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A Comparative Study of Estimation Methods for Nakagami Distribution in Reliability Analysis

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Abstract

In this study, the Nakagami distribution is examined in the context of reliability analysis, focusing on key reliability measures. Various estimation techniques for the distribution parameters are explored and compared. A novel approach for deriving different estimators is introduced. Asymptotic confidence intervals for the parameters are constructed based on both MLE and log-MLE methods. In addition, hypothesis testing procedures are developed for different scenarios. The performance of the proposed estimation methods is assessed through a comprehensive Monte Carlo simulation study. Finally, the applicability of these methods is demonstrated using a real data set, providing clarity and practical insight into the estimation process.

Key words: Nakagami distribution; Reliability; Classical methods; Markov chain Monte Carlo.

AMS Subject Classifications: 62N05, 62E20 , 62F03 , 62F05, 62F10.

1. Introduction and preliminaries

Reliability function is the probability that a system performs its intended function without any failure at time t under the prescribed conditions. So, if we suppose that lifetime of an item or any system is denoted by the random variate X then the reliability is $R(t) = Pr(X > t)$. The other important measure of reliability is $P = Pr(X > Y)$ and this represents reliability of X (random strength) subject to Y (random stress). This is known as the reliability of an item under the stress strength set up. This measure is very useful to find the reliability of an item in no time like in case we want to test the reliability of an electric wire. Various authors have conducted estimation and testing of the reliability measure $R(t)$ and P considering different distributions. For literature, one can refer Pugh (1963), Basu (1964), Tong (1974, 1975), Johnson (1975), Sathe and Shah (1981), Chao (1982). Chaturvedi and Surinder (1999) developed the inferential procedures for testing these reliability measures of exponential distribution. Awad and Gharraf (1986) estimated P in case of Burr distribution. Tyagi and Bhattacharya (1989) and, Chaturvedi and Rani (1998) done esti-

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mation related to Maxwell and generalized Maxwell distributions respectively. Chaturvedi and Pathak (2012) derived inferential procedures for exponentiated Weibull and Lomax distributions. Chaturvedi and Rani (1997) and Chaturvedi and Tomer (2003) draw inferences for $R(t)$ and P for the families of lifetime distributions which are very useful as they cover many distributions as particular cases. Chaturvedi and Vyas (2018) have done estimation of $R(t)$ and P for three parameter Burr distribution under different censoring schemes.
In the present communication, a very important distribution known as the Nakagami distribution is taken into consideration which is most useful in communication engineering. It was Nakagami (1960) who proposed this distribution which models the fading of radio signals by name as Nakagami-m distribution with shape parameter m.
If a random variable (rv) X follows the Nakagami distribution with shape parameter $\alpha \geq 0.50$ and scale parameter $\lambda > 0$ then its probability density function (P.d.f) is as follows

$$f(x;\alpha,\lambda) = \frac{2}{\Gamma\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha} x^{2\alpha-1} exp\left(-\frac{\alpha}{\lambda}x^2\right); \quad x > 0, \alpha \geq 0.5, \lambda > 0. \tag{1}$$

Hereafter, we denote Nakagami distribution by $ND(\alpha,\lambda)$, where shape parameter α is known and scale parameter λ is unknown. The corresponding cumulative distribution function (cdf) of $ND(\alpha,\lambda)$ is given by,

$$F(x) = \frac{1}{\Gamma\alpha}\Gamma\left(\frac{\alpha}{\lambda}x^2, \alpha\right); \quad x > 0, \alpha \geq 0.5, \lambda > 0. \tag{2}$$

where $\Gamma(x,a) = \int_0^x t^{a-1}e^{-t}dt$ is the lower incomplete gamma function.
The reliability function of $ND(\alpha,\lambda)$ is

$$R(t) = 1 - \frac{1}{\Gamma\alpha}\Gamma\left(\frac{\alpha}{\lambda}t^2, \alpha\right); \quad t > 0, \alpha \geq 0.5, \lambda > 0 \tag{3}$$

The failure rate of $ND(\alpha,\lambda)$ is

$$h(t) = \frac{\frac{2}{\Gamma\alpha}\left(\frac{\alpha}{\lambda}\right)^{\alpha} t^{2\alpha-1} exp\left(-\frac{\alpha}{\lambda}t^2\right)}{1 - \frac{1}{\Gamma\alpha}\Gamma\left(\frac{\alpha}{\lambda}t^2, \alpha\right)}; \quad t > 0, \alpha \geq 0.5, \lambda > 0. \tag{4}$$

1.1. Relations with other distribution

1. For $\alpha = 0.5$, $ND(\alpha,\lambda)$ reduces to Half Normal distribution.

2. With $\alpha = 1, ND(\alpha,\lambda)$ becomes Rayleigh distribution.

3. If rv Y is distributed as $Gamma(k,\lambda)$ with shape k and scale λ then $\sqrt{Y}$ follows $ND(k, k\lambda)$.

4. If Z follows chi-square with parameter 2α and 2α is integer-valued then $\sqrt{\frac{\lambda}{2\alpha}Z}$ is $ND(\alpha,\lambda)$ variate.

This distribution has found applications in various disciplines such as in hydrology, multimedia data traffic over networks, medical imaging studies, in modeling of seismogram envelope of high frequency etc. For review, one may see Schwartz *et al.* (2013). Using the Monte Carlo simulation technique, Abdi and Kaveh (2000) made comparison of three different estimators of Nakagami-m distribution. Cheng and Beaulieu (2001) estimated the distribution using Maximum Likelihood method. Schwartz *et al.* (2013) discussed the estimation of the shape parameter using improved maximum likelihood estimation and also gave some distributional properties.

The main aim of this paper is to develop point estimation and hypotheses testing procedures for two measures of reliability *viz.*, $R(t)$ and P. In Section 2, we present point estimation when shape parameter is known but scale parameter is unknown. Uniformly Minimum Variance Unbiased Estimators (U.M.V.U.Es), Maximum Likelihood Estimators(M.L.Es) and Moment estimators have been found in this section. Section 3 comprises of point estimation when both scale and shape parameters are unknown. Asymptotic confidence intervals are developed for the parameters in Section 4. Section 5 is devoted for developing testing procedures for testing different hypotheses. In Section 6, we present the simulation study using Monte Carlo techniques with Section 6.1 devoted for the case when shape parameter α is known and scale parameter λ is unknown, Section 6.2 for the case when both α and λ are unknown and Section 6.3 for hypotheses testing . In Section 7, a real data study is performed and finally the paper is concluded in Section 8.

2. Point estimation when shape parameter is known

Let us take a random sample $X_1, X_2, \ldots, X_n$ from the model (1) having size n. Taking α to be known, the likelihood function of the parameter λ given the sample observations $\underline{x}$ comes out to be

$$L(\lambda|\underline{x}) = \prod_{i=1}^{n} f(x_i, \lambda) = \left(\frac{2\alpha^{\alpha}}{\Gamma\alpha}\right)^{n} \frac{1}{\lambda^{\alpha n}} \prod_{i=1}^{n} {x_i}^{2\alpha-1} exp\left(-\frac{\alpha}{\lambda}\sum_{i}^{n} {x_i}^2\right) \tag{5}$$

Theorem 1: For $q \in (-\infty, \infty)$, $q \neq 0$, U.M.V.U.E of λ^q is

$$\tilde{\lambda^q} = \begin{cases} \left\{\frac{\Gamma(n\alpha - q)}{\Gamma(n\alpha)}\right\} S^q & ; n\alpha > q \\ 0 & ; \text{Otherwise} \end{cases} \tag{6}$$

Proof: From the likelihood (5) and factorization theorem Rohtagi and Saleh (2012, pp.361) it can be easily obtained that $S = \sum_{i=1}^{n} {x_i}^2$ is a sufficient statistic for λ and the $P.d.f$ of S is

$$f_s(S|\lambda) = \frac{S^{n\alpha-1}}{\Gamma(n\alpha)\lambda^{n\alpha}} exp\left(-\frac{S}{\lambda}\right) \tag{7}$$

From (7), since the distribution of S belongs to the exponential family, it is also complete Rohtagi and Saleh (2012, pp.367).

