.e., $2, 3, 4, 5, 6$ would be declared as 'passed. Hence the identification of unit 1 as 'possessed' is complete and perfect. Similar is the argument with any other unit. What is the underlying property of this GTD ? A GTD is said to be t-complete if for any collection of t units from the total collection of n units [$n > t$], the 'union' of all GTs [of the GTD], each of which *excludes* all of these t units, when examined, includes all the *remaining* $(n-t)$ units of the collection. Once this happens, one can argue that the GTD has the property of t-completeness. This then induces the capability of identification of upto and including t 'possessed' units out of n units in the collection. In the above example, 1-completeness property holds for the suggested GTD and hence the claim.

Immediately after this completeness property was introduced, an improvement over it was suggested in Saha and Sinha (1981). It was claimed that the number of GTs could be cut down at least by one - if one started with the above solution! Herein the concept and property of 'Detecting Power of Order t' of a GTD - abbreviated as $DP(t)$ - was introduced by Saha and Sinha (1981). This latter concept has an easy interpretation and it has been found quite useful. We will elaborate it below.

We now turn back to the above example and this time we exclude $GT(IV)$. Can we still claim that the GTD is acceptable ? The answer is 'yes'. To see this, we may proceed as follows.

Note that a GTD involving m GTs and n units can be viewed as one generating an Incidence Matrix of order $m \times n$. We write $A = ((a_{ij}))$ where $a_{ij} = 1$, if the i^{th} GT includes the unit labeled $j; = 0$, otherwise; $i = 1, 2, ..., m; j = 1, 2, ..., n$. A GTD with incidence matrix A is said to have a detecting power of order 1, i.e., $DP(1)$ iff the columns of the matrix A are all distinct as vectors of order $m \times 1$. Note that upon implementation of the GTD, the outcome for each GT will be either 'possessed' [coded by a plus sign, say] or 'passed' [coded by a minus sign, say]. With m tests, the total number of possible outcome vectors is given by 2^m. If we assume that there is at the most one 'possessed' unit out of n units, then the zero-count will be identified by the result vector $(-, -, ..., -)$, an m-ple. All others [i.e., possibility of each one of the n units being identified as 'possessed'] must arise out of the remaining $2^m - 1$ result vectors. This establishes one of the early results which states that the minimum number of GTs m and the given number of units n are related by the inequality : $2^m \geq (1 + n)$, in case there is at most one 'possessed' unit; $2^m \geq [1 + n + n(n - 1)/2]$, in case there are at the most 2 'possessed' units and so on. It is readily seen that GTDs with $DP(1)$ and $DP(2)$ property serve as GTDs in situations wherein it is apriori known that there is at the most one or there are at the most two 'possessed' units. We will elaborate on these aspects later.

Remark 1. It thus transpires that for $2^{(k-1)} \leq n < 2^k$ and with at most one 'possessed' unit in this collection of n units, the minimum number of GTs needed for identification is exactly equal to k. This result has been known since the work of Renyi (1961) in a different context. See also Erdos et al. (1982, 1985). Its generalization for two 'possessed' units is a fairly recent observation.

In the example above, we have assumed existence of at the most one 'possessed' unit out of 6 units. Hence, $m = 3$ is just fine. The GTD suggested above corresponds to the A-matrix :

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

However, we may as well drop $GT(IV)$ and display the others in the form of the $A-$matrix as follows :

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The verification that this reduced GTD involving 3 GTs serves the purpose is straightforward. Both the above are GTDs with $DP(1)$ property.

Remark 2. We can even include one more unit, say unit labeled 7 and extend the matrix arising out of 3 GTs by one more column :(1 1 1). It will still pass as a GTD with $DP(1)$ property.

Remark 3. If it is known that there is *exactly* one 'possessed' unit, we can further extend the list by an 8^{th} unit and extend the matrix of 3 GTs by an addition of the column : (000). That means, the 8^{th} unit is kept on 'reserve' and when the 3 GTs [involving the remaining 7 units] indicate 'all passed' by signalling the result vector $(- - -)$, we conclude that the 8^{th} unit is 'possessed', without even testing it ! This also renders the GTD as one with $DP(1)$ property.

Remark 4. The above, by no means, suggests that the solution is unique with 3 GTs. Here is another solution with $DP(1)$ property involving 6 units and 3 GTs:

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Remark 5. If we decide to utilize 4 GTs, then we can enhance the number of units to be tested to a maximum of 15, again presupposing that there is at the most one 'possessed' unit in the collection. To accommodate 16 units, we need to assume that there is exactly one 'possessed' unit in the lot. The resulting GTD has $DP(1)$ property.

