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# Some pairwise additive cyclic BIB designs

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**Abstract**

The existence of additive balanced incomplete block (BIB) designs and pairwise additive BIB designs with $\lambda = 1$ has been discussed through direct and recursive constructions in Matsubara et al. (2006, 2013, 2014) and Sawa et al. (2007). In this paper, cyclic designs are considered and then the existence of pairwise additive cyclic BIB designs is investigated for $\lambda = 1$.

*Key words:* Incidence matrix; BIB design; Additive cyclic BIB design; Pairwise additive cyclic BIB design; Nested BIB design; Perpendicular array; Cyclic difference matrix; Cyclic relative difference family.

# 1 Introduction

A *balanced incomplete block* (BIB) *design* is a system $(V, \mathcal{B})$ with $v$ points and $b$ blocks each containing $k$ different points, each point appearing in $r$ different blocks and any two different points appearing in exactly $\lambda$ blocks (Raghavarao, 1988). This is denoted by BIBD$(v, b, r, k, \lambda)$ or B$(v, k, \lambda)$. Let $\boldsymbol{N} = (n_{ij})$ be the $v \times b$ incidence matrix of a BIB design, where $n_{ij} = 1$ or 0 for all $i$ $(= 1, ..., v)$ and $j$ $(= 1, ..., b)$, according as the $i$th point occurs in the $j$th block or otherwise. Hence the incidence matrix $\boldsymbol{N}$ satisfies the conditions: (i) $\sum_{j=1}^{b} n_{ij} = r$ for all $i$, (ii) $\sum_{i=1}^{v} n_{ij} = k$ for all $j$, (iii) $\sum_{j=1}^{b} n_{ij} n_{i'j} = \lambda$ for all $i, i'$ $(i \neq i') = 1, ..., v$.

For a BIB design $(V, \mathcal{B})$, let $\sigma$ be a permutation on $V$. For a block $B = \{v_1, \ldots, v_k\} \in \mathcal{B}$ and a permutation $\sigma$ on $V$, let $B^\sigma = \{v_1^\sigma, \ldots, v_k^\sigma\}$. When $\mathcal{B} = \{B^\sigma | B \in \mathcal{B}\}$, $\sigma$ is called an *automorphism* of $(V, \mathcal{B})$. If there exists an automorphism $\sigma$ of order $v = |V|$, then the BIB design is said to be *cyclic*. For a cyclic $\mathrm{B}(v, k, \lambda)$, the set $V$ of points can be identified with $Z_v = \{0, 1, \ldots, v-1\}$. The *block orbit* containing $B \in \mathcal{B}$ is a set of distinct blocks $B + i = \{v_1 + i, \ldots, v_k + i\}$ (mod $v$) for $i \in Z_v$. A block orbit is said to be *full* or *short* according as whether $|\{B + i | 0 \le i \le v-1\}| = v$ or not. If a short block orbit contains a block $\{0, v/k, 2v/k, \ldots, (k-1)v/k\}$, then it is said to be *regular*. Choose an arbitrary block from each block orbit and call it an *initial block*. Then any $\mathrm{B}(2t, 2, 1)$, $t \in \mathcal{N}$, must have a short initial block $\{a, a+t\}\mathrm{PC}(t)$ with an arbitrary $a \in Z_{2t}$. This fact will be used in Sections 6 and 7. Throughout the paper, $\mathrm{PC}(s)$ means a partial cycle of order $s$, i.e., only $0, 1, \ldots, s-1$ are to be added to the initial block.

Let $s = v/k$, where $s$ need not be an integer unlike other parameters. A set of $\ell$ $\mathrm{BIBD}(v, b, r, k, \lambda)$ is called $\ell$ *pairwise additive BIB designs*, denoted by $\ell$ $\mathrm{PAB}(v, k, \lambda)$, if $\ell$ corresponding incidence matrices $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_\ell$ $(2 \le \ell \le s)$ of the BIB design satisfy that $\boldsymbol{N}_{i_1} + \boldsymbol{N}_{i_2}$ is the incidence matrix of a $\mathrm{BIBD}(v^* = v = sk$, $b^* = b$, $r^* = 2r$, $k^* = 2k$, $\lambda^* = 2r(2k-1)/(sk-1))$ for any distinct $i_1, i_2 \in \{1, 2, ..., \ell\}$. When $\ell = s$, this is called *additive BIB designs* (Matsubara et al., 2006 and Sawa et al., 2007), denoted by $\mathrm{AB}(v, k, \lambda)$. Furthermore, in $\ell$ $\mathrm{PAB}(v, k, \lambda)$, if $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_\ell$ are cyclic and the $j$th initial block of $\boldsymbol{N}_{i_1} + \boldsymbol{N}_{i_2}$ is a set-union of the $j$th initial blocks of $\boldsymbol{N}_{i_1}$ and $\boldsymbol{N}_{i_2}$ for any distinct $i_1, i_2 \in \{1, 2, ..., \ell\}$ and $1 \le j \le \lceil b/v \rceil$, then the $\ell$ $\mathrm{PAB}(v, k, \lambda)$ is said to be *cyclic*, denoted by $\ell$ $\mathrm{PACB}(v, k, \lambda)$, where $\lceil x \rceil$ means the smallest integer $y$ such that $x \le y$. When $\ell = s$, this is called *additive cyclic BIB designs*, denoted by $\mathrm{ACB}(v, k, \lambda)$.

In $\mathrm{PAB}(v, k, \lambda)$, it is known (Sawa et al., 2007) that $2\lambda/(k-1)$ is a positive integer, which implies $k = 2$ or $3$ when $\lambda = 1$. Some classes of $\ell$ $\mathrm{PAB}(v, k, \lambda)$ have been constructed by use of direct and recursive methods in Matsubara et al. (2006, 2013, 2014) and Sawa et al. (2007). In particular, it has been shown (Matsubara and Kageyama, 2014) that there are 3 $\mathrm{PAB}(v, 2, 1)$ for any $v \ge 6$. However, for example, the existence of $\ell$ $\mathrm{PAB}(12, 2, 1)$ with $\ell \in$

$\{4, 5, 6\}$ is not known in literature.

The purpose of this paper is devoted to provide some constructions of 2 PACB$(v, 2, 1)$ and to show the existence of 2 PACB$(v, 2, 1)$ for (i) any odd integer $v(\geq 5)$ such that $\gcd(v, 9) \neq 3$ and (ii) $v = 2^m t$ with any integer $m(\geq 2)$ and any odd integer $t(\geq 1)$ such that $m \not\equiv 1 \pmod 4$ and $\gcd(t, 27) \neq 3, 9$, and to prove the nonexistence of $\ell$ PACB$(v, 2, 1)$ for $(\ell, v) = (5, 12), (6, 12), (2, 4m + 2)$ with any positive integer $m$.

# 2 Preliminaries

In this section, some arrays with four rows will be introduced for direct and recursive constructions of 2 PACB$(v, 2, 1)$, and some properties of the arrays will be discussed.