Now, from (7), we have

$$E[S^{-q}] = \frac{1}{\Gamma(n\alpha)\lambda^{n\alpha}} \int_0^\infty S^{n\alpha-q-1} exp\left(-\frac{S}{\lambda}\right) dS$$
$$= \left\{\frac{\Gamma(n\alpha - q)}{\Gamma(n\alpha)}\right\} \frac{1}{\lambda^q}$$

and the theorem holds on using Lehmann-Scheffe theorem Rohtagi (1976, pp.357). □

Theorem 2: The U.M.V.U.E of the reliability function is

$$\tilde{R}(t) = \begin{cases} 1 - I_{\frac{t^2}{S}}[\alpha, (n-1)\alpha] & ; t^2 < \frac{S}{\alpha} \\ 0 & ; \text{Otherwise} \end{cases} \tag{8}$$

where $I_x(p, q) = \frac{1}{\beta(p,q)} \int_0^x y^{p-1}(1-y)^{q-1}\, dy; 0 \le y \le 1, x < 1, p, q > 0$ is the incomplete beta function.

Proof: Let us define a random variable as

$$V = \begin{cases} 1, & X_1 > t \\ 0, & \text{Otherwise} \end{cases} \tag{9}$$

which is based on a single observation and is an unbiased estimator of $R(t)$. Using Rao-Blackwellization and (9), we have

$$\tilde{R}(t) = E(V|S)$$
$$= P\left(X_1 > t|S\right)$$
$$= P\left(v_1 > \frac{t^2}{S}\right), \; say; \tag{10}$$

where $v_1 = \frac{{X_1}^2}{S}$. From (7), we see that v_1 follows beta distribution of first kind with parameters $[\alpha, (n-1)\alpha]$. Applying Basu's theorem, from (10), we have

$$\tilde{R}(t) = 1 - P\left(v_1 \le \frac{t^2}{S}\right)$$
$$= 1 - \frac{\beta[\frac{t^2}{S}; \alpha, (n-1)\alpha]}{\beta[\alpha, (n-1)\alpha]} \tag{11}$$

and the theorem holds. □

Corollary 2.1: The Reliability estimate of the distribution for which $\alpha = 1$ is

$$\tilde{R}(t) = \begin{cases} \left(1 - \frac{t^2}{S}\right)^{n-1} & ; t^2 < S \\ 0 & ; \text{Otherwise} \end{cases} \tag{12}$$

Corollary 2.2: The U.M.V.U.E of sampled $P.d.f$ (1) at a specified point x is :

$$\tilde{f}(x;\lambda) = \begin{cases} \left(\frac{\alpha}{S}\right)^{\alpha} \frac{x^{2\alpha-1}}{\beta[\alpha,(n-1)\alpha]} \left(1 - \frac{\alpha x^2}{S}\right)^{(n-1)\alpha-1} & ; x^2 < \frac{S}{\alpha_1} \\ 0 & ; \text{Otherwise} \end{cases} \tag{13}$$

Let us take two independent random variables X and Y with P.d.fs $f(x,\alpha_1,\lambda_1)$ and $f(y,\alpha_2,\lambda_2)$ respectively, where

$$f(x;\alpha_1,\lambda_1) = \frac{2}{\Gamma\alpha_1}\left(\frac{\alpha_1}{\lambda_1}\right)^{\alpha_1} x^{2\alpha_1-1} exp\left(-\frac{\alpha_1}{\lambda_1}x^2\right); \quad x > 0, \alpha_1 \geq 0.5, \lambda_1 > 0. \tag{14}$$

and

$$f(y;\alpha_2,\lambda_2) = \frac{2}{\Gamma\alpha_2}\left(\frac{\alpha_2}{\lambda_2}\right)^{\alpha_2} x^{2\alpha_2-1} exp\left(-\frac{\alpha_2}{\lambda_2}y^2\right); \quad y > 0, \alpha_2 \geq 0.5, \lambda_2 > 0. \tag{15}$$

Now draw a random sample $X_1, X_2, \ldots, X_n$ from $f(x;\alpha_1,\lambda_1)$ and random sample $Y_1, Y_2, \ldots, Y_m$ from $f(y;\alpha_2,\lambda_2)$. Denote $S = \sum_{i=1}^{n} {x_i}^2$ and $T = \sum_{i=1}^{m} {y_i}^2$.

Theorem 3: The U.M.V.U.E of P is

$$\tilde{P} = \begin{cases} \frac{1}{2\beta[\alpha_2,(m-1)\alpha_2]} \int_0^{\frac{\alpha_2 S}{\alpha_1 T}} \{1 - I_{\frac{Tz}{\alpha_2 S}}[\alpha_1,(n-1)\alpha_1]\} z^{\alpha_2-1}(1-z)^{(m-1)\alpha_2-1}\, dz \\ \qquad ; \left(\frac{S}{\alpha_1}\right)^{\frac{1}{2}} \leq \left(\frac{T}{\alpha_2}\right)^{\frac{1}{2}} \\ \frac{1}{2\beta[\alpha_2,(m-1)\alpha_2]} \int_0^{1} \{1 - I_{\frac{Tz}{\alpha_2 S}}[\alpha_1,(n-1)\alpha_1]\} z^{\alpha_2-1}(1-z)^{(m-1)\alpha_2-1}\, dz \\ \qquad ; \left(\frac{S}{\alpha_1}\right)^{\frac{1}{2}} > \left(\frac{T}{\alpha_2}\right)^{\frac{1}{2}} \end{cases}$$

Proof:

Proceeding as in case of proving Corollary 2, we can rewrite U.M.V.U.E of P in terms of $\tilde{R}(y,\lambda_1)$ as follows

$$\begin{aligned} \tilde{P} &= \int_{y=0}^{\infty}\int_{x=y}^{\infty} \tilde{f}(x;\lambda_1)\tilde{f}(y;\lambda_2)\,dx\,dy \\ &= \int_{y=0}^{\infty} \tilde{R}(y;\lambda_1)\tilde{f}(y;\lambda_2)\,dy \end{aligned}$$

Now, using Theorem 2, we have

$$\begin{aligned} \tilde{P} = &\int_0^{min\left[\left(\frac{S}{\alpha_1}\right)^{\frac{1}{2}},\left(\frac{T}{\alpha_2}\right)^{\frac{1}{2}}\right]} \left[1 - I_{\frac{y^2}{S}}(\alpha_1,(n-1)\alpha_1)\right] \\ &\left(\frac{\alpha_2}{T}\right)^{\alpha_2} \frac{y^{2\alpha_2-1}}{\beta[\alpha_2,(m-1)\alpha_2]}\left(1 - \frac{\alpha_2 y^2}{T}\right)^{(m-1)\alpha_2-1} dy \\ = &\frac{\left(\frac{\alpha_2}{T}\right)^{\alpha_2}}{\beta[\alpha_2,(m-1)\alpha_2]} \int_0^{min\left[\left(\frac{S}{\alpha_1}\right)^{\frac{1}{2}},\left(\frac{T}{\alpha_2}\right)^{\frac{1}{2}}\right]} \\ &\left[1 - I_{\frac{y^2}{S}}(\alpha_1,(n-1)\alpha_1)\right] y^{2\alpha_2-1}\left(1 - \frac{\alpha_2 y^2}{T}\right)^{(m-1)\alpha_2-1} dy \end{aligned} \tag{16}$$

