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On the Likelihood Ordering and Tail Behavior of Certain Classes of Skew Normal Distributions

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Abstract

The scalar skew normal distribution introduced has been used to model asymmetric types of data sets which are unimodal in nature. In order to provide more flexibility in modelling, several classes of asymmetric normal distributions that can accomodate plurimodal data sets has been appeared. Throughout the present note, we provide a comparison between asymmetric normal distributions based on certain stochastic orderings as well as their tail behavior.

Key words: Plurimodality; Probability density function; Skewness; Stochastic ordering; Tail behavior.

AMS Subject Classifications: MSC 60E05, MSC 60E10

1. Introduction

The normal distribution is the basis of many statistical works and it has a unique position in statistical theory and applications. The importance of normality is that, many of the distributions including sampling distributions can be approximated by normal distribution. Also, the distributions corresponding to most of the natural processes are either normally distributed or can be approximated to normal, so it is an important tool for analyzing all types of numerical data. But it is noticed that in many of the practical situations, the data set exhibit a departure from normality, and consequently unrestricted usage of normal distribution creates several types of errors. Even though to handle normal distribution is an easy task, it is not an appropriate tool to handle the quantities which are near to normal but skewed naturally.

Consequently, there has been an increased interest in constructing classes of distributions that can account skewness. Azzalini (1985) initiated a methodology to introduce skewness in normal distribution and termed this class of distributions as “the skew normal distribution (SND)”. The SND has been further studied by Liseo (1990), Ball and Mankiw

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(1995), Azzalini and Dalla-Valle (1996), and several others. A detailed account of the SND is available in the books due to Genton (2004) and Azzalini (2014). Following Azzalini (1985), several generalizations came forward. For example, see Kim (2005), Sharafi and Behboodian (2008), Jamalizadeh *et al.* (2008, 2009), Gupta *et al.* (2013)etc. These generalizations can be mainly classified into three broad classes of asymmetric normal distributions those contains the existing types of SND and some of their modified versions. In this direction, Kumar and Anusree (2011, 2013, 2015) considered three classes of asymmetric normal distributions that can accomodate plurimodality and skewness.

In many practical situations, the actuarial and financial data more often exhibits asymmetrically distributed structures with extreme values resulting in tails which are heavier than that of the normal and the tails turn out to behave differently. Tail behaviour of scalar skew normal distribution has been studied in detail by Capitanio (2010),Xin *et al.* (2013). Also, for the asymmetric normal distributions tail behavior and its properties has not been deeply explored, in this context through the present note, we attempt to compare these classes of distributions with their corresponding sub-classes based on some stochastic ordering and nature of tail behavior. The paper is organized as follows. Section 2 contains the description of various asymmetric normal distributions. In Section 3, stochastic ordering of classes of distributions were discussed in detail and the tail behavior in practice of the theoretical results is analyzed in Section 4 and in Section 5, a numerical discussion is carried out.

2. The classes of skew normal distributions

Azzalini (1985) defined the SND as given below.

Definition 1: A random variable X is said to have SND with skewness parameter $\lambda \in R = (-\infty, \infty)$ denoted by $SND(\lambda)$, if its probability density function (p.d.f.) is of the following form, for $x \in R$.

$$g_1(x;\lambda) = 2f(x)F(\lambda x), \tag{1}$$

where $f(\cdot)$ and $F(\cdot)$ are respectively the p.d.f. and cumulative distribution function (c.d.f.) of a standard normal variate and this class of distribution owns the strict inclusion of the normal distribution. The density is strongly unimodal and is not suitable for the analysis of plurimodal situations. Buccianti (2005) remarked that normal and skew normal models are not adequate for describing the situations of plurimodality. There are several such phenomena which cannot be described by either the normal or the skew normal distributions. In order to accommodate plurimodality, Kumar and Anusree (2011) developed a class of asymmetric normal distribution as a modified version of the $SND(\lambda)$, namely "the modified skew normal distribution (MSND)". They developed the MSND as a generalized mixture of standard normal and skew normal distributions, through the following p.d.f.,

$$g_1{}^*(x;\lambda,\alpha) = \frac{2}{\alpha+2} f(x)\left[1+\alpha F(\lambda x)\right], \tag{2}$$