For two vectors of the same order and involving the elements $0, 1$, we define the 'Boolean Sum"[BS] as the vector addition subject to replacing $x + y$ by max (x, y). Thus, $(1101) + (1010) = (1111)$ is the BS. This notion can be extended to define the BS of any number of vectors.

Remark 6 . For the case of at the most 1 'possessed', the GTD must be so formed that incidence matrix A of order $m \times n$ necessarily has all distinct columns [in the sense of vectors]. This is the $DP(1)$ Property mentioned above. For the case of at the most 2 'possessed' units, A must have : Property (i)- all columns must be distinct, Property (ii) - BS arising out of all pairs of columns must be distinct. It is easy to argue that the Property (ii) implies the Property (i). Hence GTD with $DP(2)$ Property is also a GTD with $DP(1)$ Property.

3 Harder Combinatorial Problems for GTDs with DP(1)

Katona (1966) took up the following non-trivial extension of the problem of $DP(1)$ formulation of the GTD problem. We first illustrate the problem and the solution with an example. This time, again suppose there are 15 units with at the most one 'possessed unit. Our consideration for $DP(1)$ suggests that we need 4 GTs. We can easily display the underlying matrix A. At this point, we may mention that Das and Roy Choudhury (1987) had used the simple concept of Yates' factorial representation in a 2^m factorial experiment to develop the GTs. Here $m = 4$ and we use the standard letter-representation $[a, b, ab; c, ac, bc, abc; d, ad, bd, abd, cd, acd, bcd, abcd]$. These letter-combinations are identified as the 15 units in the order $1, 2, \ldots\ldots, 15$. Next, the GTs are formed by considering the letters - one at a time - and collecting all those units with the letter in common in their letter-representations. Thus the GTs are formed as follows [row 1 formed of use of the letter a, and so on]:

$$A = \begin{pmatrix} 1&0&1&0&1&0&1&0&1&0&1&0&1&0&1 \\ 0&1&1&0&0&1&1&0&0&1&1&0&0&1&1 \\ 0&0&0&1&1&1&1&0&0&0&0&1&1&1&1 \\ 0&0&0&0&0&0&0&1&1&1&1&1&1&1&1 \end{pmatrix}.$$

It may now be noted that each GT involves exactly 8 of the units. Katona (1966) addressed the question of cutting down the size of the GTs to a lesser number, say 7, or 6, or 5, or even 4 - without sacrificing on the property of $DP(1)$. In the process, the number of GTs will increase. The question is to develop a strategy to resolve this problem. We first show the solutions to each of the GT sizes shown above.

(i) GTD with DP(1) Property - 5 GTs - each of exact size 7

$$A = \begin{pmatrix} 1&0&0&0&0&1&0&0&1&1&1&0&1&1&0 \\ 0&1&0&0&0&1&1&0&0&1&0&1&0&1&1 \\ 0&0&1&0&0&1&1&1&0&0&1&0&1&0&1 \\ 0&0&0&1&0&0&1&1&1&0&1&1&0&1&0 \\ 0&0&0&0&1&0&0&1&1&1&0&1&1&0&1 \end{pmatrix}.$$

(ii) GTD with DP(1) Property - 5 GTs - each of exact size 6

$$A = \begin{pmatrix} 1&0&0&0&0&1&0&0&0&1&1&0&0&1&1 \\ 0&1&0&0&0&1&1&0&0&0&1&1&0&0&1 \\ 0&0&1&0&0&0&1&1&0&0&1&1&1&0&0 \\ 0&0&0&1&0&0&0&1&1&0&0&1&1&1&0 \\ 0&0&0&0&1&0&0&0&1&1&0&0&1&1&1 \end{pmatrix}.$$

(iii) GTD with DP(1) Property - 5 GTs - each of exact size 5

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

(iv) GTD with DP(1) Property - 6 GTs - each of exact size 4

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

We now state the existence result, as established by Katona (1966). Note that we are asking for a GTD possessing $DP(1)$ property and having GTs, each of size not exceeding a given number $k \leq n/2$. This is as good as asking for a matrix of order $m \times n$ for a given n, by suitably choosing the smallest integer m with the property that no row sum exceeds the given number k, along with the stipulation that all column vectors of order $m \times 1$ are distinct. In application, m stands for the number of GTs for n units. The existence of such a matrix with the desired property is ensured if and only if

$$(i) \;\; mk = \sum_i i s_i; (ii) \;\; n = \sum_i s_i; (iii) \;\; s_i \leq m_{c_i}.$$

The construction depends on use of cyclic permutations of suitably chosen 'initial' sets of columns. The above solutions have been reached at by using the principle of cyclic permutation.

4 GTDs with Further Restrictions

We now examine the solution to the GTD for $n = 15$ units in $m = 4$ GTs with $DP(1)$ property from a different angle. We reproduce the solution below.