A *nested BIB design*, denoted by NB$(v; b_1, b_2; k_1, k_2)$, is a triple $(V, \mathcal{B}_1, \mathcal{B}_2)$ with $v$ points, $|V| = v$, and two systems of blocks, $|\mathcal{B}_i| = b_i$, $i = 1, 2$, such that (i) the first system is nested within the second, i.e., each block in $\mathcal{B}_2$ is partitioned into $e$ subblocks of size $k_1$ and the resulting subblocks form $\mathcal{B}_1$, here, $b_1 = eb_2$ and $k_2 = ek_1$, (ii) $(V, \mathcal{B}_i)$ is a BIB design with $v$ points and $b_i$ blocks of $k_i$ points each (cf. Preece, 1976). A nested BIB design $(V, \mathcal{B}_1, \mathcal{B}_2)$ is said to be *cyclic*, denoted by CNB$(v; b_1, b_2; k_1, k_2)$, if both of $(V, \mathcal{B}_1)$ and $(V, \mathcal{B}_2)$ are cyclic with respect to the same automorphism $\sigma : i \longmapsto i + 1 \pmod v$ (cf. Jimbo, 1993).

Denote $Z_v^* = \{1, 2, \ldots, v - 1\}$. Let a multiset $Z^*_{(v,\lambda)} = \{1, \ldots, 1, 2, \ldots, 2, \ldots, v - 1, \ldots, v - 1\}$ that contains each element of $Z_v^*$ $\lambda$ times, $Z'_v$ be a subset of $Z_v^*$ such that $Z_v^* = \{a, -a | a \in Z'_v\} \pmod v$ (hence $|Z'_v| = (v - 1)/2$), and a multiset $Z'_{(v,\lambda)}$ be a subset of $Z^*_{(v,\lambda)}$ such that $Z^*_{(v,\lambda)} = \{a, -a | a \in Z'_{(v,\lambda)}\} \pmod v$ (hence $|Z'_{(v,\lambda)}| = (v - 1)\lambda/2$).

A *perpendicular array* (PA), denoted by PA$_d(g, v)$, is a matrix of $g$ rows and $d\binom{s}{2}$ columns with entries from $Z_v$ such that each column has $g$ distinct entries, and each set of 2 rows contains each set of 2 distinct entries of $Z_v$ as a column precisely $d$ times (cf. Bierbraner, 2007). When $d = 1$, it is simply written as PA$(g, v)$. If $(t_1 + 1, t_2 + 1, \ldots, t_g + 1)^T \pmod v$ is a column of the PA$(g, v)$ for any column $(t_1, t_2, \ldots, t_g)^T$ of the PA$(g, v)$, then the PA$(g, v)$ is said to be *cyclic*, denoted by CPA$(g, v)$. Choose an arbitrary column from each set of columns $(t_1 + i, t_2 + i, \ldots, t_g + i)^T$, $1 \le i \le g$, and call it an *initial column*.

Now, let $v$ be an odd integer and $(v-1)/2$ initial columns of a CPA$(4, v)$ be $(a(1,n), a(2,n), a(3,n),\ a(4,n))^T$, $1 \le n \le (v-1)/2$, with $a(m,n) \in Z_v$ for $1 \le m \le 4$. Then it is shown that

$$Z'_v = \{a(i,n) - a(j,n) \pmod v | 1 \le n \le \frac{v-1}{2}\} \tag{2.1}$$

for each of $1 \le i < j \le 4$.

A *cyclic difference matrix* on $Z_v$, denoted by CDM$(4, v)$, is defined as a $4 \times v$ array $(a(m,n))$, $a(m,n) \in Z_v$, $1 \le m \le 4$, that satisfies

$$Z_v = \{a(i,n) - a(j,n) \pmod v | 1 \le n \le v\} \tag{2.2}$$

for each of $1 \le i < j \le 4$ (cf. Ge, 2005).

Next, let $v \equiv 1 \pmod 4$ and $(v-1)/4$ initial blocks of a CNB$(v; (v-1)/2, (v-1)/4; 2, 4)$ which contains no short orbit be $\{a(1,n), a(2,n) | a(3,n),\ a(4,n)\}$, $1 \le n \le (v-1)/4$, with $a(m,n) \in Z_v$ for $1 \le m \le 4$. Then it is shown that

$$Z'_v = \{a(1,n) - a(2,n), a(3,n) - a(4,n) | 1 \le n \le \frac{v-1}{4}\}, \tag{2.3}$$

$$Z'_{(v,2)} = \bigcup_{i \in \{1,2\}, j \in \{3,4\}} \{a(i,n) - a(j,n) | 1 \le n \le \frac{v-1}{4}\}. \tag{2.4}$$

For $v \equiv 1$ (or 0) $\pmod 4$, a *whist tournament* Wh$(v)$ is a schedule of $v(v-1)/4$ games $(a, b, c, d)$, where the unordered pairs $\{a, c\}, \{b, d\}$ are called *partners*, the pairs $\{a, b\}$, $\{c, d\}$ *opponents of the first kind,* and the pairs $\{a, d\}, \{b, c\}$ *opponents of the second kind*, such that (i) the games are arranged into $v$ (or $v-1$) rounds of $(v-1)/4$ (or $v/4$) games each, (ii) each player plays in exactly one game in each round, (iii) each player has every other player as a partner exactly once, and (iv) each player has every other player as an opponent exactly twice. A *triplewhist tournament* TWh$(v)$ is a Wh$(v)$ with, in the stead of the above condition (iv), another condition (v) such that each player has every other player as an opponent of the first kind exactly once, and that of the second kind exactly once (cf. Anderson and Finizio, 2007).

Note (Anderson and Finizio, 2007) that a necessary and sufficient condition for the existence of a Wh($v$) is $v \equiv 0, 1 \pmod 4$. When $v \equiv 1 \pmod 4$, the Wh($v$) is said to be *Z-cyclic* if the players are the elements of $Z_v$ and the round $i+1$ is obtained from the round $i$ by adding 1 (mod $v$) to each element. Since each of players and each of games can be seen as an element and a block, respectively, it can be seen that any $Z$-cyclic Wh($v$) with $v \equiv 1 \pmod 4$ itself shows a CNB($v; v(v-1)/2, v(v-1)/4; 2, 4$) with initial blocks $\{a, c|b, d\}$. On the other hand, $Z$-cyclic Wh($v$) with $v \equiv 0$ (mod 4) are introduced in Anderson and Finizio (2007). However, since its point set is $\{0, 1, \ldots, v-2\} \cup \{\infty\}$ and each of initial blocks is developed on $Z_{v-1}$, such designs cannot be cyclic. Hence this case is not discussed in this paper.

