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Stress strength reliability under right truncated exponential and power function strength with normal stress

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Abstract

The problem of estimating strength reliability R = P(Y>X) of a component plays an important role in reliability analysis, where stress X is imposed on a component having strength Y. Stress-strength reliability has been discussed in Birnbaum (1956), Lloyed and Lipow (1962) to the recent work of Rehman, Ullah and Singh (2000) and Srivastava (2005) etc. Usually the studies have been carried out about the evaluation of R. In this article the problem of strength reliability is considered in a different perspective. Instead of finding R for a given set of distributions of stress and strength, we have found the required parametric values of the assumed distributions so that a desired level of strength reliability may be achieved. We assume normal stress with two different strength distributions of a component namely: right truncated exponential distribution and power function distribution. Numerical results under both the cases are derived. We find that for some values of the parameter of the power function strength distribution, normal stress produces higher reliability of a component then the exponential stress considered by Alam and Roohi (2003).

Key words: Stress-strength model; Reliability; Estimation; Disaster.

1 Introduction

If Y denotes the strength of a component and X the stress imposed on it, the reliability of the component is defined as

$$R = P(Y > X). \tag{1.1}$$

Such a reliability of a component is known as its strength reliability. The determination of stress-strength reliability was first considered by Birnbaum (1956).

Lloyed and Lipow (1962) describe an application where *X* is the maximum chamber pressure generated by the ignition of a solid propellant in a rocket engine. Estimation of R under independent stress-strength model was considered by Downton(1973), Beg and Sing(1979), Nandi and Aich(1994), Basu(1981), Reiser and Guttman (1986), Rehman, Ullah and Singh(2000), Srivastava(2005) and many others. Usually the studies have been carried out about the evaluation of R under given set of probability distributions of stress placed on the component and strength of the component. The performance of a component is reflected in the parameters of the probability distribution of the strength of the component. Alam and Roohi (2003) have assumed that it is possible to re-design/ re-assemble the component so as to bring the parameters of the strength distribution of the component at a desired level.

They have proposed a method to obtain the required values of the parameters of the strength distribution of the component, so that a desired level of strength reliability may be achieved. They have used power function distribution for the strength of a component facing an exponential stress.

In this paper we consider a normal stress with two strength distributions independent of distribution of stress, namely: (i) right truncated exponential distribution and (ii) the power function distribution with finite range of strength. The aim of the paper is to determine the values of the parameter of the strength distribution for the given tolerance level, so that desired levels of strength reliability of a component may be achieved. Such values are presented in the form of tables. The main results are derived in section 3 and 4, followed by an illustrative example in section 4 and conclusions in section 5.

2 Derivation of the results under power function stress

Let stress *X* and strength *Y* respectively have the probability density function (pdf)

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x, \mu < \infty, \sigma > 0 \tag{2.1}$$

and

$$g(y) = \left(\frac{a}{\theta}\right)\left(\frac{y}{\theta}\right)^{a-1}, \quad 0 < y < \theta,\ a > 0 \tag{2.2}$$

Since the upper limit of *Y* is θ, *Y* can not exceed *X* if *X* exceeds θ. Hence it is necessary to find P($X > \theta$) first. It will indicate the change of utter failure of the strength *Y* against the stress X. We may regard such an eventuality as a ‘disaster’.

Here

$$P(X > \theta) = 1 - \Phi\left(\frac{\theta - \mu}{\sigma}\right)$$

$$= 1\text{-}\Phi(k) = \alpha \quad \text{(say)}, \qquad (2.3)$$

where $k = \frac{\theta - \mu}{\sigma}$ and $\Phi(k)$ is the cumulative distribution function of the standard normal distribution at point k.

Of course this probability should be very small which implies that for fixed θ and known μ and σ, k can be determined, which has to be large positive real number. Clearly, chances of disaster recedes as θ increases with respect to the standard deviation (σ) of the stress. Alternatively we may find the required values of k for a fixed α. The values of k for given tolerance levels α can be obtained using the table of the area under the standard normal distribution (See, Fisher (1953)). Table 1 gives the values of k for selected tolerance level α.