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

It is found that not all the 15 units are equally included in the GTs. There are 4 units each included only once, 6 units each included twice, 4 units each included thrice and one unit included in all GTs. What if we want to restrict the 'inclusion size' of the individual units? We may wish to make it 'uniform' for all units, or, at least, put an upper limit for all. In terms of the incidence matrix A, we are now referring to the individual column totals and wish to restrict each column total to a number less than the number of GTs i.e., the number of rows of the matrix. For 15 units, if the column totals are not to exceed 2 each, we need 5 GTs. See solution (iii) in the previous section. For not exceeding 3 each, we still need 5 GTs. See solution (ii) in the previous section. All these refer to GTs with $DP(1)$ property. Katona's work relates to the row-total restrictions. When both restrictions apply, one has to develop the GTs a bit carefully.

5 GTDs with DP(2) Property : Design Considerations

The rest of the paper deals with GTDs having DP(2) property. In effect, there are n units and it is apriori known that at the most 2 of these units are 'possessed'. We are required to suggest minimum number of non-adaptive GTs in order to ascertain the status of all the n units. Another approach would be to maximize the number of units n that can be accommodated under this formulation for a given number of GTs, say m. This latter formulation was adopted by Raghavarao and his co-authors [1987a, 1987b, 1990, 1994]. It has been asserted by Saha and Sinha (1981) that for a collection of $n \leq 5$ units, a GTD with $DP(2)$ property calls for exactly n GTs. That means, for $n \leq 5$, one-at-a-time testing can not be replaced by any GT in case 2 units are likely to be 'possessed.

The $DP(2)$ property with minimum number of GTs is harder to achieve. At this stage, Remark 6 is relevant. It calls for the properties of the incidence matrix A [of order $m \times n$ where m denotes the number of GTs and n denotes the number of units under group testing]: Property (i)- all columns of A must be distinct, Property (ii) - BS arising out of all pairs of columns of A must be distinct. This necessarily implies that m and n are related by the inequality : $2^m \geq 1+n+n_{c_2}$. However, Property (ii) is hard to achieve with the smallest conceivable value of m. For example, for $n = 5$, we need $m = 5$ GTs which is as good as individual testing. For $n = 6$, we need $m = 5$ GTs with $DP(2)$ property and this is only marginally better than individual testing.

We will now describe below the results of Raghavarao and his collaborators [Weideman, Vakil, Parnes].

Result 1. For a given number m of GTs having $DP(2)$ property, the maximum number of units n to be accommodated in the GTD can not exceed $m(m+1)/6$, provided each unit is allowed to be included in 2 or 3 GTs.

The actual construction of GTDs rests on use of subclasses of Group Divisible Designs with $\lambda_1 = 0, \lambda_2 = 1$. Vide Raghavarao (1971).

Here is an example for $m = 6, n = 7$

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Here is another example for $m = 8, n = 12$.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark 7. It may be noted that in the above two examples, no two GTs have more than one unit in common. In fact, this property is shared by all GTDs covered under Result 1.

Remark 8. It is further to be noted that the units do appear twice or thrice in the GTDs. The case of exactly two appearances of each unit in the GTDs was studied by Vakil et al (1990).

Remark 9. Weideman-Raghavarao (1987b) specifically discussed the cases of $m \neq 0$, mod(6) and $m \neq 2$, mod(6), by use of appropriate subclasses of group divisible designs.

Remark 10. In a subsequent paper, Vakil and Parnes (1994) mentioned about the cases of $m = 1, 3, 5 \mod(6)$. We close this section with an example from this paper.

Example of a GTD having $DP(2)$ property involving $n = 9$ units and $m = 7$ GTs such that (i) no unit appears in more than 3 GTs, and (ii) each GT involves exactly 3 units.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

6 GTDs with DP(2) Property : Further Considerations

There have been some attempts towards construction of GTDs with DP(2) property, using graphs and other combinatorial systems. We refer to Rao et al. (2006a) [RRS2006 - for brevity] for details. We first state some basic results which have been otherwise discussed before. For formal proofs, we refer to RRS2006.

Result 2 [RRS2006]: A necessary and sufficient condition for a matrix $A_{m\times n}$ to be the design matrix of a *d-complete* design is that for $j = 1, 2, \cdots, n$, j^{th} column α_j of A satisfies

$$\alpha_j \not\leq \alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_d} \;\; \forall \, j_1, j_2, \cdots, j_d \neq j \;,$$

where the summation is Boolean.