in which $\lambda \in R = (-\infty, \infty)$ and $\alpha \geq -1$. In case of the $SND(\lambda)$, for moderate values of λ nearly all the mass accumulates either on the positive side or on the negative side depending on the sign of the parameterλ. This can be accounted as one of the limitation of the $SND(\lambda)$. In such cases, $SND(\lambda)$closely resembles the half-normal density, with a nearly linear shape in the side with smaller mass. To partially mitigate such a limitation, Arellano-Valle *et al.* (2004)introduced a family of asymmetric normal distributions which they termed as "the skew generalized normal distribution (SGND)" that contains the SND as a special case and it exhibits a better behavior, particularly at the sides with smaller mass. Arellano-Valle *et al.* (2004) defined the SGND as given below.

Definition 2: A random variable Y is said to have SGND with parameters $\lambda_1 \in R = (-\infty, \infty)$ and $\lambda_2 \geq 0$, if its p.d.f. takes the following form, for $y \in R$.

$$g_2(y; \lambda_1, \lambda_2) = 2f(y)F\left(\frac{\lambda_1 y}{\sqrt{1+\lambda_2 y^2}}\right) \tag{3}$$

Following SGND and SND, Kumar and Anusree (2015) proposed a model called "the modified skew generalized normal distribution (MSGND)", which is capable for describing asymmetric as well as plurimodal situations. Kumar and Anusree (2015) defined the MSGND through the p.d.f.

$$g_2{}^*(y; \lambda_1, \lambda_2, \alpha) = \frac{2}{\alpha+2}f(y)\left[1 + \alpha F\left(\frac{\lambda_1 y}{\sqrt{1+\lambda_2 y^2}}\right)\right], \tag{4}$$

in which $\alpha \geq -1$, $\lambda_1 \in R = (-\infty, \infty)$ and $\lambda_2 \geq 0$. Another class of SND was due to Kim (2005), which he defined as follows:

Definition 3: A random variable Z is said to have "two-piece skew normal distribution (TPSND)" with parameter λ, if its p.d.f. takes the following form, for $z \in R = (-\infty, \infty)$.

$$g_3(z; \lambda) = \frac{2\pi \ f(x)F(\lambda|z|)}{[\pi + 2\tan^{-1}(\lambda)]} \tag{5}$$

Note that the TPSND is found to be bimodal for all values of its parameters. The TPSND has been further studied by authors such as Jamalizadeh *et al.* (2012), Salehi *et al.* (2014). From a practical point of view, based on the sign of the shape parameter λ, there exists a symmetric behavior on either sides of the origin, which is a main drawback of the TPSND. To overcome this limitation, Kumar and Anusree (2013) onsidered a modified class of two-piece skew normal distribution, through the p.d.f. for$z \in R$, $\lambda \in R$ and $\rho \in [-1, 1]$.

$$g_3^*(z; \lambda, \rho) = \begin{cases} Cf(z)F(\lambda z), \ z < 0 \\ Cf(y)F(\lambda\rho z), \ z \geq 0, \end{cases} \tag{6}$$

where $C = 2\pi\left[\pi - \tan^{-1}(\lambda) + \tan^{-1}(\rho\lambda)\right]^{-1}$. A distribution with p.d.f. (5) we denoted as TPSND (λ) and a distribution with p.d.f. (6) we denoted as MTPSND (λ, ρ).

3. Stochastic ordering

Here, first we present the definition of likelihood ratio order relation

Definition 4: Let X and Y be two absolutely continuous random variables with p.d.f. f(.) and g(.) respectively. Then X is said to be larger than Y on the basis of likelihood ratio ordering, if $\frac{f(x)}{g(x)}$ is non-decreasing, as x increases.

Thus, for comparing different types of asymmetric normal distributions discussed in this paper, here we obtain the following theorems.

Theorem 1: Let X_1 be a continuous random variable following the MSND with p.d.f.(2) and U be a normal variate. Then X_1 is larger than U on the basis of likelihood ratio ordering if for all $\lambda > 0$, $F(\lambda x_1) \leq F(\lambda u)$.