Now from (16), when $\left(\frac{S}{\alpha_1}\right)^{\frac{1}{2}} \leq \left(\frac{T}{\alpha_2}\right)^{\frac{1}{2}}$

$$\tilde{P} = \frac{1}{2\beta[\alpha_2,(m-1)\alpha_2]} \int_0^{\frac{\alpha_2 S}{\alpha_1 T}} \{1 - I_{\frac{Tz}{\alpha_2 S}}[\alpha_1,(n-1)\alpha_1]\} z^{\alpha_2 - 1} (1-z)^{(m-1)\alpha_2 - 1}\, dz$$

and we have the first assertion.

Furthermore, for $\left(\frac{S}{\alpha_1}\right)^{\frac{1}{2}} > \left(\frac{T}{\alpha_2}\right)^{\frac{1}{2}}$,

$$\tilde{P} = \frac{1}{2\beta[\alpha_2,(m-1)\alpha_2]} \int_0^{1} \{1 - I_{\frac{Tz}{\alpha_2 S}}[\alpha_1,(n-1)\alpha_1]\} z^{\alpha_2 - 1} (1-z)^{(m-1)\alpha_2 - 1}$$

and this proves the second assertion. □

Corollary 3.1: U.M.V.U.E of P when $\alpha_1 = \alpha_2 = 1$ is given by

$$\tilde{P} = \begin{cases} \frac{1}{2\beta[1,m-1]} \int_0^{\frac{S}{T}} \{1 - I_{\frac{Tz}{S}}[1,(n-1)]\}(1-z)^{(m-2}\, dz; & S^{\frac{1}{2}} \leq T^{\frac{1}{2}} \\ \frac{1}{2\beta[1,m-1]} \int_0^{1} \{1 - I_{\frac{Tz}{S}}[1,(n-1)]\}(1-z)^{(m-2)}\, dz; & S^{\frac{1}{2}} > T^{\frac{1}{2}} \end{cases}$$

We provide M.L.E. of λ^q, $R(t)$ and P under the assumption that α is known in the following given theorems.
From (5), M.L.E of λ is

$$\hat{\lambda} = \frac{S}{n} \tag{17}$$

Theorem 4: The M.L.E. of λ^q is

$$\hat{\lambda}^q = \left(\frac{S}{n}\right)^q \tag{18}$$

Theorem 5: The M.L.E. of $R(t)$ is given by

$$\hat{R}(t) = 1 - \frac{1}{\Gamma\alpha}\Gamma\left(\frac{n\alpha t^2}{S}, \alpha\right) \tag{19}$$

We obtain M.L.E. of sampled $P.d.f$ with the help of Theorem 5 in the following corollary. This will be used to obtain M.L.E. of P.

Corollary 5.1: The M.L.E. of $f(x;\lambda)$ at a specified point x is

$$\hat{f}(x;\lambda) = \frac{2}{\Gamma\alpha}\left(\frac{n\alpha}{S}\right)^{\alpha} x^{2\alpha-1} exp\left(-\frac{n\alpha x^2}{S}\right) \tag{20}$$

Theorem 6: The M.L.E. of P is

$$\hat{P} = 1 - \frac{1}{\Gamma\alpha_1\Gamma\alpha_2}\int_{z=0}^{\infty} z^{\alpha_2-1}e^{-z}\,\Gamma\left(\frac{n\alpha_1 zT}{m\alpha_2 S}, \alpha_1\right) dz \tag{21}$$

Proof: We know that

$$\hat{P} = \int_{y=0}^{\infty}\int_{x=y}^{\infty} \hat{f}(x;\lambda_1)\hat{f}(y;\lambda_2)\,dx\,dy$$
$$= \int_{y=0}^{\infty} \hat{R}(y;\lambda_1)\hat{f}(y;\lambda_2)\,dy$$

Now using (20) and Theorem 5, we have

$$\hat{P} = \int_{y=0}^{\infty}\left[1 - \frac{1}{\Gamma\alpha_1}\Gamma\left(\frac{n\alpha_1 y^2}{S}, \alpha_1\right)\right]\frac{2}{\Gamma\alpha_2}\left(\frac{m\alpha_2}{T}\right)^{\alpha_2}$$
$$.y^{2\alpha_2-1}exp\left(-\frac{m\alpha_2 y^2}{T}\right)dy$$

Substituting $\frac{m\alpha_2 y^2}{T} = z$ and solving for the above integral, we get the desired result. □
Next, we provide moment estimators for the parameters. For this, below given theorem provides the r^{th} moment generating function of the distribution.

Theorem 7: For $r = 1, 2, 3\ldots$, the moment generating function r^{th} is given by

$$u_r = E(X^r) = \frac{\Gamma(\alpha + \frac{r}{2})}{\Gamma(\alpha)}\left(\frac{\lambda}{\alpha}\right)^{\frac{r}{2}} \tag{22}$$

From (22), we have

$$u_1 = \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha)}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2}}$$

and

$$u_2 = \frac{\Gamma(\alpha + \frac{2}{2})}{\Gamma(\alpha)}\left(\frac{\lambda}{\alpha}\right)^{\frac{2}{2}} = \lambda$$

Equating the population moments with the sample moments, we have

$$\hat{\lambda}_m = \frac{S}{n} \tag{23}$$

Using (23), the moment estimator $\hat{\alpha}_m$ of α is obtained by the solution of

$$\bar{X} - \frac{\Gamma(\alpha + \frac{1}{2})}{\Gamma(\alpha)}\sqrt{\left(\frac{\hat{\lambda}_m}{\alpha}\right)} = 0 \tag{24}$$

uniroot function in R-software is used for finding the roots of the above equation.

3. Point estimation when shape parameter is unknown

Now we discuss the case when both the parameters are unknown. The log-likelihood function of the parameters α and λ given the sample observations $\underline{x}$ is:

$$l(\lambda|\underline{x}) = nlog(2\alpha^n) - nlog(\Gamma\alpha) - n\alpha log(\lambda) + \sum_{i=1}^{n} log({x_i}^{2\alpha-1}) - \frac{\alpha}{\lambda}\sum_{i}^{n}{x_i}^2$$

The M.L.E of α is given by the solution of the following equation

$$\frac{\partial l}{\partial \alpha} = \frac{n^2}{\alpha} - n\Psi_0(\alpha) - nlog(\lambda) + 2\sum_{i=1}^{n} log(x_i) - \frac{1}{\lambda}\sum_{i=0}^{n} {x_i}^2 = 0 \quad (25)$$

where Ψ_0 is a polygamma function of order zero and $\Psi_0(\alpha) = \frac{\partial log\Gamma(\alpha)}{\partial \alpha}$ is diagamma function.

$$\frac{\partial l}{\partial \lambda} = -\frac{n\alpha}{\lambda} + \frac{\alpha}{\lambda^2}\sum_{i=1}^{n} {x_i}^2 = 0 \quad (26)$$

$$\implies \hat{\lambda} = \frac{S}{n}$$

Since (25) do not have a closed form solution, therefore any iterative procedure have to be use to compute M.L.E.