Furthermore, let initial blocks of a $Z$-cyclic TWh($4t+1$) be $\{a(1,n),\ a(2,n)|a(3,n),\ a(4,n)\}$, $1 \le n \le t$, with $a(m,n) \in Z_v$ for $1 \le m \le 4$. Then it is shown that

$$\begin{aligned} Z'_v &= \{a(1,n)-a(2,n), a(3,n)-a(4,n)|1 \le n \le t\} && (2.5)\\ &= \{a(1,n)-a(3,n), a(2,n)-a(4,n)|1 \le n \le t\} && (2.6)\\ &= \{a(1,n)-a(4,n), a(2,n)-a(3,n)|1 \le n \le t\}. && (2.7) \end{aligned}$$

Also, letting $v$ be an odd integer and $(v-1)/2$ initial blocks of $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ of 2 PACB($v, 2, 1$) be $\{a(1,n),\ a(2,n)\}$ and $\{a(3,n),\ a(4,n)\}$, $1 \le n \le (v-1)/2$, with $a(m,n) \in Z_v$ for $1 \le m \le 4$, respectively, it follows that

$$\begin{aligned} Z'_v &= \{a(1,n)-a(2,n)|1 \le n \le \frac{v-1}{2}\} && (2.8)\\ &= \{a(3,n)-a(4,n)|1 \le n \le \frac{v-1}{2}\}, && (2.9)\\ Z'_{(v,4)} &= \bigcup_{i \in \{1,2\}, j \in \{3,4\}} \{a(i,n)-a(j,n)|1 \le n \le \frac{v-1}{2}\}. && (2.10) \end{aligned}$$

Finally, a *special array* on $Z_v$ for an odd integer $v$, denoted by SA($4, v$), is defined as a $4 \times (v-1)/2$ array $(a(m,n))$, $a(m,n) \in Z_v$,

$1 \leq m \leq 4$, that satisfies

$$Z'_v = \{a(1,n) - a(2,n) | 1 \leq n \leq \frac{v-1}{2}\} \tag{2.11}$$

$$= \{a(3,n) - a(4,n) | 1 \leq n \leq \frac{v-1}{2}\}, \tag{2.12}$$

$$Z^*_v = \{a(1,n) - a(3,n), a(2,n) - a(4,n) | 1 \leq n \leq \frac{v-1}{2}\} \tag{2.13}$$

$$= \{a(1,n) - a(4,n), a(2,n) - a(3,n) | 1 \leq n \leq \frac{v-1}{2}\}. \tag{2.14}$$

Then it follows from (2.8) to (2.14) that the initial blocks $\{a(1,n), a(2,n)\}$ and $\{a(3,n), a(4,n)\}$ yield 2 PACB$(v, 2, 1)$.

# 3 Direct construction

Some direct constructions are presented in this section. It has been shown (Matsubara and Kageyama, 2013) that the existence of a PA$(g, v)$ implies the existence of $\lfloor g/2 \rfloor$ PAB$(v, 2, 1)$, where $\lfloor x \rfloor$ means the greatest integer $y$ such that $y \leq x$. Similarly, a class of 2 PACB$(v, 2, 1)$ can be constructed through CPA$(4, v)$ as the following shows.

**Theorem 3.1** The existence of a CPA$(g, v)$ implies the existence of $\lfloor g/2 \rfloor$ PACB$(v, 2, 1)$.

*Proof.* Let initial columns of the CPA$(g, v)$ be $(a(1,n), \ldots, a(g,n))^T$ with $1 \leq n \leq (v-1)/2$. It can be shown by (2.1) that the following incidence matrices yield the required $\lfloor g/2 \rfloor$ PACB$(v, 2, 1)$:

$$\begin{array}{rcll} \boldsymbol{N}_1 & : & \{a(1,n), a(2,n)\} & \bmod v \\ \boldsymbol{N}_2 & : & \{a(3,n), a(4,n)\} & \bmod v \\ \vdots & & \vdots & \\ \boldsymbol{N}_{\lfloor \frac{g}{2} \rfloor} & : & \{a(2\lfloor \frac{g}{2} \rfloor - 1, n), a(2\lfloor \frac{g}{2} \rfloor, n)\} & \bmod v \end{array}$$

for $1 \leq n \leq (v-1)/2$. $\square$

**Lemma 3.2** Let $v \geq 5$ and $\gcd(v, 6) = 1$. Then there exists a CPA$(4, v)$.

*Proof.* Since $v \geq 5$ and $\gcd(v, 6) = 1$ imply that

$$\{2i|1 \leq i \leq \frac{v-1}{2}\} = \{3i|1 \leq i \leq \frac{v-1}{2}\} = Z'_v,$$

the following columns on $Z_v$ can be seen to be initial columns of the required CPA$(4, v)$.

$$(0, i, 2i, 3i)^T,\ 1 \leq i \leq \frac{v-1}{2}.$$

□

**Theorem 3.3** Let $v \geq 5$ and $\gcd(v, 6) = 1$. Then 2 PACB$(v, 2, 1)$ exist.

*Proof.* Because of the existence of a CPA$(4, v)$ shown by Lemma 3.2, the proof is complete by applying Theorem 3.1. □

Next, some individual examples (Examples 3.4, 3.6, 3.7, 3.8, 3.9 and 3.10 next) which can be obtained by use of a computer are provided. Each of such examples cannot be obtained by the construction methods presented in this paper.

In each example except for Example 3.5, the following procedure of checking PACB properties will be taken. For two incidence matrices $\boldsymbol{N}_1$ with initial blocks $B_h^{(1)} = \{a_h, b_h\}$ and $\boldsymbol{N}_2$ with initial blocks $B_h^{(2)} = \{c_h, d_h\}$ for $1 \leq h \leq \lfloor v/2 \rfloor$, it can be checked that $\boldsymbol{N}_1 + \boldsymbol{N}_2$ with initial blocks $\{a_h, b_h, c_h, d_h\}$ is the incidence matrix of a B$(v, 4, 6)$. In fact, (i) for a full initial block $\{a, b\} \cup \{c, d\}$, let a multiset $\Delta_f(\{a, b\}, \{c, d\}) = \{e - f, f - e|e \in \{a, b\}, f \in \{c, d\}\}$ on $Z_v$, and (ii) for a short initial block $\{a, b\} \cup \{c, d\}$, let a multiset $\Delta_s(\{a, b\}, \{c, d\}) = \{a - f, f - a|f \in \{c, d\}\}$ on $Z_v$. If every nonzero element of $Z_v$ occurs 4 times in $\bigcup_{h=1}^{(v-1)/2} \Delta_f(B_h^{(1)}, B_h^{(2)})$ for an odd integer $v$ or every nonzero element of $Z_v$ occurs 4 times in $\bigcup_{h=1}^{(v-2)/2} \Delta_f(B_h^{(1)}, B_h^{(2)}) \cup \Delta_s(B_{v/2}^{(1)}, B_{v/2}^{(2)})$ for an even integer $v$, then $\boldsymbol{N}_1 + \boldsymbol{N}_2$ forms the incidence matrix of a B$(v, 4, 6)$. In Example 3.5, PACB properties can be checked by a similar procedure as above for $\boldsymbol{N}_i + \boldsymbol{N}_j$ with any $1 \leq i < j \leq 4$.

These symbols $\Delta_f$ for *full* initial blocks and $\Delta_s$ for *short* initial blocks will play an important role in Section 7.

**Example 3.4** Developing the following blocks on $Z_4$ yields an ACB(4,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{0,1\},\{0,2\}\mathrm{PC}(2) \quad \bmod 4\\ \boldsymbol{N}_2 &: \{2,3\},\{1,3\}\mathrm{PC}(2) \quad \bmod 4.\end{aligned}$$

**Example 3.5** Developing the following blocks on $Z_8$ yields an ACB(8,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\}\mathrm{PC}(4) \quad \bmod 8\\ \boldsymbol{N}_2 &: \{4,5\},\{1,3\},\{2,5\},\{2,6\}\mathrm{PC}(4) \quad \bmod 8\\ \boldsymbol{N}_3 &: \{3,6\},\{6,7\},\{1,7\},\{1,5\}\mathrm{PC}(4) \quad \bmod 8\\ \boldsymbol{N}_4 &: \{2,7\},\{4,5\},\{4,6\},\{3,7\}\mathrm{PC}(4) \quad \bmod 8.\end{aligned}$$

Here, the additive property is pointed out in Matsubara et al. (2006), while the cyclic property is checked directly.