Proof: Let X_1 follows MSND with p.d.f.(2) and $g_U(x)$ be the p.d.f. of U. Then by definition 4, the ratio

$$\frac{2}{(\alpha+2)f(x)} f(x)[1+\alpha F(\lambda x)]$$

is non decreasing only if for $x_1 < x_2$

$$\begin{aligned} \frac{2f(x_1)[1+\alpha F(\lambda x_1)]}{(\alpha+2)f(x_1)} &\leq \frac{2f(x_2)[1+\alpha F(\lambda x_2)]}{(\alpha+2)f(x_2)} \\ F(\lambda x_1) &\leq F(\lambda x_2), \textit{for all } \lambda > 0. \end{aligned}$$

□

Theorem 2: Let X_1 be a continuous random variable following the MSND with p.d.f.(2) and X follows $SND(\lambda)$. Then X_1 is said to be larger than X on the basis of likelihood ratio ordering if $F(\lambda x) \leq F(\lambda x_1)$ for all $\lambda > 0$.

Proof: Let X_1 follows MSND with p.d.f.(2) and $g_X(x)$ be the p.d.f. of X. Then by definition 4, the ratio $\frac{[1+\alpha F(\lambda x)]}{(\alpha+2)F(\lambda x)}$ is non decreasing only if for $x_1 < x_2$

$$\begin{aligned} \frac{2f(x_1)[1+\alpha F(\lambda x_1)]}{(\alpha+2)2f(x_1)F(\lambda x_1)} &\leq \frac{2f(x_2)[1+\alpha F(\lambda x_2)]}{(\alpha+2)2f(x_2)F(\lambda x_2)} \\ F(\lambda x_2) &\leq F(\lambda x_1), \textit{for all } \lambda > 0. \end{aligned}$$

□

Theorem 3: Let Y_1 be a random variable following MSGND with parameters $\lambda_1 \in R$, $\lambda_2 \geq 0$ and $\alpha \geq -1$ and U be a normal variate. Then Y_1 is said to be larger than U on the basis of likelihood ratio ordering if for all $\lambda_1 > 0, \lambda_2 \geq 0$, $F\left(\frac{\lambda_1 y_1}{\sqrt{1+\lambda_2 y_1^2}}\right) \leq F\left(\frac{\lambda_1 u}{\sqrt{1+\lambda_2 u^2}}\right)$.

Proof: Y_1 follows MSGND with p.d.f. (4) and $g_U(x)$ be the p.d.f. of U. Then by definition 4, the ratio $\frac{2}{(\alpha+2)} f(x)[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2)}}\right)]/f(x)$ is non decreasing only if for $x_1 < x_2$,

$$\begin{aligned} \frac{2}{(\alpha+2)}\left[1+\alpha F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)\right] &\leq \frac{2}{(\alpha+2))}\left[1+\alpha F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)\right] \\ \left[1+\alpha F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)\right] &\leq \left[1+\alpha F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)\right] \\ F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right) &\leq F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right), \textit{for all } \lambda_1 > 0,\ \lambda_2 \geq 0. \end{aligned}$$

□

Theorem 4: Let Y_1 be a random variable following MSGND with parameters $\lambda_1 \in R$, $\lambda_2 \geq 0$ and $\alpha \geq -1$ and Y follows SGND(λ_1, λ_2). Then Y_1 is said to be larger than Y on the basis of likelihood ratio ordering if for all $\lambda_1 > 0$, $\lambda_2 \geq 0$, $F\left(\frac{\lambda_1 y}{\sqrt{1+\lambda_2 y^2}}\right) \leq F\left(\frac{\lambda_1 y_1}{\sqrt{1+\lambda_2 y_1^2}}\right)$

Proof: Y_1 follows MSGND with p.d.f. (4) and $g_Y(y)$ be the p.d.f. of Y. Then by definition 4, the ratio

$$\frac{\frac{2}{(\alpha+2)} f(y_1)\left[1+\alpha F\left(\frac{\lambda_1 y_1}{\sqrt{1+\lambda_2 y_1^2}}\right)\right]}{2f(y)F\left(\frac{\lambda_1 y}{\sqrt{1+\lambda_2 y^2}}\right)}$$

is non-decreasing only if for $x_1 < x_2$,

$$\frac{2}{(\alpha+2)F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)}\left[1+\alpha F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)\right] \leq \frac{2}{(\alpha+2)F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)}\left[1+\alpha F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)\right]$$

$$\begin{aligned}\left[\alpha+\frac{1}{F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)}\right] &\leq \left[\alpha+\frac{1}{F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)}\right]\\ F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right) &\leq F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right),\end{aligned}$$

for all $\lambda_1 > 0, \lambda_2 \geq 0$.