Result 3 [RRS2006]: A necessary and sufficient condition for a matrix $A_{m\times n}$ to be the design matrix of a GTD that has the property $DP(d)$ i.e., *detecting power of order* d is that

$$Ax = Ay \Longrightarrow x = y$$

for all Boolean vectors x, y of order $n \times 1$ and of weight $\leq d$, where the weight of a Boolean vector is the number of 1's in it.

For the case of GTDs having $DP(1)$ property, we have more or less hinted on the following result before.

Result 4 [RRS2006]: Suppose $2^{(p-1)} \leq n < 2^p$ and let $A_{p\times n}$ be the matrix with j^{th} column as the $p - bit$ binary representation of the number j, for $j = 1, 2, \cdots, n$. Then the GTD with A as the design matrix has 1-detecting power involving n units and p tests.

Corollary 1 . If it is known that there is exactly one defective the above design can also work for $n = 2^p$, with the design matrix obtained by appending a null column to the above matrix A.

Result 5 [RRS2006]: Let $A_{m\times n}$ be the matrix with j^{th} column as the m-bit binary representation of the number j, for $j = 1, 2, ..., n$. Then the GTD with A as the design matrix has $DP(1)$ property for the values of n satisfying $2^{m-1} \leq n < 2^m$.

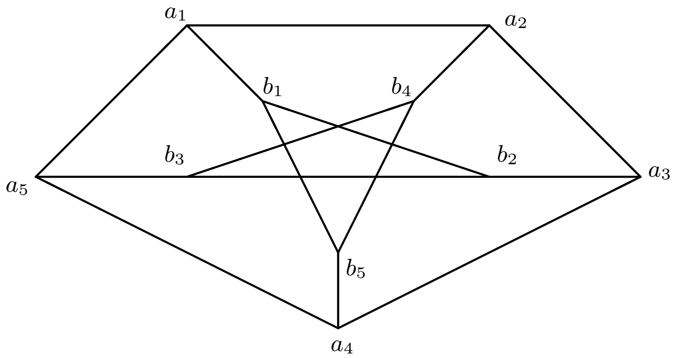
Result 6 [RRS2006]: For the case of $n = 2^q$ and $d = 2$, any GTD needs at least $2q$ tests.

We now present some results indicating use of graphs in this context. We refer to any standard book on graph theory for the basic definitions. Vide Harary (1988), for example. Recall Remark 6 with reference to GTDs with $DP(2)$ property.

Result 7 [RRS2006]: The incidence matrix of any graph with no multiple edges and containing neither triangles nor quadrilaterals is a design matrix of a GTD for $d = 2$.

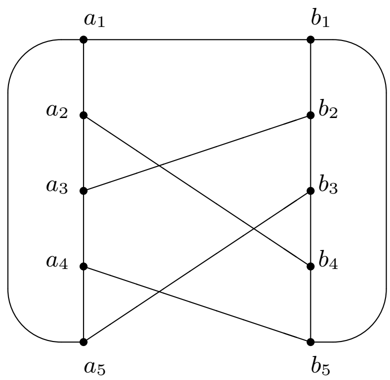
Such graphs are referred to as GTD-graphs. GTD-graphs based on Petersen Graphs [PGs] and Generalized PGs have been studied in RRS2006. Below we illustrate this with an example from RRS2006.

The following graph $P(10, 15)$ with 10 vertices and 15 edges is known as Petersen Graph [PG].



This graph has no triangle and no quadrilateral, and each vertex has degree 3. By appealing to the Result 6 [RRS2006] above, we can claim that the above PG results in a GTD involving 15 units and 10 GTs, having $DP(2)$ property. The vertices represent the 10 group tests and the edges represent the 15 units of study. Each vertex generates exactly three edges and these represent the three units contained in the GT representing that vertex. [This result is not the best known. Weideman and Raghavarao(1987) furnish an example of a GTD with $DP(2)$ property for 15 units in 9 GTs.]

The Petersen Graph shown above has an equivalent graphical representation given below.



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From this, it is possible to construct Generalized Petersen Graphs $P(2n, 3n)$ with $2n$ vertices and $3n$ edges for $n \geq 5$, except for $n = 6$ as there is no such graph for $n = 6$. We refer to RRS2006 for the constructional details.

Again, it has been demonstrated that this GPG has neither a triangle nor a quadrilateral and hence it can be used as a GTD with DP(2) property.

Further, considering two $P(2n, 3n)$ graphs G_1 and G_2, a method of 'combining' the two has also been explained in RRS2006. The suggested method enables one to retain the $DP(2)$ property of the resulting GPG. More than that, a set of rules has been given for identifying the defective items from the results of a GTD based on GPGs.

Vakil (1990) used design-based techniques to construct optimal GTDs in certain cases which include a GTD for 175 units and 50 tests for the case $d = 2$. RRS2006 provided a GTD for the same parameter values using GPGs. We do not enter into any details.