Theorem 8: The M.L.E. of $R(t)$ is given by:

$$\hat{R}(t) = 1 - \frac{1}{\Gamma\hat{\alpha}}\Gamma\left(\frac{n\hat{\alpha}t^2}{S}, \hat{\alpha}\right) \quad (27)$$

Corollary 8.1: The M.L.E. of $f(x; \alpha, \lambda)$ at a specified point x is

$$\hat{f}(x; \alpha, \lambda) = \frac{2}{\Gamma\hat{\alpha}}\left(\frac{n\hat{\alpha}}{S}\right)^{\hat{\alpha}} x^{2\hat{\alpha}-1} exp\left(-\frac{n\hat{\alpha}x^2}{S}\right) \quad (28)$$

Theorem 9: The M.L.E. of P is

$$\hat{P} = 1 - \frac{1}{\Gamma\hat{\alpha_1}\Gamma\hat{\alpha_2}}\int_{z=0}^{\infty} z^{\hat{\alpha_2}-1}e^{-z}\,\Gamma\left(\frac{n\hat{\alpha_1}zT}{m\hat{\alpha_2}S}, \hat{\alpha_1}\right) dz \quad (29)$$

4. Asymptotic confidence intervals

The Confidence Intervals (C.I) can be obtained by using the variance-covariance matrix of the M.L.Es of the parameters. The asymptotic variance-covariance matrix of $\hat{\eta} = (\hat{\alpha}, \hat{\lambda})$ is the inverse of the following Fisher Information matrix

$$I(\eta) = -E\begin{bmatrix} \frac{\partial^2 l}{\partial^2 \alpha} & \frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ \frac{\partial^2 l}{\partial \lambda \partial \alpha} & \frac{\partial^2 l}{\partial^2 \lambda} \end{bmatrix}$$

This is very cumbersome to obtain the exact distributions of the M.L.Es and the alternative is to use the observed Fisher information matrix which is

$$I(\hat{\eta}) = \begin{bmatrix} -\frac{\partial^2 l}{\partial^2 \alpha} & -\frac{\partial^2 l}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 l}{\partial \lambda \partial \alpha} & -\frac{\partial^2 l}{\partial^2 \lambda} \end{bmatrix}$$

Thus, we have observed variance-covariance matrix as

$$I^{-1}(\hat{\eta}) = \begin{bmatrix} \hat{V}ar(\hat{\alpha}) & \hat{C}ov(\hat{\alpha}, \hat{\lambda}) \\ \hat{C}ov(\hat{\lambda}, \hat{\alpha}) & \hat{V}ar(\hat{\lambda}) \end{bmatrix}$$

Assuming asymptotic normality of the M.L.Es, confidence intervals for α and λ are constructed. Let $\hat{\sigma}^2(\hat{\alpha})$ and $\hat{\sigma}^2(\hat{\lambda})$ be the estimated variances of α and λ. Then the two sided equal tail asymptotic $100(1-\delta)\%$ confidence intervals for the parameters α and λ are $\left(\hat{\alpha} \pm Z_{\frac{\delta}{2}}\hat{\sigma}(\hat{\alpha})\right)$ and $\left(\hat{\lambda} \pm Z_{\frac{\delta}{2}}\hat{\sigma}(\hat{\lambda})\right)$, respectively, where $Z_{\frac{\delta}{2}}$ is the $\left(\frac{\delta}{2}\right)^{th}$ percentile of the standard normal distribution. The coverage probabilities (CP) are given as,

$$CP_{\alpha} = P\left[\left|\frac{\hat{\alpha}-\alpha}{\hat{\sigma}(\hat{\alpha})}\right| \leq Z_{\frac{\delta}{2}}\right]$$

and

$$CP_{\lambda} = P\left[\left|\frac{\hat{\lambda}-\lambda}{\hat{\sigma}(\hat{\lambda})}\right| \leq Z_{\frac{\delta}{2}}\right]$$

The asymptotic C.I based on log(M.L.E) has better coverage probability as reported by Meeker and Escober (1998). An approximate $100(1-\delta)\%$ C.I for $log(\alpha)$ and $log(\lambda)$ are

$$\left\{log(\hat{\alpha}) \pm Z_{\frac{\delta}{2}}\hat{\sigma}[log(\hat{\alpha})]\right\} \text{ and } \left\{log(\hat{\lambda}) \pm Z_{\frac{\delta}{2}}\hat{\sigma}[log(\hat{\lambda})]\right\},$$

where $\hat{\sigma}^2[log(\hat{\alpha})]$ and $\hat{\sigma}^2[log(\hat{\lambda})]$ are the estimated variance of $log(\alpha)$ and $log(\lambda)$ respectively, and are approximated by

$$\hat{\sigma}^2[log(\hat{\alpha})] = \tfrac{\hat{\sigma}^2(\hat{\alpha})}{\hat{\alpha}^2} \text{ and } \hat{\sigma}^2[log(\hat{\lambda})] = \tfrac{\hat{\sigma}^2(\hat{\lambda})}{\hat{\lambda}^2}.$$

Hence, approximate $100(1-\delta)\%$ C.I for α and λ are

$$\left(\hat{\alpha}e^{\pm Z_{\frac{\delta}{2}}\frac{\hat{\sigma}(\hat{\alpha})}{\hat{\alpha}}}\right) \text{ and } \left(\hat{\lambda}e^{\pm Z_{\frac{\delta}{2}}\frac{\hat{\sigma}(\hat{\lambda})}{\hat{\lambda}}}\right).$$

5. Testing of statistical hypotheses

Under this section, we consider the following three cases of hypothesis testing.

1. $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$, when α is known.
2. $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$, when α is known.
3. $H_0 : P = P_0$ versus $H_1 : P \neq P_0$, when $\alpha_1 = \alpha_1$ is known.

Testing $H_0 : \lambda = \lambda_0$ against $H_1 : \lambda \neq \lambda_0$ is considered to be the most important. From (5), we can have the likelihood of observing λ given the sample observations $\underline{x}$ as

$$L(\lambda|\underline{x}) = \left(\frac{2\alpha^{\alpha}}{\Gamma\alpha}\right)^n \frac{1}{\lambda^{\alpha n}} \prod_{i=1}^{n} {x_i}^{2\alpha-1} exp\left(-\frac{\alpha}{\lambda}\sum_{i}^{n} {x_i}^2\right) \tag{30}$$

Under H_0, we have

$$\sup_{\Theta_0} L(\lambda; \underline{x}, \alpha) = \left(\frac{2\alpha^{\alpha}}{\Gamma\alpha}\right)^n \frac{1}{\lambda_0^{\alpha n}} \prod_{i=1}^{n} {x_i}^{2\alpha-1} exp\left(-\frac{\alpha}{\lambda_0}\sum_{i}^{n} {x_i}^2\right); \quad \Theta_0 = \{\lambda : \lambda = \lambda_0\} \tag{31}$$

and

$$\sup_{\Theta} L(\lambda; \underline{x}, \alpha) = \left(\frac{2\alpha^{\alpha}}{\Gamma\alpha}\right)^{n} \left(\frac{n}{S}\right)^{\alpha n} \prod_{i=1}^{n} {x_i}^{2\alpha-1} exp\Bigg(-\frac{n\alpha}{S}\sum_{i}^{n} {x_i}^2\Bigg); \quad \Theta_0 = \{\lambda : \lambda > 0\} \tag{32}$$