□

Theorem 5: Let Y_1 be a random variable following MSGND with parameters $\lambda_1 \in R$, $\lambda_2 \geq 0$ and $\alpha \geq -1$ and X_1 follows *MSND* (λ, α). Then Y_1 is said to be larger than X_1 on the basis of likelihood ratio ordering if $A(x_1, y_1) \leq B(x_1, y_1)$ where

$$\begin{aligned}A(x_1,y_1) &= \alpha F(\lambda x_1) + \alpha F\left(\frac{\lambda_1 y_1}{\sqrt{1+\lambda_2 y_1^2}}\right) + \alpha^2 F\left(\frac{\lambda_1 y_1}{\sqrt{1+\lambda_2 y_1^2}}\right) F(\lambda x_1) \text{ and}\\ B(y_1,x_1) &= \alpha F(\lambda y_1) + \alpha F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right) + \alpha^2 F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right) F(\lambda x_1)\end{aligned}$$

Proof: Y_1 follows MSGND with p.d.f. (4) and $g_{X_1}(x)$ be the p.d.f. of X_1. Then by definition 4, the ratio

$$\frac{\frac{2}{(\alpha+2)} f(x)\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}{2f(x)F(\lambda x)}$$

is non-decreasing only if for $x_1 < x_2$,

$$\frac{\frac{2f(x_1)}{(\alpha+2)}\left[1+\alpha F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)\right]}{\frac{2f(x_1)}{(\alpha+2)}[1+\alpha F(\lambda x_1)]} \leq \frac{\frac{2f(x_2)}{(\alpha+2)}\left[1+\alpha F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)\right]}{\frac{2f(x_2)}{(\alpha+2)}[1+\alpha F(\lambda x_2)]}$$

$\left[1+\alpha F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)\right]\left[1+\alpha F\left(\lambda x_2\right)\right] \leq \left[1+\alpha F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)\right]\left[1+\alpha F\left(\lambda x_1\right)\right]$
$\alpha F\left(\lambda x_2\right)+\alpha F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right)+\alpha^2 F\left(\frac{\lambda_1 x_1}{\sqrt{1+\lambda_2 x_1^2}}\right) F\left(\lambda x_2\right) \leq \alpha F\left(\lambda x_1\right)+\alpha F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right)+$
$\alpha^2 F\left(\frac{\lambda_1 x_2}{\sqrt{1+\lambda_2 x_2^2}}\right) F\left(\lambda x_1\right)$ holds for all $\lambda_1 \geq 0, \lambda_2 \geq 0, \alpha \geq -1$, and $\lambda \geq 0$. □

Theorem 6: Let Z_1 be a random variable following MTPSND with parameters $\lambda \in R$, $\rho \in [-1,1]$, and U be a normal variate. Then Z_1 is said to be larger than U on the basis of likelihood ratio ordering if for $\lambda \geq 0$ *and* $\rho \epsilon [0,1]$.

$$\begin{cases} F\left(\lambda z_1\right) \leq F\left(\lambda u\right), z_1<0, u<0 \\ F\left(\rho\lambda z_1\right) \leq F\left(\rho\lambda u\right), z_1>0, u>0 \end{cases}$$