7 GTDs with DP(2) Property : Sequential Two-stage Multi-state GTDs

In this section we briefly discuss some results related to adaptive i.e., sequential group testing designs with DP(2) property. These are to be found in Das and Roychoudhuri (1987) and Rao et al [RRS2006].

Result 8 [**RRS2006**] : For the case $d = 2$ and for n of the form p^q , there exists a two-stage adaptive GTD with pq tests in the first stage and at most $q - 1$ tests in the second stage.

Proof. We number the $n(= p^q)$ units serially from 0 to $p^q - 1$ and represent them as q-digit numbers with base p. For $i = 1, 2, \cdots, q$ and $j = 0, 1, 2, \cdots, p - 1$ form groups $G_j^{(i)}$ where $G_j^{(i)}$ contains all those items with j as the i^{th} digit in their representation. Observe that for each $i, G_i^{(0)}, G_i^{(1)}, \cdots, G_i^{(p-1)}$ form a partition of the p^q items, each containing p^{q-1} items.

In the first stage we test all the pq groups $G_j^{(i)}$. If no test shows positive, we conclude that there is no defective item. Otherwise for each i either one or two of the values of j, $G_j^{(i)}$ results in positive. However if for all i only for one value of j, $G_j^{(i)}$ is positive, then there exists exactly one possessed unit and it can be identified easily. If there is at least one i for which there are two values of j such that $G_j^{(i)}$ tests positive, then there are two defective items and we go to second stage.

Observe that for each i there can be at most two values of j, for which $G_j^{(i)}$ is positive. Let i_1 and i_2 denote these values of j, if there are two values; and for some i, if there is a single is positive, we will give both i_1 and i_2 the same value j. This gives us $2q$ digits having two digits (need not be distinct) for each of the q digit positions giving a maximum of 2^q numbers that represent the 2^q units that contain the two possessed units. Without loss of generality let i_1 and i_2 be different for $i = 1$. From the above 2^q units form the group G with those items having numbers with first digit as 1_1. Observe that this group contains $2^{(q-1)}$ units of which exactly one unit is possessed and that can be identified using $q1$ tests, by Corollary above. This in turn identifies the q digits that represent its number. Then the remaining q digits give the number of the other defective item. Thus we need $q - 1$ tests at the second stage making the total number of tests needed $pq + q - 1$. This completes the proof.

In passing, we may mention the following interesting result due to Das and Roy Choudhury (1987).

Result 9 [$\mathbf{D - RC}$]. There exists a two-stage adaptive GTD which needs at most $(k + 3)$ tests for $k(k + 1)/2$ units containing at the most 2 possessed units.

The following result is taken from RRS(2006a).

Result 10 [**RRS2006**]. There exists a multistage adaptive GTD which needs a maximum of $2p$ group tests to identify all the possessed units from a given set of $n = 2^p$ items containing a maximum of 2 possessed units.

Proof. Number the $n(= 2p)$ units serially from 0 to $2^p - 1$ and represent them as $p - bit$ binary numbers. Form groups $G_1^{(0)}$ and $G_1^{(1)}$, where $G_1^{(j)}$, for $j = 0, 1$, contains all those items with j as the first bit in their binary representations. Observe that $G_1^{(0)}$ and $G_1^{(1)}$ form a partition of 2^p units each containing $2^p - 1$ units. Conduct the first pair of tests on $G_1^{(0)}$ and $G_1^{(1)}$. If both result in negative, we can conclude that there is no defective. If both test results are positive then there are two defective items. However, if only one

result, say $G_1^{(j_1)}$, is positive then there could be one or two possessed units, in which case we conduct tests on the next pair $G_2^{(0)}$ and $G_2^{(1)}$, where $G_2^{(j)}$ for $j = 0, 1$, contains all those units with j_1 as the first bit and j as the second bit in their binary representations. Thus we continue testing pairs as long as exactly one of them results in positive. Let $(r+1)^{st}$ be the first pair of tests that results in both positive. At this stage, we have the following situation.
The first r pairs of tests resulted in $G_1^{(j_1)}, G_2^{(j_2)} \cdots, G_r^{(j_r)}$ positive and at the $(r+1)^{st}$ stage both $G_{r+1}^{(0)}$ and $G_{r+1}^{(1)}$ resulted in positive. Now, consider the two groups, the first group containing all those items having representation with the first $(r+1)$ bits as $j_1, j_2, \cdots, j_r, 0$ and the second group containing all those items having the first $(r+1)$ bits as $j_1, j_2, \cdots, j_r, 1$. Observe that each of these groups contains exactly $2^{(p-r-1)}$ units with exactly one possessed unit which can be identified using $(p-r-1)$ tests by Corollary above. Thus at this last stage we need $2^{(p-r-1)}$ tests to identify both the defective items. Hence, as $2(r+1)$ tests are used in the first $(r+1)$ stages, a total of $2p$ tests are enough to identify all the defective items.