Therefore, the Likelihood Ratio (L.R) is given by

$$\begin{aligned} \phi(\underline{x}) &= \frac{\sup_{\Theta_0} L(\lambda; \underline{x}, \alpha)}{\sup_{\Theta} L(\lambda; \underline{x}, \alpha)} \\ &= \left(\frac{S}{n\lambda_0}\right)^{n\alpha} exp\left[-\alpha\left(\frac{S}{\lambda_0} - n\right)\right] \end{aligned} \tag{33}$$

From Right Hand Side (R.H.S) of above equation, it is clear that first term is an increasing whereas the second term is monotonically decreasing function in S. As $2\frac{S}{\lambda_0} \sim \chi^2_{(2n)}$, where $\chi^2_{(2n)}$ is the Chi-Square statistics with $2n$ degrees of freedom, the critical region is given by

$$\{0 < S < \gamma_0\} \cup \{\gamma_0^i < S < \infty\},$$

where the constants γ_0 and γ_0^i are obtained such that

$$P\left[\chi^2_{(2n)} < 2\frac{\gamma_0}{\lambda_0} \quad or \quad 2\frac{\gamma_0^i}{\lambda_0} < \chi^2_{(2n)}\right] = \varepsilon$$

Thus,

$$\gamma_0 = \frac{\lambda_0 \chi^2_{(2n)}\left(1 - \frac{\varepsilon}{2}\right)}{2}$$

and

$$\gamma_0^i = \frac{\lambda_0 \chi^2_{(2n)}\left(\frac{\varepsilon}{2}\right)}{2}$$

where ε is the probability of type I error.

The second important hypothesis is $H_0 : \lambda \leq \lambda_0$ versus $H_1 : \lambda > \lambda_0$. For $\lambda_1 > \lambda_2$, we have from (5)

$$\frac{L(\lambda_1|\underline{x})}{L(\lambda_2|\underline{x})} = \left(\frac{\lambda_2}{\lambda_1}\right)^{n\alpha} exp\left[-S\left(\frac{\alpha}{\lambda_1} - \frac{\alpha}{\lambda_2}\right)\right] \tag{34}$$

From (34), we can see $L(\lambda, \underline{x})$ has Monotone Likelihood Ratio (M.L.R) in S. Thus, the Uniformly Most Powerful Critical Region (U.M.P.C.R) for testing $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda > \lambda_0$ is given as Lehmann (1959, pp.88)

$$\phi(\underline{x}) = \begin{cases} 1, & S \leq \gamma_0^{ii} \\ 0, & \text{Otherwise.} \end{cases}$$

where, γ_0^{ii} is obtained such that $P\left[\chi^2_{(2n)} < 2\frac{\gamma_0^{ii}}{\lambda_0}\right] = \varepsilon$. Therefore,

$$\gamma_0^{ii} = \frac{\lambda_0 \chi^2_{(2n)}(1 - \varepsilon)}{2}$$

It can be seen that for two independent random variables X and Y with $\alpha_1 = \alpha_2 = 1$,

$$P = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Let us test $H_0 : P = P_0$ against $H_1 : P \neq P_0$. Thus, H_0 is equivalent to $\lambda_1 = k\lambda_2$, where $k = \frac{P_0}{1-P_0}$. Therefore, $H_0 : \lambda_1 = k\lambda_2$ and $H_1 : \lambda_1 \neq k\lambda_2$.
Under H_0, we can have

$$\hat{\lambda}_1 = \frac{S + Tk}{(n+m)}$$

and

$$\hat{\lambda}_2 = \frac{S + Tk}{k(n+m)}$$

Thus, for C (a generic constant), we have the likelihood of observing λ_1 and λ_2 as

$$L(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) = \frac{C}{{\lambda_1}^n {\lambda_2}^m} exp\left[-\left(\frac{S}{\lambda_1} + \frac{T}{\lambda_2}\right)\right] \tag{35}$$

Thus,

$$\sup_{\Theta_0} L(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) = C\left[\frac{k(n+m)}{S+Tk}\right]^{n+m} exp\left[-(n+m)\right]; \quad \Theta_0 = \{\lambda_1, \lambda_2 : \lambda_1 = k\lambda_2\} \tag{36}$$

and

$$\sup_{\Theta} L(\lambda_1, \lambda_2 | \underline{x}, \underline{y}) = C\left(\frac{n}{S}\right)^n \left(\frac{m}{T}\right)^m exp\left[-(n+m)\right]; \quad \Theta = \{\lambda_1, \lambda_2 : \lambda_1 > 0, \lambda_2 > 0\} \tag{37}$$

From (36) and (37), the Likelihood ratio criterion is

$$\phi(\underline{x}, \underline{y}) = \frac{C\left(\frac{S}{T}\right)^n}{\left(1 + \frac{S}{Tk}\right)^{n+m}} \tag{38}$$

Let us denote the F- statistic with (a, b) degrees of freedom by $F_{a,b}(.)$. As

$$\frac{S}{T} \sim \frac{n\lambda_1}{m\lambda_2} F_{(2n,2m)},$$

the critical region is

$$\left\{\frac{S}{T} < \gamma_2 \, or \, \frac{S}{T} > {\gamma_2}^i\right\},$$

where γ_2 and ${\gamma_2}^i$ are obtained such that

$$P\left\{\frac{nk\gamma_2}{m} < F_{2n,2m} \cup \frac{nk{\gamma_2}^i}{m} > F_{2n,2m}\right\} = \varepsilon$$

Thus, we have $\gamma_2 = \frac{nk}{m} F_{2n,2m}\left(1 - \frac{\varepsilon}{2}\right)$ and ${\gamma_2}^i = \frac{nk}{m} F_{2n,2m}\left(\frac{\varepsilon}{2}\right)$.

6. Simulation study

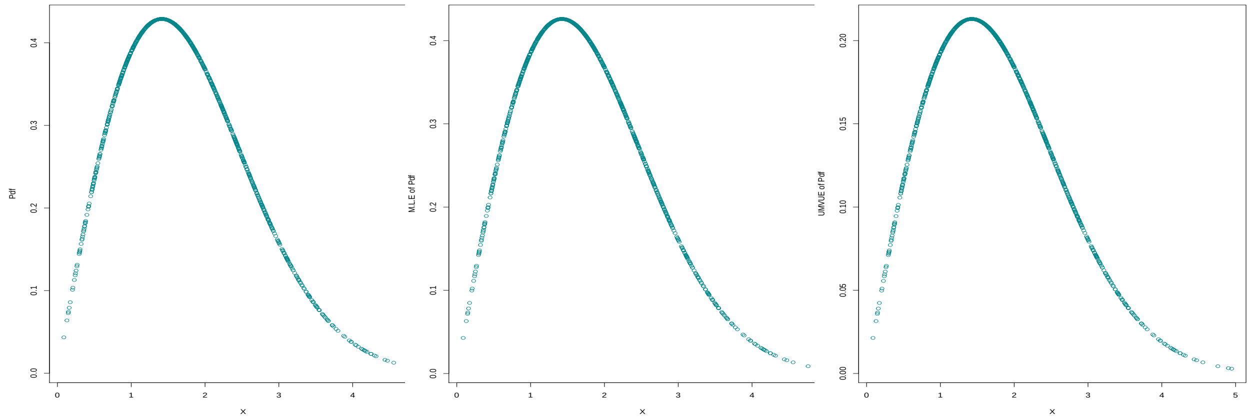
For validating the results obtained theoretically in Section 2 and Section 4, we, firstly present results which are based on Monte Carlo simulation technique. We have computed Mean Square Error (M.S.E) for comparison purpose. All the analyses have been done using R 3.4.3 software R Core Team (2013).