Proof: Z_1 follows MTPSND with p.d.f. (6) and $g_{\mathrm{U}}(x)$ be the p.d.f. of U. Then by definition 4, the ratio

$$\begin{cases} \frac{Cf(x)F(\lambda x)}{f(x)}, x<0 \\ \frac{Cf(x)F(\rho\lambda x)}{f(x)}, x \geq 0 \end{cases}$$

is decreasing only if for $x_1 < x_2$

$$\begin{cases} \frac{cf(x_1)F(\lambda x_1)}{f(x_1)} \leq \frac{cf(x_2)F(\lambda x_2)}{f(x_2)}, & x_1<0,\ x_2<0,\ x_1<x_2 \\ \frac{cf(x_1)F(\rho\lambda x_1)}{f(x_1)} \leq \frac{cf(x_2)F(\rho\lambda x_2)}{f(x_2)}, & x_1>0,\ x_2>0,\ x_1<x_2 \end{cases}$$

$$\begin{cases} F\left(\lambda x_1\right) \leq F\left(\lambda x_2\right), & x_1<0,\ x_2<0,\ x_1<x_2 \\ F\left(\rho\lambda x_1\right) \leq F\left(\rho\lambda x_2\right), & x_1>0,\ x_2>0,\ x_1<x_2 \end{cases}$$

for all $\lambda \geq 0$ and $\rho \in [0,1]$.

□

Theorem 7: Let Z_1 be a random variable following MTPSND with parameters $\lambda \in R$, $\rho \in [-1,1]$, and Z follows TPSND (λ). Then Z_1 is said to be larger than Z on the basis of likelihood ratio ordering if for any $\lambda \geq 0$ and $\rho \in [0,1]$,

$$\begin{cases} F(\lambda z_1) \leq F(\lambda z), z_1<0, z<0 \\ F(\rho\lambda z_1) \leq F(\rho\lambda z), z_1>0, u<0 \end{cases}$$

Proof: Z_1 follows MTPSND with p.d.f. (6) and $g_{\mathrm{Z}}(x)$ be the p.d.f. of Z. Then by definition 4, for $x_1 < x_2$ the ratio

$$\begin{cases} \frac{Cf(x_1)F(\lambda x_1)}{f(x_1)} \leq \frac{Cf(x_2)F(\lambda x_2)}{f(x_2)} \\ \frac{Cf(x_1)F(\rho\lambda x_1)}{f(x_1)} \leq \frac{Cf(x_2)F(\rho\lambda x_2)}{f(x_2)} \end{cases}$$

$$\begin{cases} F(\lambda x_1) \leq F(\lambda x_2),\ x_1<0, \quad x_2<0, x_1<x_2 \\ F(\rho\lambda x_1) \leq F(\rho\lambda x_2),\ x_1>0, \quad x_2>0, x_1<x_2 \end{cases}$$

for any $\lambda \geq 0$ and $\rho \in [0,1]$. □

4. Tail behavior of the distribution

Tail behaviors are discussed with reference to Tse (2009).

Definition 5: Let X and Y be two absolutely continuous random variables with probability density functions f(x) and g(x) respectively. Then, the tail behavior of the two distributions can be considered as the limiting ratio of their densities. That is, tail of the numerator density will be thinner (or thicker) than the denominator density as the ratio approaches to zero (or infinity).

Now we obtain the following theorem's, which are helpful for the comparison of different types of asymmetric normal distribution based on the nature of their tail behaviour.

Theorem 8: Let X follows $SND(\lambda)$ with p.d.f (1) and X_1 follows MSND (λ, α) with p.d.f. (2). Then X_1 has thinner tail than that of X, when $\alpha \to -1$ and $\lambda \neq 0$.

Proof: The limiting ratios of the densities of the variables X_1 and X is given by

$$\lim_{x\to\infty} \frac{g_{X_1}(x)}{g_X(x)} = \lim_{x\to\infty} \frac{\frac{2}{\alpha+2} f(x)[1+\alpha F(\lambda x)]}{2f(x)F(\lambda x)}$$

$$= \lim_{x\to\infty} \frac{[1+\alpha F(\lambda x)]}{(\alpha+2)F(\lambda x)}$$
$$= \frac{1+\alpha}{2+\alpha}, for \lambda \neq 0 \tag{7}$$

Thus, the right hand side expression of (7) tends to zero only when $\alpha \to -1$. □

Theorem 9: Let Y_1 follows MSGND $(\lambda_1, \lambda_2, \alpha)$ with p.d.f. (4) and Y follows SGND (λ_1, λ_2) with p.d.f. (3). Then (i) Y_1 has thinner tail than that of Y if $\alpha \to -1$, $\lambda_1 \to \infty$ or $\alpha \to -1$, $\lambda_2 \to 0$ and (ii) Y_1 has thicker tail than that of Y if $\alpha \to -1$ and $\lambda_1 \to -\infty$ or $\alpha \to 0 \; and \; \lambda_1 \to -\infty$.