**Example 3.6** Developing the following blocks on $Z_9$ yields 2 PACB(9,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\} \quad \bmod 9\\ \boldsymbol{N}_2 &: \{2,3\},\{4,6\},\{2,6\},\{5,8\} \quad \bmod 9.\end{aligned}$$

**Example 3.7** Developing the following blocks on $Z_{12}$ yields 2 PACB(12,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\}\mathrm{PC}(6) \quad \bmod 12\\ \boldsymbol{N}_2 &: \{3,6\},\{4,9\},\{2,4\},\{7,8\},\{2,6\},\{1,7\}\mathrm{PC}(6) \quad \bmod 12.\end{aligned}$$

**Example 3.8** Developing the following blocks on $Z_{15}$ yields 2 PACB(15,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\\ &\quad\ \{0,7\} \quad \bmod 15\\ \boldsymbol{N}_2 &: \{5,10\},\{4,13\},\{1,2\},\{8,12\},\{10,12\},\{9,12\},\\ &\quad\ \{1,8\} \quad \bmod 15.\end{aligned}$$

**Example 3.9** Developing the following blocks on $Z_{16}$ yields 2 PACB(16,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\{0,7\},\\ &\quad \{0,8\}\mathrm{PC}(8) \quad \mathrm{mod}\ 16\\ \boldsymbol{N}_2 &: \{3,8\},\{3,9\},\{4,5\},\{7,10\},\{1,15\},\{8,12\},\{5,12\},\\ &\quad \{3,11\}\mathrm{PC}(8) \quad \mathrm{mod}\ 16.\end{aligned}$$

**Example 3.10** Developing the following blocks on $Z_{24}$ yields 2 PACB(24,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{0,1\},\{0,2\},\{0,3\},\{0,4\},\{0,5\},\{0,6\},\{0,7\},\{0,8\},\\ &\quad \{0,9\},\{0,10\},\{0,11\},\{0,12\}\mathrm{PC}(12) \quad \mathrm{mod}\ 24\\ \boldsymbol{N}_2 &: \{15,18\},\{1,7\},\{5,6\},\{11,16\},\{21,8\},\{2,10\},\{11,15\},\\ &\quad \{13,15\},\{3,10\},\{6,15\},\{13,23\},\{2,14\}\mathrm{PC}(12) \quad \mathrm{mod}\ 24.\end{aligned}$$

Next, take the following $4 \times 13$ array quoted from Abel and Ge (2005):

$$\begin{pmatrix} 18 & 24 & 8 & 9 & 5 & 25 & 16 & 7 & 21 & 12 & 17 & 2 & 20\\ 19 & 26 & 11 & 13 & 10 & 4 & 23 & 15 & 3 & 22 & 1 & 14 & 6\\ 2 & 1 & 26 & 17 & 8 & 11 & 22 & 16 & 20 & 9 & 7 & 23 & 18\\ 4 & 21 & 3 & 14 & 24 & 19 & 12 & 25 & 5 & 15 & 6 & 10 & 13 \end{pmatrix} = (a(m,n)),$$

say. This is an SA$(4,27)$ that satisfies (2.11) to (2.14). Hence the following example can be further presented.

**Example 3.11** Developing the following blocks on $Z_{27}$ yields 2 PACB(27,2,1):

$$\begin{aligned}\boldsymbol{N}_1 &: \{a(1,n),a(2,n)\} \quad \mathrm{mod}\ 27\\ \boldsymbol{N}_2 &: \{a(3,n),a(4,n)\} \quad \mathrm{mod}\ 27\end{aligned}$$

for $1 \le n \le 13$.

**Remark 3.12** The existence of ACB$(v,2,1)$ is only known for $v = 4,8$ as in Examples 3.4 and 3.5. In this paper, the nonexistence of ACB$(v,2,1)$ for $v = 12$ and $v \equiv 2 \pmod 4$ will be discussed in Section 7.

# 4 Construction by cyclic nested BIB designs

At first it is pointed out that there are some classes of CNB$(4t+1; 2t(4t+1), t(4t+1); 2, 4)$ as follows.

**Lemma 4.1** (Anderson and Finizio, 2007) For $3 \le t \le 37$, there exists a CNB$(4t+1; 2t(4t+1), t(4t+1); 2, 4)$.

**Lemma 4.2** (Anderson and Finizio, 2005) For any integer $m \geq 2$, there exists a CNB$(3^{2m};\ 3^{2m}(3^{2m}-1)/2,\ 3^{2m}(3^{2m}-1)/4; 2, 4)$.

On the other hand, it has been shown (Matsubara and Kageyama, 2014) that 3 PAB$(v, 2, 1)$ can be constructed by an NB$(v; v(v-1)/2, v(v-1)/6; 2, 6)$. Similarly, it will be shown that 2 PACB$(v, 2, 1)$ can be obtained from a cyclic nested BIB design.

**Theorem 4.3** The existence of a CNB$(v; v(v-1)/2, v(v-1)/4; 2, 4)$ implies the existence of 2 PACB$(v, 2, 1)$.

*Proof.* Let the $n$th initial blocks of the CNB$(v; (v-1)/4, (v-1)/2; 2, 4)$ be

$$\{a(1,n), a(2,n) | a(3,n), a(4,n)\},\ 1 \le n \le \frac{v(v-1)}{4}.$$

Then it follows from (2.3) and (2.4) that the following incidence matrices yield the required 2 PACB$(v, 2, 1)$:

$$\begin{array}{lcll} \boldsymbol{N}_1 & : & \{a(1,n), a(2,n)\}, \{a(3,n), a(4,n)\} & \mod v \\ \boldsymbol{N}_2 & : & \{a(3,n), a(4,n)\}, \{a(1,n), a(2,n)\} & \mod v \end{array}$$

for $1 \le n \le v(v-1)/4$. $\square$

Hence Theorem 4.3 with Lemma 4.2 can produce the following.

**Corollary 4.4** There are 2 PACB$(3^{2m}, 2, 1)$ for any integer $m \geq 2$.

# 5 Construction by cyclic relative difference family

A $(vg, g, k, \lambda)$ *cyclic relative difference family*, denoted by $(vg, g, k, \lambda)$-CDF, is a family $\mathcal{F}$ of $k$-subsets of $Z_{vg}$ with the property that a multiset of differences $\cup_{B\in\mathcal{F}}\Delta B$ is $Z^*_{(vg,\lambda)}\backslash vZ^*_{(vg,\lambda)}$, where $\Delta B = \{x_i - x_j, x_j - x_i | 0 \leq i < j \leq k-1\}$ for $B = \{x_0, \ldots, x_{k-1}\}$ and $vZ^*_{(vg,\lambda)}$ is a multiset which contains each element of $\{v, 2v, \ldots, vg\}$ $\lambda$ times (cf. Chang, 2004).

Some classes of $(vg, g, 4, 1)$-CDF are known as follows.