6.1. When shape parameter is known

For acquiring the performance of $\tilde{\lambda}^q$ and $\hat{\lambda}^q$, we have generated 1000 random samples from (1) of different sizes $n = (20, 30, 40, 60)$ with $\alpha = (0.8, 0.9, 1.0)$. We have computed average $\tilde{\lambda}^q$, $\hat{\lambda}^q$, corresponding average biases and M.S.E, and approximate $(1-\delta)100\%$, where $\delta = 0.05$, confidence intervals for λ^q. As $q \in (-\infty, \infty)$, $q \neq 0$, we choose a negative and a positive power of q to have better look into the performance of the estimators. For $q = -1$ and $q = 1$, results are given in Table **??**. The $1^{st}, 2^{nd}, 3^{rd}$ row represents average estimates, average bias, M.S.E and 4^{th} row represents the confidence interval. From Table **??**, we can infer that for negative values of q, U.M.V.U.E is performing better than the M.L.E but for positive value of q M.L.E is performing better than U.M.V.U.E. It can be seen that as the value of sample size is increasing M.S.E is decreasing for both the estimators. The length of the confidence interval is shorter for both estimators in all cases which means it is more informative. So, U.M.V.U.E should be preferred if we want to estimate the negative power of λ and for positive power, we should opt for M.L.E. It is interesting to note here that for $\alpha = 1$ and $q = -1$ both estimators are yielding the same results for all values of sample sizes.

Now, to acquire and compare the performance of the two estimators of $R(t)$, 1000 random samples are generated from (1) of different sizes $n = (10, 20, 30, 40, 60)$ with $\alpha = 3$ and $\lambda = 0.5$. Taking values of $t = (0.10, 0.15, 0.20, 0.25, 0.35)$, $\tilde{R}(t)$ and $\hat{R}(t)$, corresponding biases, M.S.E and approximate $(1-\delta)100\%$ C.I have been calculated. The obtained results are presented in Table 2 where $I^{st}, 2^{nd}, 3^{rd}$ row represents average estimates, average bias, M.S.E and 4^{th} row represents the C.I.
Looking at M.S.E values in Table 2, we can say that performance of M.L.E of $R(t)$ is better than that of U.M.V.U.E of $R(t)$. Performance of estimators is decreasing with the increase in time t as the M.S.E values are increasing. Estimators tends to perform better in case of large sample sizes. Table 3 presents Moment estimators $\hat{\alpha}_m$ and $\hat{\lambda}_m$ of the parameters α and λ are given for different values of n=(500,1000,1500) and different set of the parameters $(\alpha, \lambda) = (0.6, 0.8), (1.5, 0.8)$ and $(1.5, 1.0)$. The moment estimator and M.L.E of λ are equal and both the estimators are the functions of the sufficient statistics. So, both M.L.E and Moment estimators are equally efficient and works good.

In order to investigate how well estimators of P performs, 1000 random samples are generated from (14) and (15) of sizes $(n, m) = (5, 10), (10, 5)$ and $(10, 10)$ with $\alpha_1 = 0.5$ and $\alpha_2 = 10$, and $(\lambda_1, \lambda_2) = (3, 5), (3, 6), (4, 5), (4, 6)$. The obtained results are presented in Table 4 where $I^{st}, 2^{nd}, 3^{rd}$ row represents average estimates, average bias, M.S.E and 4^{th} row represents the confidence interval. Data in table 4 reveals that M.L.E of P gives better estimates than U.M.V.U.E of P for all combinations of $(\lambda_1,\ \lambda_2)$ and (n, m).



(a) P.d.f **(b) M.L.E** **(c) U.M.V.U.E**

Figure 1: Estimates of Sampled Probability Density Function

The estimates of P.d.f obtained in Section 2 are plotted in Figure 1. From the figure, we can see that the estimates of P.d.f fits well to the actual model.

6.2. When both scale and shape parameters are unknown

For obtaining the estimate of $R(t)$ when both scale and shape parameters are unknown, we have generated first random sample of size $n = 15$ from (1) with $\alpha_1 = 5$ and $\lambda_1 = 4$. Let it be X- population or random strength X given as

X- Population: 2.049508, 1.697911, 2.057258, 2.093914, 1.830376, 2.299230, 1.030369, 1.352851, 1.910835, 2.518206, 1.717836, 2.318662, 1.932082, 1.436255, 2.776821.

The M.L.Es of α and λ comes out to be $\hat{\alpha_1} = 5.512470$ and $\hat{\lambda}_1 = 3.937005$. For $t = 0.2$, actual $R(t) = 0.9999869$ and $\hat{R}(t) = 0.9999869$.

Now, for estimation of P, we have generated another random sample say Y population or random stress Y from (1) of size $m = 10$ with $\alpha_2 = 4$ and $\lambda_2 = 3$. The sample is

Y Population: 1.263758, 1.816875, 1.346044, 2.083317, 2.489531, 1.119266, 1.714329, 1.912815, 1.371682, 2.496376. The M.L.Es of α and λ comes out to be $\hat{\alpha_2} = 4.118107$ and $\hat{\lambda}_2 = 3.321188$. For $t = 0.2$, actual $R(t) = 0.9964545$ and $\hat{R}(t) = 0.9994916$. The actual $P = 0.4363503$ and the M.L.E of P comes out to be $\hat{P} = 0.4696917$. All the estimates can be seen validating the theoretical results obtained.

6.3. Hypothesis testing

This section comprises of checking the validity of the hypothesis testing procedures developed in section 5. Firstly, we test the hypothesis $H_0 : \lambda = \lambda_0 = 4$ against $H_1 : \lambda = \lambda_0 \neq 4$. For this we have generated a random sample of size 50 from (1) with $(\lambda = 4, \alpha = 5)$, given by

Sample 1 : 1.8715040, 2.4957160, 1.3041026, 1.0625339, 1.9552509, 1.8412767, 1.4635787, 1.6677863, 2.1402472, 1.6651901, 1.4523474, 2.4088220, 1.6413565, 2.2162550, 1.6001383, 2.0236934, 2.0894237, 1.7744711, 2.0995504, 2.9366243, 2.3269415, 1.6324515, 1.5328350, 0.9560068, 2.4759661, 2.0723630, 2.2769360, 1.3536968, 2.0298724, 2.4644942, 2.0113171, 1.6845441, 1.8919575, 2.5608773, 1.9408668, 1.8201857, 2.3742209, 2.1374813, 2.5166206,

2.4151387, 1.4238698, 1.4754821, 1.6192035, 2.1958351, 1.7966403, 2.2790533, 2.0138617, 1.4063136, 1.8715380, 1.6387806.
Now, using chi-square table at $\varepsilon = 5\%$ Level of Significance (LOS), we obtained $\gamma_0 = 259.12$ and $\gamma_0^i = 148.44$. From the sample we have $S = 192.5144$. Here, it can be seen that the value of S is not lying in the critical region. So, we do not have enough evidence to reject the null hypothesis at 5% LOS.

Consider the above sample 1 again for testing $H_0 : \lambda = \lambda_0 \leq 4$ against $H_1 : \lambda = \lambda_0 > 4$ at 5% LOS, we obtained $\gamma_0^{ii} = 148.68$. As $S = 192.5144$ is not lying in the critical region so we do not have sufficient evidence in support of alternate hypothesis. Thus, we do not reject the null hypothesis.