Proof: Case (i): The limiting ratios of the densities of the variables Y_1 and Y can be written as

$$\lim_{x\to\infty} \frac{h_{Y_1}(x)}{h_Y(x)} = \lim_{x\to\infty} \frac{\frac{2}{\alpha+2} f(x)\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}{2f(x)F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}$$

$$= \frac{1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}{(\alpha+2)F\left(\frac{\lambda_1}{\sqrt{\lambda_2}}\right)}$$

$$\to \frac{F\left(\frac{-\lambda_1 x}{\sqrt{(1+\lambda_2 x^2}}\right)}{F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2)}}\right)}, \; as \; \alpha \to -1$$
$$\to 0, \; as \; \lambda_1 \to \infty$$

Thus, Y_1 is thinner than that of Y if $\alpha \to -1$ and $\lambda_1 \to \infty$.
In a similar way,

$$\lim_{x\to\infty}\frac{h_{Y_1}(x)}{h_Y(x)} = \frac{1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2}}\right)}{(\alpha+2)F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2)}}\right)}$$

$$= \frac{1+\alpha}{(\alpha+2)}, as \lambda_2 \to 0$$

$$\to 0,\ as\ \alpha \to -1$$

Thus, Y_1 is thinner than that of Y if $\alpha \to -1$ and $\lambda_2 \to 0$.
Case (ii): The limiting ratios of the densities of the variables Y_1 and Y can be written as

$$\lim_{x\to\infty}\frac{h_{Y_1}(x)}{h_Y(x)} = \lim_{x\to\infty}\frac{\frac{2}{\alpha+2}f(x)\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}{2f(x)F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}$$

$$= \frac{1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2)}}\right)}{(\alpha+2)F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2)}}\right)}$$

$$\to \frac{F\left(\frac{-\lambda_1 x}{\sqrt{(1+\lambda_2 x^2}}\right)}{F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2)}}\right)},\ as\ \alpha \to -1$$

$$\to \infty,\ as\ \lambda_1 \to -\infty$$

Thus, Y_1 is thicker than that of Y if $\alpha \to -1$ and $\lambda_1 \to -\infty$.
Similarly

$$\lim_{x\to\infty}\frac{h_{Y_1}(x)}{h_Y(x)} = \lim_{x\to\infty}\frac{\frac{2}{\alpha+2}f(x)\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}{2f(x)F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}$$

$$\to \frac{1}{2F\left(\frac{\lambda_1 x}{\sqrt{(1+\lambda_2 x^2)}}\right)},\ as\ \alpha \to 0$$

$$\to \infty,\ as\ \lambda_1 \to -\infty$$

Thus, Y_1 is thicker than that of Y if $\alpha \to 0$ and $\lambda_1 \to -\infty$. □

Theorem 10: Let Z_1 follows MTPSND (λ, ρ) with p.d.f.(6) and Z follows TPSND (λ) with p.d.f.(5). Then Z_1 is thinner than Z if $\rho \to 1$ *and* $\lambda \to -\infty$ and is thicker than Z if $\rho \to -1$ *and* $\lambda \to -\infty$.

Proof: The limiting ratios of the densities of the variables Z_1 and Z can be written in the following way, in which $C = 2\pi[\pi + 2\tan^{-1}(\lambda)]^{-1}$and $C_0 = 2\pi[\pi - \tan^{-1}(\lambda) + \tan^{-1}(\rho\lambda)]^{-1}$.