8 Comparison of GTDs with DP(2) Property

It so happens that the study of GTDs P(2) property has turned out to be exciting and intriguing. By now we find three different approaches towards construction of GTDs with DP(2) property : Non-adaptive GTDs by Raghavarao-Weideman-Vakil-Parnes (R-W-V-P), Two-stage adaptive GTDs by Das-RoyChoudhury (D-RC) and Two-stage/Multi-stage GTDs by Rao-Rao-Sinha (R-R-S). Their results overlap and, at times, one improves over the other!

Here are some illustrative examples.

Example 1. Consider $n = 25$. Here R-W-V-P provides a non-adaptive solution with $m = 12$, D-RC provides a two-stage adaptive solution with $m = 10$ and R-R-S provides a multi-stage solution with $m = 11$. We describe the D-RC solution in the Appendix.

Example 2. Consider $n = 32$. Here R-W-V-P provides a non-adaptive solution with $m = 11$, D-RC provides a two-stage adaptive solution with $m = 11$ and R-R-S provides a Multi-stage solution with $m = 10$. We describe the R-R-S solution in the Appendix.

Example 3. Consider $n = 35$. Here R-W-V-P provides a non-adaptive solution with $m = 14$, D-RC provides a two-stage adaptive solution with $m = 11$ and R-R-S provides a multi-stage solution with $m = 13$.

Remark 11. As expected, non-adaptive GTDs tend to incorporate slightly larger number of GTs. Such designs, by nature, are by far more interesting and difficult to reach out with relatively large number of test units.

9 Concluding Remarks

There have been other approaches to resolve the identification problems with reference to $DP(2)$ property. Use of Petersen Graphs and Generalized Petersen Graphs has been mentioned before. We may also mention about use of Tactical Configurations, including Steiner Triple Systems and Hanani′s Quadruple and Quintuple Systems in Hanani (1961) and in Raghavarao (1971), Raghavarao and Weideman (1987a, 1987b) and Vakil et al. (1990).

An altogether different but analogous identification problem is related to the above formulation of GT. Assume that there is an infinite population of units - each being passed or possessed with respect to a qualitative feature. We are interested in ascertaining the unknown proportion (say, p) of possessed units. We start with a random sample of, say n units and wonder if we should test them individually [one-at-a-time -leading to usual binomial testing] or employ some sort of GT scheme to cut down the number of tests (which is otherwise n). For the case of $n = 2$ units, as against one-at-a-time testing, one may consider a sequential testing procedure such as (a) testing both the units together with a chance of $(1-p)^2$ of being declared as 'passed' as a GT. Otherwise, one would test unit labeled, say 1 next and stop if it tested 'passed'. That would imply that unit labeled 2 would not need any testing but could be declared as 'possessed'. On the other hand, if the unit labeled 1 tested 'possessed', one would still need to test unit labeled 2 to see if this was 'possessed' or 'passed'. This simple-minded alternative strategy would lead to the number of GTs to be $1, 2, 3$ with respective chances q^2, pq, p where $q = 1 - p$. This would mean that expected number of GTs for this strategy would be $q^2 + 2pq + 3p$ and this would be less than 2 [the number of tests in one-at-a-time testing plan] iff $p < (3 - \sqrt{5})/2 = 0.3820$! This result is due to Ungar (1960). There is a long history for this type of GT strategies, known as Binomial Group Testing Strategies [BGTS]. The idea is to start with one master sample or even a fewer group tests and execute the search and examine the results and decide on the next courses of action [in terms of formation of new group tests] until the status of each unit in the collection is ascertained. Then the question of 'inference' on p is to be handled in a probabilistic sense. One could also stop after ascertaining the status of each unit as above. That would correspond to Sequential Group Testing Strategy [SGTS] and it is combinatorial in nature. We have discussed some such SGTS above. We did not make any attempt to discuss BGTS which is itself extremely fascinating area of research. Some of the early references are Dorfman (1943), Sterrett (1957), Sobel and Groll (1959, 1966), Sobel (1960, 1968), Kumar and Sobel (1971).