**Lemma 5.1** (Chang and Miao, 2003, Chang, 2004) There exist a $(243, 27, 4, 1)$-CDF and a $(2^{s+4}, 2^s, 4, 1)$-CDF for any integer $s \geq 2$.

On the other hand, 2 PACB$(v, 2, 1)$ will be obtained from some cyclic relative difference family.

**Theorem 5.2** The existence of a $(vg, g, 4, 1)$-CDF and 2 PACB$(g, 2, 1)$ implies the existence of 2 PACB$(vg, 2, 1)$.

*Proof.* Let 4-subsets of the $(vg, g, 4, 1)$-CDF on $Z_{vg}$ be

$$\{a_i, b_i, c_i, d_i\},\ 1 \leq i \leq \frac{(v-1)g}{12}$$

and let initial blocks of 2 PACB$(g, 2, 1)$ on $vZ_{vg} = \{0, v, 2v, \ldots, (g-1)v\}$ be

$$\begin{aligned}
\boldsymbol{N}_1 &: \{x_j, y_j\},\ 1 \leq j \leq \lfloor\frac{g}{2}\rfloor \\
\boldsymbol{N}_2 &: \{z_j, w_j\},\ 1 \leq j \leq \lfloor\frac{g}{2}\rfloor.
\end{aligned}$$

Then it follows that developing the following initial blocks on $Z_v$ yields the required 2 PACB$(v, 2, 1)$:

$$\begin{aligned}
\boldsymbol{N}_1 &: \{a_i, b_i\}, \{a_i, c_i\}, \{a_i, d_i\}, \{c_i, b_i\}, \{b_i, d_i\}, \{d_i, c_i\}, \\
&\quad \{x_j, y_j\} \quad \bmod vg \\
\boldsymbol{N}_2 &: \{c_i, d_i\}, \{d_i, b_i\}, \{b_i, c_i\}, \{d_i, a_i\}, \{c_i, a_i\}, \{b_i, a_i\}, \\
&\quad \{z_j, w_j\} \quad \bmod vg
\end{aligned}$$

for $1 \leq i \leq (v-1)g/12$ and $1 \leq j \leq \lfloor g/2 \rfloor$. □

Finally, a class of 2 PACB$(v, 2, 1)$ can be given by use of Theorem 5.2.

**Theorem 5.3** There are 2 PACB$(2^m, 2, 1)$ for any positive integer $m \not\equiv 1 \pmod 4$.

*Proof.* There exist $(2^{s+4}, 2^s, 4, 1)$-CDF for $s \geq 2$ on account of Lemma 5.1. Hence Theorem 5.2 shows that the existence of 2 PACB$(2^s, 2, 1)$ implies the existence of 2 PACB$(2^{s+4}, 2, 1)$ for $s \geq 2$. On the other hand, 2 PACB$(2^t, 2, 1)$ with $t = 2, 3$ and $4$ are obtained as in Examples 3.4, 3.5 and 3.9. Thus, the proof is complete. □

**Remark 5.4** If 2 PACB$(2^5, 2, 1)$ could be constructed, then the condition on $m$ in Theorem 5.3 would be removed.

# 6 Recursive construction

At first an existence of cyclic difference matrices is reviewed.

**Lemma 6.1** (Ge, 2005) There exists a CDM$(4, v)$ for any odd integer $v \geq 5$ and $\gcd(v,\, 27) \neq 9$.

Some recursive constructions of cyclic BIB designs with some regular short orbits and of cyclic nested BIB designs with no short orbit are provided by using a CDM$(g, v)$ in Jimbo (1993). Next, some similar methods are presented.

**Theorem 6.2** Let $v \geq 5$ and $v' \geq 5$ be odd integers. Then the existence of 2 PACB$(v, 2, 1)$, 2 PACB$(v', 2, 1)$ and a CDM$(4, v')$ implies the existence of 2 PACB$(vv', 2, 1)$.

*Proof.* Let sets of initial blocks of 2 PACB$(v, 2, 1)$ and 2 PACB$(v', 2, 1)$ be

$$\{\{x_i^{(h)}, y_i^{(h)}\} | 1 \leq i \leq \frac{v-1}{2}\}, \{\{z_j^{(h)}, w_j^{(h)}\} | 1 \leq j \leq \frac{v'-1}{2}\}, h = 1, 2,$$

respectively. Let $A$ be the CDM$(4, v')$ with $a(m, n)$ as the $(m, n)$-entry for $1 \leq m \leq 4$ and $1 \leq n \leq v'$. Then, since each block orbit of 2 PACB$(v, 2, 1)$ and 2 PACB$(v', 2, 1)$ is full for odd integers $v \geq 5$ and $v' \geq 5$, it can be shown by (2.2) that the following two incidence matrices yield the required 2 PACB$(v, 2, 1)$ on $Z_{vv'}$:

$$\begin{aligned} \boldsymbol{N}_1 &: \{x_i^{(1)} + a(1,n)v, y_i^{(1)} + a(2,n)v\}, \{z_j^{(1)}v, w_j^{(1)}v\} \mod vv' \\ \boldsymbol{N}_2 &: \{x_i^{(2)} + a(3,n)v, y_i^{(2)} + a(4,n)v\}, \{z_j^{(2)}v, w_j^{(2)}v\} \mod vv' \end{aligned}$$

for $1 \leq i \leq (v-1)/2, 1 \leq j \leq (v'-1)/2$ and $1 \leq n \leq v'$. □

Now, Theorem 6.2 produces a new class of 2 PACB$(v, 2, 1)$.

**Corollary 6.3** There are 2 PACB$(3^m, 2, 1)$ for any integer $m \geq 2$.

*Proof.* Examples 3.6 and 3.11 show the existence of 2 PACB$(9, 2, 1)$ and 2 PACB$(27, 2, 1)$, respectively. Corollary 4.4 and Lemma 6.1 can produce 2 PACB$(3^{2s}, 2, 1)$ with $s \geq 2$ and a CDM$(4, 27)$, respectively. Now, for $v = 9 \cdot 27, 3^{2s} \cdot 27$ $(s \geq 2)$, Theorem 6.2 yields 2 PACB$(v, 2, 1)$. Hence 2 PACB$(v, 2, 1)$ can be constructed for $v = 3^2, 3^3, 3^{2s}, 3^5, 3^{2s+3}$ with $s \geq 2$. □

**Theorem 6.4** Let $v \geq 5$ and $\gcd(v, 6) = 1$. Then there are 2 PACB$(3^m v,\ 2,\ 1)$ for any integer $m \geq 2$.

*Proof.* Because of the existence of 2 PACB$(3^m, 2, 1)$ and 2 PACB$(v, 2, 1)$ shown by Corollary 6.3 and Theorem 3.3, the proof is complete by applying Theorem 6.2 with the CDM$(4, v)$ given by Lemma 6.1. □

Unfortunately, the above method of construction can be applied only for 2 PACB$(v,\ 2, 1)$ with no short orbit, that is, for $v$ being an odd integer. Note that the recursive construction given in Jimbo (1993) of cyclic BIB designs with some regular short orbits cannot also be applied for the construction of 2 PACB$(v, 2, 1)$ with an even integer $v$. Next, another recursive construction of 2 PACB $(2t,\ 2, 1)$ with short initial blocks will be considered.