$$\lim_{x\to\infty}\frac{g_{Z_1}(x)}{g_Z(x)} = \lim_{x\to\infty}\frac{C_0 f(x)F(\rho\lambda x)}{Cf(x)F(\lambda x)}$$

$$= \frac{\pi + 2\tan^{-1}(\lambda)}{\pi - \tan^{-1}(\lambda) + \tan^{-1}(\rho\lambda)}$$

$$\rightarrow 0\ ,\ as\ \rho \rightarrow 1\ and\ \lambda \rightarrow -\infty$$

Thus, Z_1 is thinner than Z if $\rho \rightarrow 1$ *and* $\lambda \rightarrow -\infty$. Also,

$$\lim_{x\rightarrow\infty} \frac{g_{Z_1}(x)}{g_Z(x)} = \frac{\pi + 2\tan^{-1}(\lambda)}{\pi - \tan^{-1}(\lambda) + \tan^{-1}(\rho\lambda)}$$

$$\rightarrow \infty\ ,\ as\ \rho \rightarrow -1\ and\ \lambda \rightarrow \infty$$

Thus, Z_1 is thicker than Z if $\rho \rightarrow -1$ *and* $\lambda \rightarrow \infty$. □

Theorem 11: Let X_1 follows MSND (λ, α) with p.d.f. (2) and Y_1 follows MSGND $(\lambda_1, \lambda_2, \alpha)$ with p.d.f.(4). Then X_1 is thinner than Y_1 when $\alpha \rightarrow -1$.

Proof: The limiting ratios of the densities of the variables X_1 and Y_1 is given by

$$\lim_{x\rightarrow\infty} \frac{g_{X_1}(x)}{h_{Y_1}(x)} = \lim_{x\rightarrow\infty} \frac{\frac{2}{\alpha+2} f(x)[1+\alpha F(\lambda x)]}{\frac{2}{\alpha+2} f(x)\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}$$

$$= \frac{1+\alpha F(\lambda x)}{1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x_2{}^2}}\right)}$$

$$\rightarrow 0,$$

when $\alpha \rightarrow -1$. Thus, X_1 is thinner than Y_1 for $\alpha \rightarrow -1$. □

Theorem 12: Let X_1 follows MSND (λ, α) with p.d.f.(2) and Z_1 follows MTPSND (λ, ρ) with p.d.f.(6). Then X_1 is thinner than Z_1 either if $\alpha \rightarrow 0$ or $\alpha \rightarrow -1$ and $\rho \rightarrow -1$.

Proof: The limiting ratios of the densities of the variables X_1 and Z_1 is

$$\lim_{x\rightarrow\infty} \frac{g_{X_1}(x)}{g_{Z_1}(x)} = \lim_{x\rightarrow\infty} \frac{\frac{2}{\alpha+2} f(x)[1+\alpha F(\lambda x)]}{C f(x) F(\rho\lambda x)}$$

$$= \frac{[\pi - \tan^{-1}(\lambda) + \tan^{-1}(\rho\lambda)]}{\pi(\alpha+2)}$$

$$\rightarrow 0,$$

when $\alpha \rightarrow 0$ $or\alpha \rightarrow -1$ and $\rho \rightarrow -1$. Thus, then X_1 is thinner than Z_1 when $\alpha \rightarrow 0$ $or\alpha \rightarrow -1$ and $\rho \rightarrow -1$. □

Theorem 13: Let Y_1 follows MSGND $(\lambda_1, \lambda_2, \alpha)$ with p.d.f.(4) and Z_1 follows MTPSND (λ, ρ) with p.d.f.(6). Then Y_1 is thinner than Z_1 if for $\lambda_1 \rightarrow \infty$ either $\alpha \rightarrow -1,\ \rho \rightarrow 1$ or $\alpha \rightarrow -1,\ \rho \rightarrow -1$.

Proof: The limiting ratios of the densities $h_{Y_1}(x)$ and $k_{Z_1}(x)$ of the variables $Y_1$and Z_1 is

$$\lim_{x\to\infty}\frac{h_{Y_1}(x)}{k_{Z_1}(x)}=\lim_{x\to\infty}\frac{\frac{2}{\alpha+2}f(x)\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}{Cf(x)F(\rho\lambda x)}$$

$$=\frac{[\pi-\tan^{-1}(\lambda)+\tan^{-1}(\rho\lambda)]\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}{\pi(\alpha+2)F(\rho\lambda x)}$$

$$=\begin{cases}\dfrac{1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)}{(\alpha+2)F(\lambda x)}, \ as\ \rho\to 1\\ \dfrac{\left[\pi-2\tan^{-1}(\lambda)\right]\left[1+\alpha F\left(\frac{\lambda_1 x}{\sqrt{1+\lambda_2 x^2}}\right)\right]}{\pi(\alpha+2)F(-\lambda x)}, as\ \rho\to -1\end{cases}$$