Now, to test $H_0 : P = P_0 = 0.5$ against $H_1 : P = P_0 \neq 0.5$, we have generated two random samples X_i and Y_i of sizes $n = 12$ and $m = 10$ from the distribution with $\lambda_1 = \lambda_2 = 4$ and $\alpha_1 = \alpha_2 = 1$ given by X= 0.6960666, 1.9268595, 2.1383461, 1.4266733, 1.8846088, 2.1335468, 2.0400911, 0.2361899, 3.8670944, 1.0444884, 1.5116124, 1.0438313 and Y=0.843277, 2.642273, 2.342635, 1.710249, 2.558351, 2.145001, 1.933631, 2.861752, 3.867026, 1.158481. From the two samples, we get $\frac{S}{T} = 0.766934$. Using F-table at 5% LOS, we computed $\gamma_2 = 2.7924$ and $\gamma_2^i = 0.498$. Thus, we do not reject the null hypothesis on the basis above information.

7. Real data analysis

Now we present two real data set to understand and illustrate the procedures discussed in the previous sections broadly.

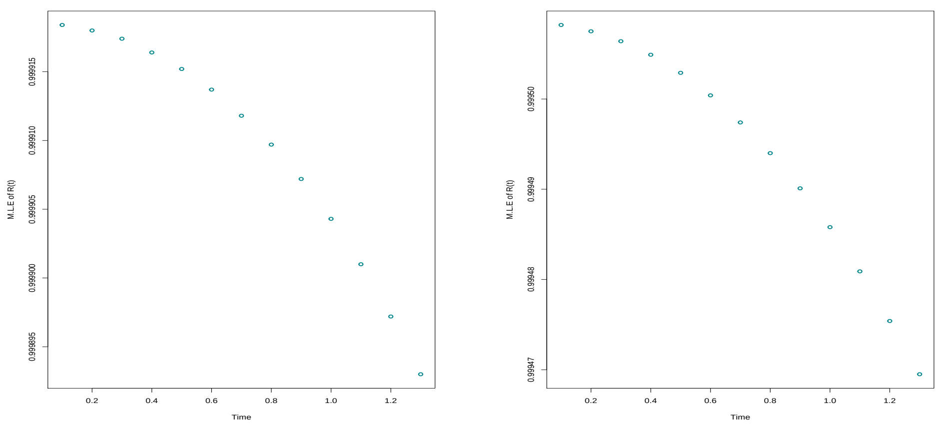
7.1. First data set

The data set has been taken from Lawless (2003, pp.267). This was originally reported by Schafft *et al.* (1987). This data represents the hours to failure of 59 conductors of 400-micrometer length. The specimens are put on a test with same temperature and current density and they all ran to failure at a certain high temperature with current density.
X-Population: 6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496,4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.491, 5.923.
Kumar *et al.* (2017) used this data set and found that Nakagami distribution fits well to the data with M.L.Es as $\hat{\alpha} = 4.8336$ and $\hat{\lambda} = 51.2823$. For $t = (0.1, 0.2, 0.3, 0.4, 0.5)$ we have computed $R(t) = (0.9985447, 0.9985376, 0.9985258, 0.998509, 0.9984872)$ and their M.L.Es are $\hat{R}(t) = (0.9999184, 0.999918, 0.9999174, 0.9999164, 0.9999152)$.

7.2. Second data set

The second data set given below is taken from Murthy *et al.* (2004, pp.180)(2004, pp.180). This data represents 50 items that are put on use at t=0 and failure times are in recorded (in weeks). The data set is
Y-Population: 0.013, 0.065, 0.111, 0.111, 0.163, 0.309, 0.426, 0.535, 0.684, 0.747, 0.997,

1.284, 1.304, 1.647, 1.829, 2.336, 2.838, 3.269, 3.977, 3.981, 4.520, 4.789, 4.849, 5.202, 5.291, 5.349, 5.911, 6.018, 6.427, 6.456, 6.572, 7.023, 7.291, 7.087, 7.787, 8.596, 9.388, 10.261, 10.713, 11.658, 13.006, 13.388, 13.842, 17.152, 17.283, 19.418, 23.471, 24.777, 32.795, 48.105. Mudasir and Ahmed (2017) used this data set for analysis and comparison purpose in case of weighted Nakagami distribution. The M.L.Es of α and λ came out to be $\hat{\alpha} = 4.1$ and $\hat{\lambda} = 144.2292$. For $t = (0.1, 0.2, 0.3, 0.4, 0.5)$, $R(t) = (0.9966496, 0.9966451, 0.9966375, 0.9966269, 0.9966132)$ and their M.L.Es are $\hat{R}(t) = (0.9995082, 0.9995075, 0.9995064, 0.9995049, 0.9995029)$. The MLE estimate of $R(t)$ for both data sets is plotted in Figure 2. From the figure, it can



(a) Population X **(b) Population Y**

Figure 2: M.L.Es of R(t)

be seen that in both cases the survival is very high at initial time but as the time increases survival probability goes on decreasing.

To evaluate M.L.E of P, first data set is taken as X population and second set as Y population. Actual P came out to be $P = 0.7377018$ and its M.L.E is $\hat{P} = 0.703271$.

8. Conclusion

This paper presents estimation and testing procedures for the reliability functions of the Nakagami distribution. A new, simpler technique for obtaining Uniformly Minimum Variance Unbiased Estimators (UMVUEs) and Maximum Likelihood Estimators (MLEs) of $R(t)$ and P is introduced, requiring no explicit forms of the parametric functions. In addition to these estimators, moment estimators for the parameters are derived. The efficiency of MLEs and moment estimators is compared through simulations, showing similar performance as both are functions of the sufficient statistic. Hypothesis testing is also performed, with real data analysis on strength (X) and stress (Y) datasets.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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ANNEXURE