**Theorem 6.5** The existence of 2 PACB$(2t, 2, 1)$ and an SA$(4, v)$ implies the existence of 2 PACB$(2tv, 2, 1)$ for any integer $t \geq 2$ and any odd integer $v \geq 5$ with $\gcd(t, v) = 1$.

*Proof.* Let a set of initial blocks of 2 PACB$(2t, 2, 1)$ on $Z_{2t}$ be

$$\{\{x_i^{(h)}, y_i^{(h)}\} | 1 \le i \le t-1\} \cup \{z^{(h)}, z^{(h)} + t\}\mathrm{PC}(t),\ h = 1, 2,$$

and let columns of the SA$(4, v)$ on $Z_v$ be

$$(a(1,n), a(2,n), a(3,n), a(4,n))^T,\ 1 \le n \le \frac{v-1}{2}.$$

Then it can be shown by (2.11) to (2.14) that developing the following initial blocks on $Z_{2t} \times Z_v$ yields 2 PAB$(2tv, 2, 1)$ with $2tv$ elements denoted by $(z, w)$ for $0 \le z \le 2t-1$ and $0 \le w \le v-1$:

$$\begin{array}{rcl}
\boldsymbol{N}_1 & : & \{(x_i^{(1)}, 0), (y_i^{(1)}, 0)\}, \{(0, a(1,n)), (0, a(2,n))\}, \\
& & \{(x_i^{(1)}, a(1,n)), (y_i^{(1)}, a(2,n))\}, \\
& & \{(x_i^{(1)}, a(2,n)), (y_i^{(1)}, a(1,n))\}, \\
& & \{(z^{(1)}, a(1,n)), (z^{(1)} + t, a(2,n))\}, \\
& & \{(z^{(1)}, 0), (z^{(1)} + t, 0)\}\mathrm{PC}\,(t, v) \quad \mathrm{mod}\ (2t, v) \\
\boldsymbol{N}_2 & : & \{(x_i^{(2)}, 0), (y_i^{(2)}, 0)\}, \{(0, a(3,n)), (0, a(4,n))\}, \\
& & \{(x_i^{(2)}, a(3,n)), (y_i^{(2)}, a(4,n))\}, \\
& & \{(x_i^{(2)}, a(4,n)), (y_i^{(2)}, a(3,n))\}, \\
& & \{(z^{(2)}, a(3,n)), (z^{(2)} + t, a(4,n))\}, \\
& & \{(z^{(2)}, 0), (z^{(2)} + t, 0)\}\mathrm{PC}\,(t, v) \quad \mathrm{mod}\ (2t, v)
\end{array}$$

where $1 \le i \le t-1$, $1 \le n \le (v-1)/2$ and PC$(t, v)$ means a partial cycle of order $tv$ on $Z_{2t} \times Z_v$, i.e., only $(s, u)$, $0 \le s \le t-1$ and $0 \le u \le v-1$ are to be added to the initial block.

Since $\gcd(2t, v) = 1$, the required 2 PACB$(2tv, 2, 1)$ on $Z_{2tv}$ can be constructed by corresponding the element $j$ to $(z, w)$ for $0 \le j \le 2tv - 1$, where $j \equiv z \pmod{2t}$ and $j \equiv w \pmod{v}$. □

The following example illustrates Theorem 6.5 with $t = 2$ and $v = 5$.

**Example 6.6** Consider an SA$(4, 5)$ on $Z_5$:

$$\begin{pmatrix} 0 & 1 & 4 & 2 \\ 0 & 2 & 3 & 4 \end{pmatrix}^T.$$

Since there are 2 PACB$(4, 2, 1)$ as in Example 3.4, 2 PACB$(20, 2, 1)$ are provided by developing the following initial blocks on $Z_4 \times Z_5$:

$$\begin{aligned}
\boldsymbol{N}_1 \;:\;& \{(0,0),(1,0)\},\{(0,0),(0,1)\},\{(0,0),(0,2)\},\{(0,0),(1,1)\},\\
& \{(0,0),(1,2)\},\{(0,1),(1,0)\},\{(0,2),(1,0)\},\{(0,0),(2,1)\},\\
& \{(0,0),(2,2)\},\{(0,0),(2,0)\}\mathrm{PC}(2,5) \quad \mathrm{mod}\ (4,5)\\
\boldsymbol{N}_2 \;:\;& \{(2,0),(3,0)\},\{(0,4),(0,2)\},\{(0,3),(0,4)\},\{(2,4),(3,2)\},\\
& \{(2,3),(3,4)\},\{(2,2),(3,4)\},\{(2,4),(3,3)\},\{(1,4),(3,2)\},\\
& \{(1,3),(3,4)\},\{(1,0),(3,0)\}\mathrm{PC}(2,5) \quad \mathrm{mod}\ (4,5).
\end{aligned}$$

Hence 2 PACB$(20, 2, 1)$ on $Z_{20}$ can be obtained by corresponding the element $j$ to $(z, w)$ for $0 \le j \le 19$, where $j \equiv z \pmod 4$ and $j \equiv w \pmod 5$. In fact, the following initial blocks on $Z_{20}$ yield 2 PACB$(20, 2, 1)$:

$$\begin{aligned}
\boldsymbol{N}_1 \;:\;& \{0,5\},\{0,16\},\{0,12\},\{0,1\},\{0,17\},\{16,5\},\\
& \{12,5\},\{0,6\},\{0,2\},\{0,10\}\mathrm{PC}(10) \quad \mathrm{mod}\ 20\\
\boldsymbol{N}_2 \;:\;& \{10,15\},\{4,12\},\{8,4\},\{14,7\},\{18,19\},\{2,19\},\\
& \{14,3\},\{9,7\},\{13,19\},\{5,15\}\mathrm{PC}(10) \quad \mathrm{mod}\ 20.
\end{aligned}$$

# 7 2 PACB$(v, 2, 1)$ with $v \equiv 2 \pmod 4$

By considering all possible combinations of initial blocks, it is easily seen that there are no 2 PACB(6,2,1). However, for a given integer $v$, whether $\ell$ PACB$(v, 2, 1)$ exist or not is a difficult problem for general $\ell \le v/2$. In this section, it will be shown that there are no 2 PACB$(v, 2, 1)$ for any $v \equiv 2 \pmod 4$ and, incidentally, no $\ell$ PACB$(12, 2, 1)$ for $\ell \in \{5, 6\}$.