$\to 0$, as $\alpha\to -1,\ \lambda_1\to\infty$

Thus, Y_1 is thinner than Z_1 if for $\lambda_1\to\infty$ either $\alpha\to -1,\ \rho\to 1$ or $\alpha\to -1,\ \rho\to -1$. □

5. Numerical discussion

The tail behaviour is influenced by its skewness parameter where one tail becomes heavier relative to the other depending shape parameter. It is very crucial in the study of rare events, risk assessment, risk management and insurance. Here for numerical illustration we have compared various distributions based on their Hill estimator value for particular values of its parameters and is shown in Table 1. In a similar manner, one can compare heavier tails using this method this value will typically be close to zero, indicating light tails or tail is not heavy. Also, the tail behaviour of classes of distribution for various simulated values of its parameter are obtained is as shown in Table 2.

Table 1: Simulated Hill Estimator values for SND and MSND, MSGND and SGND, MTPSND and TPSND, MSND and MTPSND, MSGND and MTPSND for particular values of $(\alpha=0.5,\lambda=0.5,\rho,\lambda_1=0.2$ **and** $\lambda_2=0.4)$

Distributions	Hill Estimator
MSND	0.0045
MSGND	0.002
SGND	0.0048
MTPSND	0.0098
TPSND	0.56
MSND	0.432
MTPSND	0.0029
MSGND	0.395
MTPSND	0.23

Table 2: Ratio of the densities and its behaviour for various values of its parameters

Distributions	Ratio				Behaviour
SND(0.5), MSND(0.5,-1)	0	0	0	0	X_1, X are Comparable
SND(0.5), MSND(0.5,0.55)	0.7828275	0.6817935	0.6359176	0.6169725	X_1 thinner, X thicker
SND(0.5), MSND(0.5,0.65)	0.7910227	0.6938013	0.6496566	0.6314263	X_1 thinner, X thicker
SND(0.5), MSND(0.5,0.75)	0.7986219	0.7049358	0.6623963	0.6448290	X_1 thinner, X thicker
SND(0.5), MSND(0.5,0.85)	0.8056878	0.7152889	0.6742421	0.6572911	X_1 thinner, X thicker
MSGND(0.5,0.6,-1), SGND(0.5,0.6)	0	0	0	0	Y_1, Y are Comparable
MSGND(0.5,0.6,0.55), SGND(0.5,0.6)	0.8156047	0.7709895	0.7577999	0.7525272	Y_1 thinner, Y thicker
MSGND(0.5,0.6,0.65), SGND(0.5,0.6)	0.8225630	0.7796314	0.7669395	0.7618658	Y_1 thinner, Y thicker
MSGND(0.5,0.6,0.75), SGND(0.5,0.6)	0.8290153	0.7876448	0.7754144	0.7705253	Y_1 thinner, Y thicker
MSGND(0.5,0.6,0.85), SGND(0.5,0.6)	0.8350148	0.7950959	0.7832946	0.7785770	Y_1 thinner, Y thicker
MSGND(0.5,0.6,-1), MSND(0.5,alpha)	0	0	0	0	X_1, Y_1 are Comparable
MSGND(0.5,0.6,0.55), MSND(0.5,alpha)	0.8350148	0.7950959	0.7832946	0.7785770	X_1 thinner, Y_1 thicker
MSGND(0.5,0.6,0.65), MSND(0.5,alpha)	0.8250121	0.7930989	0.7832946	0.7685571	X_1 thinner, Y_1 thicker
MSGND(0.5,0.6,0.75), MSND(0.5,alpha)	0.8250248	0.7940959	0.7852546	0.7685670	X_1 thinner, Y_1 thicker
MSGND(0.5,0.6,0.85), MSND(0.5,alpha)	0.8250314	0.7850959	0.7732946	0.7589670	X_1 thinner, Y_1 thicker

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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