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Appendix

1. Illustration of D-RC Solution of Example 1

Here we have 25 units which include at most 2 possessed units. Following D-RC, we claim that in 10 adaptive two-stage GTs, we can identify the possessed units. Towards this, we list these 25 units as follows.

unit no.	1	2	3	4	5	6	7	8	9	10	11	12	13
digit	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)	(1, 7)	(1, 8)	(2, 3)	(2, 4)	(2, 5)	(2, 6)	(2, 7)	(2, 8)

unit no.	14	15	16	17	18	19	20	21	22	23	24	25
digit	(3, 4)	(3, 5)	(3, 6)	(3, 7)	(3, 8)	(4, 5)	(4, 6)	(4, 7)	(4, 8)	(5, 6)	(5, 7)	(5, 8)

Next, suppose the units numbered 4 and 17 are possessed. Note that these two units have the representations $(1, 5)$ and $(3, 7)$ respectively in the above listing. We will explain the steps to be followed to identify these units.

At the first stage, we will form 8 GTs using the integers 1 to 8, appearing in the listing of the 25 units. For example, $GT(1)$ is derived by considering all those units whose representations involve the integer 1. That means, explicitly written,
$GT(1) : [1, 2, 3, 4, 5, 6, 7]; GT(2) : [1, 8, 9, 10, 11, 12, 13];$
$GT(3) : [2, 8, 14, 15, 16, 17, 18]; GT(4) : [3, 9, 14, 19, 20, 21, 22];$
$GT(5) : [4, 10, 15, 19, 23, 24, 25]; GT(6) : [5, 11, 16, 20, 23];$
$GT(7) : [6, 12, 17, 21, 24]; GT(8) : [7, 13, 18, 22, 25].$

We will test these GTs all at a time and naturally, we will observe that the GTs : $GT(1), GT(3)$,
$GT(5), GT(7)$ turn out to be possessed. At the next stage, it will be enough to test only two units, suitably selected. For this, using the possessed GT identifiers viz., $(1, 3, 5, 7)$, we will form six different pairs,
viz., $[(1, 3), (1, 5), (1, 7), (3, 5), (3, 7), (5, 7)]$. Note that these, in their turn, correspond to six different units. We will choose any two units with one letter common in their representations, such as $2 = (1, 3)$ and $4 = (1, 5)$. Naturally, upon testing these two units, the result will turn out to be [passed, possessed]. Once the identification of the unit/pair $4 = (1, 5)$ is done, the complimentary unit/pair $17 = (3, 7)$ will correspond to the other possessed unit. Hence the identification of units 4 and 17 will be perfect and complete. In case the pair $[(1, 3), (1, 7)]$ will be chosen for testing, the result will be [passed, passed]. Then we will first identify the only other unit having integer 1 in its representation, viz., unit $4 = (1, 5)$ as possessed. The other possessed unit will correspond to its compliment viz., $17 = (3, 7)$. Similar considerations apply for any other choices as well.

2. Illustration of RRS Solution of Example 2

Here we have $32 = 2^5 = 2^n$ units with $n = 5$. We number these 32 units serially from 0 to 31 and represent these as five digited binary numbers as follows:

unit no.	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
digit 1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
digit 2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
digit 3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
digit 4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
digit 5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

unit no.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
digit 1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
digit 2	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
digit 3	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
digit 4	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
digit 5	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Next, we form the groups $G_1^{(0)}$ with the units having the first binary digit 0 in their number, that is, the set of all units with even numbers and $G_1^{(1)}$ with the units having the first binary digit 1 in their number, that is, the set of all units with odd numbers and perform two first stage tests on these two groups. Thus we utilize 2 GTs to strat with.

Case 1: If both results are negative then there is no possessed unit.

Case 2: If both results are positive then there are two possessed units and exactly one of them is in $G_1^{(0)}$, that is, among even numbered units and the other is in $G_1^{(1)}$, that is, among odd numbered units. As each of these groups contain $2^{(n-1)} = 16$ units with exactly one possessed unit, by Corollary 1 we need $(n - 1) = 4$ second stage tests to identify the possessed unit from each group needing $2(n - 1) = 8$ tests at the second stage. So, totally we need 2 in the first stage plus $2(n-1)$ tests in the second stage, totally $2n = 10$ tests.

Case3: If one group tested positive and the other negative, the number of possessed units may be one or two. Let us assume that the group $G_1^{(0)}$ result is negative and the group $G_1^{(1)}$ result is positive. Denote the superfix digit 1 of this group by d_1. That means one or two possessed units are there and among the units with 1 as the first digit of their numbers, that is, odd numbered units. Now, form the two groups $G_2^{(0)}$ with the units having the first binary digit d_1 and the second binary digit 0 in their number, and $G_2^{(1)}$ with the units having the first binary digit d_1 and the second binary digit 1 in their number. Test these two groups. Thus in this case we need 2 tests at the second stage.