**Theorem 7.1** There are no 2 PACB$(v, 2, 1)$ for any $v \equiv 2 \pmod 4$.

*Proof.* Assume that there exist 2 PACB$(v = 2t, 2, 1)$ $(V, \mathcal{B})$ with incidence matrices $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$, where $t$ is an odd integer. Since $\mathcal{B} = \{\{v_1, v_2\} |\ v_1, v_2 \in V, v_1 \ne v_2\}$ for any B$(v, 2, 1)$, without loss of generality, let initial blocks of $\boldsymbol{N}_1$ can be

$$B_i^{(1)} = \{0, i\}, 1 \le i \le t-1,\ B_t^{(1)} = \{0, t\}\mathrm{PC}(t) \bmod 2t,$$

initial blocks of $\boldsymbol{N}_2$ can be

$$B_i^{(2)} = \{a_i, b_i\},\ B_t^{(2)} = \{c, c+t\}\,\mathrm{PC}\,(t) \mod 2t$$

and initial blocks of $\boldsymbol{N}_1 + \boldsymbol{N}_2$ can be

$$\{0, i, a_i, b_i\}, \{0, t, c, c+t\}\,\mathrm{PC}\,(t) \mod 2t,$$

where $1 \le i \le t-1$. Further let $\Delta_f(B_i^{(1)}, B_i^{(2)})$ and $\Delta_s(B_t^{(1)}, B_t^{(2)})$ be multisets on $Z_{2t}$ (see Section 3 for the meaning of notations). Then every nonzero element of $Z_{2t}$ occurs 4 times in the multiset

$$\Delta = \Delta_f(B_1^{(1)}, B_1^{(2)}) \cup \ldots \cup \Delta_f(B_{t-1}^{(1)}, B_{t-1}^{(2)}) \cup \Delta_s(B_t^{(1)}, B_t^{(2)}).$$

In other words, the number of even elements of $Z_v$ in $\Delta$ must be a multiple of 4. It is seen that exact 2 even elements occur in $\Delta_s(B_t^{(1)}, B_t^{(2)})$ and the number of even elements in each of $\Delta_f(B_i^{(1)}, B_i^{(2)})$, $1 \le i \le t-1$, is one of 0, 4 or 8. Hence the number of even elements in $\Delta$ is not a multiple of 4, which is a contradiction. $\square$

Incidentally, another nonexistence will be shown.

**Theorem 7.2** There are no 5 PACB$(12, 2, 1)$ and no ACB$(12, 2, 1)$.

*Proof.* Assume that there are 5 PACB$(12, 2, 1)$ with incidence matrices $\boldsymbol{N}_1, \ldots, \boldsymbol{N}_5$. Let $B_i^{(h)}$, $1 \le i \le 5$ and $1 \le h \le 5$, be the $i$th initial block of $\boldsymbol{N}_h$ and $B_6^{(h)} = \{c_h, c_h + 6\}$, $c_h \in Z_{12}$, $1 \le h \le 5$, be a short initial block of $\boldsymbol{N}_h$. Then, without loss of generality, we can let $c_1 = 0, c_2 = 2, c_3 = 4, c_4 = 1$ and $c_5 = 3$ by choosing an arbitrary block from the short block orbit with some replacement of subscripts.

Let $\Delta_f(B_i^{(h)}, B_i^{(h')})$ and $\Delta_s(B_6^{(h)}, B_6^{(h')})$ be multisets on $Z_v$ similarly to Section 3. Then the number of even elements in $\Delta_s(B_6^{(h)}, B_6^{(h')})$ is (i) 4 if $h, h'$ $(h \ne h') \in \{1, 2, 3\}$ and (ii) 0 if $h \in \{1, 2, 3\}$ and $h' \in \{4, 5\}$.

An initial block $B_i^{(h)}$ is said to be *even* or *odd*, according as the difference of the two elements in $B_i^{(h)}$ is even or odd. Then it is clear that each $\boldsymbol{N}_h$ includes exact 3 even initial blocks and the number of even elements in $\Delta_f(B_i^{(h)}, B_i^{(h')})$, $1 \le h < h' \le 5$, is (i) 0 or 8 if both two blocks are even and (ii) 4 if either of them is odd.

Now, every nonzero element of $Z_v$ must occur 4 times, that is, there are 24 even elements and 20 odd elements in the multiset

$$\Delta_f(B_1^{(h)}, B_1^{(h')}) \cup \ldots \cup \Delta_f(B_5^{(h)}, B_5^{(h')}) \cup \Delta_s(B_6^{(h)}, B_6^{(h')})$$

for $1 \le h < h' \le 5$. Let $G = (g(m,n))$ be a matrix of order 5 and $\boldsymbol{x}_s$, $1 \le s \le 5$, be the $s$th row vector of $G$, where $g(m,n) = 1$ or $0$, according as the $n$th initial block of $\boldsymbol{N}_m$ is even or odd. Then

$$\boldsymbol{x}_s \cdot \boldsymbol{x}_t = \begin{cases} 2 & \text{if } s = t, \\ 0, 2 & \text{if } s \in \{1,2,3\} \text{ and } t \in \{4,5\}, \\ 1 & \text{otherwise,} \end{cases} \tag{7.1}$$

where $\cdot$ is the usual inner product among row vectors.

The first three rows of $G$ satisfying (7.1) must be one of the following under some permutation of columns:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

and then the first four rows of $G$ satisfying (7.1) can be further reduced only to the following:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

It can be seen that there is no $\boldsymbol{x}_5$ such that $\boldsymbol{x}_5 \cdot \boldsymbol{x}_s = 0$ or $2$ and $\boldsymbol{x}_5 \cdot \boldsymbol{x}_4 = 1$ for any $s \in \{1,2,3\}$. This implies that there does not exist the required matrix $G$ satisfying (7.1). Hence there is no 5 PACB$(12,2,1)$ and then it is shown by the definition of additive cyclic BIB designs that there is no ACB$(12,2,1)$. $\square$

# 8 2 PACB$(v, 2, 1)$ with $v \not\equiv 2 \pmod 4$

In this section, the existence of such PACB$(v,2,1)$ will be discussed. At first, some classes of SA$(4,v)$ are provided to apply the recursive construction given in Theorem 6.5.

**Lemma 8.1** Let $v \geq 5$ and $\gcd(v, 6) = 1$. Then there exists an $\mathrm{SA}(4, v)$.

*Proof.* Since $v \geq 5$ and $\gcd(v, 6) = 1$ imply that

$$\{\pm i | 1 \leq i \leq \frac{v-1}{2}\} = \{\pm 2i | 1 \leq i \leq \frac{v-1}{2}\} = Z_v^*,$$

the following columns on $Z_v$ can be seen to form the $\mathrm{SA}(4, v)$:

$$(0, i, -i, 2i)^T,\ 1 \leq i \leq \frac{v-1}{2}.$$

□

Next, an $\mathrm{SA}(4, 4t+1)$ can be obtained from a $Z$-cyclic $\mathrm{TWh}(v)$ with $v = 4t + 1$ as in Lemma 8.2. Furthermore, an $\mathrm{SA}(4, v)$ can be obtained from a cyclic relative difference family as in Lemma 8.4. Especially, the $\mathrm{SA}(4, 81)$ and the $\mathrm{SA}(4, 243)$ constructed in Lemmas 8.3 and 8.5 below are utilized for the recursive construction of $\mathrm{SA}(4, v)$.

**Lemma 8.2** The existence of a $Z$-cyclic $\mathrm{TWh}(4t + 1)$ implies the existence of an $\mathrm{SA}(4, 4t + 1)$.

*Proof.* Let $t$ games of the $Z$-cyclic $\mathrm{TWh}(4t + 1)$ be

$$(a(1, n), a(2, n), a(3, n), a(4, n)),\ 1 \leq n \leq t.$$

Then it is shown by (2.5) to (2.7) that the following columns yield an $\mathrm{SA}(4, 4t + 1)$:

$$\begin{pmatrix} a(1,n) & a(2,n) & a(3,n) & a(4,n) \\ a(4,n) & a(3,n) & a(2,n) & a(1,n) \end{pmatrix}^T$$

for $1 \leq n \leq t$. □

Now, on account of Lemma 8.2 the $Z$-cyclic $\mathrm{TWh}(81)$ given in Abel and Ge (2005) can produce the following.