Table 1: U.M.V.U.Es and M.L.Es of λ^q for Different Values of α

$n \rightarrow$		20		30		40		60	
$\alpha \downarrow$	$q \downarrow$	$\tilde{\lambda}^q$	$\hat{\lambda}^q$	$\tilde{\lambda}^q$	$\hat{\lambda}^q$	$\tilde{\lambda}^q$	$\hat{\lambda}^q$	$\tilde{\lambda}^q$	$\hat{\lambda}^q$
		0.5671	0.7089	0.5568	0.696	0.5486	0.6857	0.5436	0.6796
		-0.0996	0.0422	-0.1098	0.0294	- 0.1181	0.019	-0.123	0.0129
0.8	-1	**0.0346**	**0.0403**	**0.0215**	**0.0221**	**0.0168**	**0.0221**	**0.0218**	**0.0106**
		(0.557,0.577)	(0.697,0.721)	(0.55,0.564)	(0.687,0.705)	(0.542,0.555)	(0.678,0.694)	(0.539,0.549)	(0.673,0.686)
		1.9853	1.4889	1.9615	1.5038	1.9284	1.4945	1.9187	1.503
		0.4853	-0.0111	0.4615	0.0038	0.4284	-0.0055	0.4187	0.003
	1	**0.4617**	**0.1274**	**0.3693**	**0.0919**	**0.2966**	**0.0679**	**0.259**	**0.0513**
		(1.956,2.015)	(1.467,1.511)	(1.937,1.986)	(1.485,1.523)	(1.908,1.949)	(1.478,1.511)	(1.901,1.937)	(1.489,1.517)
		0.6256	0.6951	0.6266	0.6962	0.6113	0.6792	0.6143	0.6826
		-0.041	0.0285	-0.0401	0.0295	-0.0554	0.0126	-0.0523	0.0159
0.9	-1	**0.0277**	**0.0329**	**0.0184**	**0.0216**	**0.0142**	**0.0143**	**0.0098**	**0.0089**
		(0.616,0.636)	(0.684,0.706)	(0.619,0.635)	(0.687,0.705)	(0.605,0.618)	(0.672,0.687)	(0.609,0.62)	(0.677,0.688)
		1.7631	1.4986	1.712	1.4838	1.6971	1.485	1.6868	1.49
		0.2631	-0.0014	0.212	-0.0162	0.1971	-0.015	0.1868	-0.01
	1	**0.2198**	**0.1088**	**0.1491**	**0.0785**	**0.114**	**0.0578**	**0.0888**	**0.0422**
		(1.739,1.787)	(1.478,1.519)	(1.692,1.732)	(1.466,1.501)	(1.68,1.714)	(1.47,1.5)	(1.672,1.701)	(1.477,1.503)
		0.7082	0.7082	0.6926	0.6926	0.6898	0.6898	0.6804	0.6804
		0.0415	0.0415	0.0259	0.0259	0.0231	0.0231	0.0137	0.0137
1.0	-1	**0.0307**	**0.0307**	**0.0169**	**0.0169**	**0.0131**	**0.0131**	**0.0086**	**0.0086**
		(0.698,0.719)	(0.698,0.719)	(0.685,0.7)	(0.685,0.7)	(0.683,0.697)	(0.683,0.697)	(0.675,0.686)	(0.675,0.686)
		1.5692	1.4908	1.5413	1.4899	1.5452	1.5066	1.5269	1.5015
		0.0692	-0.0092	0.0413	-0.0101	0.0452	0.0066	0.0269	0.0015
	1	**0.1272**	**0.1106**	**0.0804**	**0.0736**	**0.0623**	**0.0574**	**0.04**	**0.0379**
		(1.548,1.591)	(1.47,1.511)	(1.524,1.559)	(1.473,1.507)	(1.53,1.56)	(1.492,1.521)	(1.515,1.539)	(1.489,1.514)

Table 2: U.M.V.U.Es and M.L.Es of $R(t)$

$n \rightarrow$		20		40		60	
$t \downarrow$	$R(t) \downarrow$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)$
0.1	0.99294	0.999966 0.007028 **4.9e − 05** (0.999965,0.999966)	0.992928 -1e-05 **5.7e − 09** (0.992923,0.992932)	0.999966 0.007028 **4.9e − 05** (0.999965,0.999967)	0.992935 -3e-06 **2.5e − 09** (0.992932,0.992938)	0.999966 0.007028 **4.9e − 05** (0.999965,0.999966)	0.992935 -3e-06 **1.7e − 09** (0.992932,0.992937)
0.15	0.9922	0.999621 0.007419 **5.5e − 05** (0.999611,0.99963)	0.992166 -3.6e-05 **4.02e − 08** (0.992154,0.992178)	0.999627 0.007426 **5.5e − 05** (0.999621,0.999633)	0.992185 -1.6e-05 **1.67e − 08** (0.992178,0.992193)	0.999627 0.007425 **5.5e − 05** (0.999621,0.999632)	0.992189 -1.2e-056 **1.17e − 08** (0.992183,0.992196)
0.2	0.99104	0.998047 0.007011 **5e − 05** (0.998001,0.998093)	0.990963 -7.3e-05 **1.527e − 07** (0.99094,0.990987)	0.998042 0.007006 **4.9e − 05** (0.998009,0.998075)	0.990992 -4.4e-05 **7.65e − 08** (0.990975,0.991009)	0.998074 0.007038 **5e − 05** (0.998048,0.998101)	0.991019 -1.7e-05 **5.09e − 08** (0.991005,0.991033)
0.25	0.98927	0.993315 0.004043 **2.3e − 05** (0.993161,0.993469)	0.989146 -0.000126 **5.689e − 07** (0.9891,0.989192)	0.993407 0.004135 **2e − 05** (0.993305,0.99351)	0.989232 -4e-05 **2.493e − 07** (0.989201,0.989263)	0.993363 0.004091 **1.9e − 05** (0.993276,0.993449)	0.989239 -3.3e-05 **1.709e − 07** (0.989213,0.989264)
0.35	0.98261	0.960736 -0.021879 **0.000664** (0.959891,0.96158)	0.982016 -0.000599 **7.1786e − 06** (0.981854,0.982178)	0.961043 -0.021572 **0.000547** (0.960482,0.961604)	0.982313 -0.000301 **2.9177e − 06** (0.982209,0.982418)	0.961205 -0.02141 **0.000512** (0.960752,0.961657)	0.98242 -0.000195 **1.8391e − 06** (0.982336,0.982503)

Table 3: Moment estimators of α and λ

$n \rightarrow$		500		1000		1500	
$\alpha \downarrow$	$\lambda \downarrow$	$\hat{\alpha}_m$	$\hat{\lambda}_m$	$\hat{\alpha}_m$	$\hat{\lambda}_m$	$\hat{\alpha}_m$	$\hat{\lambda}_m$
0.6	0.8	0.5576	0.8025	00.5988	0.8130	0.6001	0.8164
1.5	0.8	1.4235	0.7710	1.4477	0.7796	01.4935	0.8195
1.5	1.0	1.4925	1.0162	1.4928	1.0079	1.50041	1.0017

Table 4: U.M.V.U.Es and M.L.Es of P

$\lambda_1 \rightarrow$	3		3		4		4	
$\lambda_2 \rightarrow$	5		6		5		6	
$P \rightarrow$	0.625		0.6666667		0.5555556		0.6	
$(n, m) \downarrow$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$	$\tilde{P}$	$\hat{P}$
	0.415	0.6487	0.4075	0.6443	0.4348	0.6605	0.4283	0.6567
	-0.21	0.0237	-0.2592	-0.0224	-0.1207	0.105	-0.1717	0.0567
(5,10)	**0.0447**	**8e − 04**	**0.0679**	**7e − 04**	**0.0149**	**0.0111**	**0.0299**	**0.0034**
	(0.413,0.416)	(0.6478,0.6496)	(0.406,0.409)	(0.6433,0.6453)	(0.434,0.436)	(0.6599,0.6612)	(0.427,0.43)	(0.6559,0.6574)
	0.4104	0.6505	0.403	0.6465	0.4324	0.6626	0.4269	0.6596
	-0.2146	0.0255	-0.2636	-0.0202	-0.1231	0.1071	-0.1731	0.0596
(10,5)	**0.0463**	**7e − 04**	**0.0698**	**5e − 04**	**0.0153**	**0.0115**	**0.0301**	**0.0036**
	(0.409,0.411)	(0.6499,0.6511)	(0.402,0.404)	(0.6459,0.6471)	(0.432,0.433)	(0.6622,0.663)	(0.426,0.428)	(0.6592,0.66)
	0.4111	0.6509	0.4026	0.6462	0.4325	0.6626	0.4256	0.6589
	-0.2139	0.0259	-0.0204	0.0259	-0.1231	0.1071	-0.1744	0.0588
(10,10)	**0.046**	**7e − 04**	**5e − 04**	**7e − 04**	**0.0153**	**0.0115**	**0.0306**	**0.0035**
	(0.41,0.412)	(0.6504,0.6514)	(0.402,0.404)	(0.6457,0.6468)	(0.432,0.433)	(0.6623,0.663)	(0.425,0.426)	(0.6584,0.6593)