In this case there are two possibilities. The first being both the tests result in positive in which case, each group has $2^{(n-2)} = 8$ units and exactly one possessed unit; by Corollary 1 we can find it using $(n-2) = 3$ tests. Thus we need $2(n-2)$ tests at the third stage to identify both the possessed units. So, again in this case we used 2 tests at the first stage, 2 in the second stage and $2(n-2)$ at the third stage totaling $2n = 10$ tests. Alternately, if only one of the tests result in positive, say the group $G_2^{(0)}$ results positive, we denote the superfix digit 0 of this group by d_2 and form two groups $G_3^{(0)}$ with the units having the first binary digit d_1, the second binary digit d_2 and the third binary digit 0 in their number, and $G_3^{(1)}$ with the units having the first binary digit d_1, the second binary digit d_2 and the third binary digit 1 in their number. Test these two groups. Thus in this case we need 2 tests at the third stage.

Thus we proceed until we get at some stage, say $(r+1)^{st} (\leq n)$, both the groups test positive or we finish all the n stages with only one group test resulting in positive. In the first case, we will be able to identify the two possessed units using $2n$ (2 tests at each of r stages plus 2 times $(n-r-1)$ tests at the $(r+1)^{st}$ stage) tests as described above. Alternately if only one group, say $G_n^{(1)}$ results in positive, we denote the superfix digit 1 of this group by d_n. In this case there is only one possessed unit and its number in the binary system is $d_n d_{n-1} \cdots d_2 d_1$. Again, we used $2n$ (2 tests at each of n stages)= 10 tests to identify this single possessed unit.

As an example, let us suppose that units with numbers 4 and 17 out of $32 = 2^n, n = 5$ units be possessed. Below it is shown how theses two possessed units can be identified using $2n = 10$ tests. In the following, digit numbers in a binary number are from **right to left**.

In the first stage we form groups $G_1^{(0)}$ with the set of all units having 0 as the first digit of their numbers, that is, the set of all even numbered units of which unit number 4 is a member and hence the test on this group results in positive. Similarly, we form $G_1^{(1)}$ with the set of all units having 1 as the first digit of their numbers, that is, the set of all odd numbered units of which unit number 17 is a member and hence the test on this group also results in positive. Thus each of these two groups contain $2^{(n-1)} = 2^4 = 16$ units and contain exactly one possessed unit. We illustrate (using Corollary 1) how to identify the possessed unit from $G_1^{(0)}$ using $(n-1) = 4$ tests. Similar procedure can be used to identify the possessed unit from the second group $G_1^{(1)}$.

Form groups $G_2^{(0)}$ with units $\{0, 4, 8, 12, 16, 20, 24, 28\}$ (with numbers having first two digits 00) and $G_2^{(1)}$ with units $\{2, 6, 10, 14, 18, 22, 26, 30\}$ (with numbers having first two digits 10). Notice that these two groups form a partition of $G_1^{(0)}$. Test one of these two groups, say $G_2^{(0)}$. As unit number 4 is in $G_2^{(0)}$, it will test positive and therefore we can conclude $G_2^{(1)}$ does not contain any possessed unit.

Form groups $G_3^{(0)}$ with units $\{0, 8, 16, 24\}$ (with numbers having first three digits 000) and $G_3^{(1)}$ with units $\{4, 12, 20, 28\}$ (with numbers having first three digits 100). Notice that these two groups form a partition of $G_2^{(0)}$. Test one of these two groups, say $G_3^{(0)}$. As unit number 4 is not in $G_3^{(0)}$, it will test negative and hence we can conclude that the other group $G_3^{(1)}$ contains the possessed unit.

Form groups $G_4^{(0)}$ with units $\{4, 20\}$ (with numbers having first four digits 0100) and $G_4^{(1)}$ with units $\{12, 28\}$ (with numbers having first four digits 1100). Notice that these two groups form a partition of $G_3^{(1)}$. Test one of these two groups, say $G_4^{(0)}$. As unit number 4 is in $G_4^{(0)}$, it will test positive and hence $G_4^{(1)}$ does not contain any possessed unit.

Finally, form groups $G_5^{(0)}$ with unit $\{4\}$ (with numbers having (first) five digits 00100) and $G_5^{(1)}$ with unit $\{20\}$ (with numbers having (first) five digits 10100). Notice that these two groups form a partition of $G_4^{(1)}$. Test one of these two groups, say $G_5^{(0)}$. As unit number 4 is in $G_5^{(0)}$, it will test positive and hence $G_4^{(1)}$ does not contain any possessed unit. So we have identified our single possessed unit 4 from the group $G_1^{(0)}$ using using 4 tests after the first stage.

Similarly, we can identify the single possessed unit from the group $G_1^{(1)}$ in another 4 tests. These 8 tests together with the 2 tests in the first stage constitute a total of $2n = 10$ tests using which we could identify the two possessed units.

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