**Lemma 8.3** There exists an $\mathrm{SA}(4, 81)$.

**Lemma 8.4** The existence of $(vg, g, 4, 1)$-CDF and an $\mathrm{SA}(4, g)$ implies the existence of an $\mathrm{SA}(4, vg)$.

*Proof.* Let initial blocks of the $(vg, g, 4, 1)$-CDF on $Z_{vg}$ be

$$\{a_i, b_i, c_i, d_i\},\ 1 \le i \le \frac{(v-1)g}{12}$$

and let columns of the $\mathrm{SA}(4, g)$ on $vZ_g = \{0, v, 2v, \ldots, (g-1)v\}$ be

$$(a(1,n), a(2,n), a(3,n), a(4,n))^T,\ 1 \le n \le \frac{g-1}{2}.$$

Then it follows from (2.11) to (2.14) that the following columns yield the $\mathrm{SA}(4, vg)$:

$$\begin{pmatrix} a_i & a_i & a_i & c_i & b_i & d_i & a(1,n) \\ b_i & c_i & d_i & b_i & d_i & c_i & a(2,n) \\ c_i & d_i & b_i & d_i & c_i & b_i & a(3,n) \\ d_i & b_i & c_i & a_i & a_i & a_i & a(4,n) \end{pmatrix} \quad \mathrm{mod}\ vg$$

for $1 \le i \le (v-1)g/12$ and $1 \le n \le \lfloor g/2 \rfloor$. $\square$

Thus, Lemmas 5.1 and 8.4 with the $\mathrm{SA}(4, 27)$ displayed in Section 3 can produce the following.

**Lemma 8.5** There exists an $\mathrm{SA}(4, 243)$.

Now, a class of $\mathrm{SA}(4, v)$ can be given by the recursive construction which is similar to Theorem 6.2.

**Lemma 8.6** The existence of an $\mathrm{SA}(4, v)$ and an $\mathrm{SA}(4, v')$ implies the existence of an $\mathrm{SA}(4, vv')$.

*Proof.* Let columns of the $\mathrm{SA}(4, v)$ and the $\mathrm{SA}(4, v')$ be

$$\begin{aligned} (a(1,n), a(2,n), a(3,n), a(4,n))^T, &\qquad 1 \le n \le \frac{v-1}{2}, \\ (a'(1,n'), a'(2,n'), a'(3,n'), a'(4,n'))^T, &\qquad 1 \le n' \le \frac{v'-1}{2}, \end{aligned}$$

respectively. Then the following columns yield the SA$(4, v)$:

$$\begin{pmatrix} a(1,n) & a'(1,n')v & a(1,n)+a'(1,n')v & a(1,n)+a'(2,n')v \\ a(2,n) & a'(2,n')v & a(2,n)+a'(2,n')v & a(2,n)+a'(1,n')v \\ a(3,n) & a'(3,n')v & a(3,n)+a'(3,n')v & a(3,n)+a'(4,n')v \\ a(4,n) & a'(4,n')v & a(4,n)+a'(4,n')v & a(4,n)+a'(3,n')v \end{pmatrix}$$

for $1 \leq n \leq (v-1)/2$ and $1 \leq n' \leq (v'-1)/2$. $\square$

**Lemma 8.7** There are SA$(4, 3^m)$ for any integer $m \geq 3$.

*Proof.* For $m = 3, 4, 5$, SA$(4, 3^m)$ are given as in Section 3 and Lemmas 8.3 and 8.5. Hence, applying Lemma 8.6 repeatedly with $v = 3^3, 3^4, 3^5$ and $v' = 3^3$ shows the existence of SA$(4, 3^m)$ for any integer $m \geq 3$. $\square$

Finally, the main results of this paper are established.

**Theorem 8.8** There are 2 PACB$(v, 2, 1)$ for any odd integer $v(\geq 5)$ such that $\gcd(v, 9) \neq 3$.

*Proof.* Let $v(\geq 5)$ be an odd integer such that $\gcd(v, 9) \neq 3$. When $\gcd(v, 9) = 1$, since $\gcd(v, 6) = 1$, Theorem 3.3 shows the existence of 2 PACB$(v, 2, 1)$. When $\gcd(v, 9) = 9$, we can put $v = 3^n t$ with integers $n(\geq 2)$ and $t(\geq 1)$ such that $\gcd(t, 6) = 1$. Then Corollary 6.3 and Theorem 6.4 show the existence of 2 PACB$(v, 2, 1)$. Thus, the proof is complete. $\square$

**Theorem 8.9** There are 2 PACB$(v, 2, 1)$ with $v = 2^m t$ for any integer $m(\geq 2)$ and any odd integer $t(\geq 1)$ such that $m \not\equiv 1 \pmod 4$ and $\gcd(t, 27) \neq 3, 9$.

*Proof.* Let $t(\geq 1)$ be an odd integer such that $\gcd(t, 27) \neq 3, 9$. Then we can put $t = 3^n t'$ with a nonnegative integer $n \neq 1, 2$ and an odd integer $t'(\geq 1)$ such that $\gcd(t', 6) = 1$. Now there are 2 PACB$(2^m, 2, 1)$ for any positive integer $m \not\equiv 1 \pmod 4$ (see Theorem 5.3). Also there are SA$(4, 3^n)$ for $n \geq 3$ (see Lemma 8.7). Hence Theorem 6.5 shows the existence of 2 PACB$(2^m 3^n, 2, 1)$ for any positive integer $m \not\equiv 1 \pmod 4$ and any nonnegative integer $n \neq 1, 2$. By Lemma 8.1, when $t' \geq 5$, there are SA$(4, t')$, since $\gcd(t', 6) = 1$. Hence Theorem 6.5 shows the existence of 2 PACB$(2^m 3^n t', 2, 1)$ for any integer $m (\geq 2)$ and any odd integer $3^n t' (\geq 1)$ such that $m \not\equiv 1 \pmod 4$ and $\gcd(3^n t', 27) \neq 3, 9$. Thus, the proof is complete. □

**Remark 8.10** If the existence of 2 PACB$(3p, 2, 1)$, 2 PACB$(2^5, 2, 1)$, 2 PACB$(3 \cdot 2^m, 2, 1)$ and 2 PACB$(9 \cdot 2^m, 2, 1)$ for any prime $p \geq 5$ and any integer $m \geq 2$ could be shown, then it would follow that the necessary and sufficient conditions for the existence of 2 PACB$(v, 2, 1)$ are $v \geq 4$ and $v \equiv 0, 1, 3 \pmod 4$. In fact, some individual 2 PACB$(v, 2, 1)$ for several $v$ of the above cases are obtained. For example, we can find $v = 12, 15, 24$ as in Examples 3.7, 3.8 and 3.10, and also $v = 21, 33, 57, 69, 93, 129, 141$ obtained by Lemma 4.1 and Theorem 4.3. As of today, the smallest value of $v$ such that the existence of 2 PACB$(v, 2, 1)$ is not known is $32 (= 2^5)$. Furthermore, the existence of $\ell$ PACB$(v, 2, 1)$ for some $\ell$ may be difficult to be determined.

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