Table 1: Values of k for selected tolerance level α

α	0.00005	0.0005	0.001	0.02	0.05	0.10	0.50
k	3.891	3.291	3.090	2.576	1.645	1.282	0.000

Obviously the ultimate strength capability θ must increase if we wish to have a small tolerance level. Once the value of k are fixed as per requirement, θ is fixed as $\theta = \mu + k\sigma$. Since stress and strength distributions are independent the reliability R is given by

$$R = P(Y > X)$$

$$= \int_{-\infty}^{\infty} \int_{x}^{\infty} f(x)g(y)dydx$$

Taking the transformation $Y{=}VX$, we get

$$R = \int_{-\infty}^{\infty} \int_{1}^{\infty} xf(x)g(vx)dvdx \qquad (2.4)$$

However, in this particular case, we have

$$R = \int_{-\infty}^{\theta} \int_{1}^{\frac{\mu+k\sigma}{x}} xf(x)g(vx)dvdx$$

$$= \int_{-\infty}^{\mu+k\sigma} \text{xf(x)} \int_{1}^{\frac{\mu+k\sigma}{x}} \frac{a}{(\mu+k\sigma)^{a}} x^{a-1} v^{a-1} dvdx$$

$$= \Phi(k_1) - \frac{1}{(\mu + k\sigma)^a} \int_{-\infty}^{\mu + k\sigma} x^a \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \Phi(k_1) - \frac{1}{(\mu + k\sigma)^a} \mu_a^{'}(k_1), \tag{2.5}$$

where $k_1 = \mu + k\sigma$ and $\mu_a^{'}(k_1) = a^{th}$ raw moment of $N(\mu, \sigma^2)$ distribution with right truncated at k_1.

To evaluate $\mu_a^{'}(k_1)$ we use the following recurrence relation of right truncated normal distribution.

$$a\mu_{a-1}^{'}(k_1) = \frac{1}{\sigma^2} \mu_{a+1}^{'}(k_1) + \left(\frac{k_1 - \mu}{\sigma}\right) \mu_a^{'}(k_1), \; a=2,3,....$$

with $\mu_1^{'}(k_1) = \mu - \sigma \frac{\phi(b)}{\Phi(b)}$, $b = \frac{k_1 - \mu}{\sigma} = k$

and $\mu_2(k_1)$ = variance of right truncated normal distribution truncated at k_1.

$$= \mu_2^{'}(k_1) - (\mu_1^{'}(k_1))^2$$

$$= \sigma^2 \left[1 - b\frac{\phi(b)}{\Phi(b)} - \left(\frac{\phi(b)}{\Phi(b)}\right)^2 \right],$$

where $\phi(b) = \frac{1}{\sqrt{2\pi}} e^{-b^2/2}$, $\Phi(b) = \int_{-\infty}^{b} \phi(x)dx$.

Without loss of generality we can assume μ = 0 and σ = 1 so that k_1 becomes k and (2.5) reduces to

$$R = \Phi(k) - \frac{1}{k^a} \mu_a^{'}(k) \tag{2.6}$$

Using the equation (2.6) we obtain strength reliability R for selected values a and k as shown in Table 2.

Table 2: Values of *R* for different choice of *a* and *k*.

a	*k*							
	1.2	1.5	1.8	2.1	2.4	2.7	3.0	3.3
1	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
2	0.109	0.420	0.623	0.738	0.810	0.856	0.886	0.907
3	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

3 Derivation of the results under right truncated exponential strength

Here we assume the right truncated exponential distribution for the strength of a component whose pdf is

$$g(y)=\frac{e^{-\frac{y}{\beta}}}{\beta\left(1-e^{-\frac{\theta}{\beta}}\right)},\quad 0<y<\theta,\ \beta>0 \tag{3.1}$$

and the distribution of stress *X* as normal defined in (2.1).

With the similar arguments discussed in section 2, we find *R* from equation (2.4) as $R=P(Y>X)$

$$=\int_{-\infty}^{\theta} \mathrm{xf(x)}\left\{\int_{1}^{\frac{\mu+k\sigma}{x}} \frac{e^{-vx/\beta}}{\beta\left(1-e^{-(\mu+k\sigma)/\beta}\right)}dv\right\}dx$$

$$=\frac{1}{1-e^{-(\mu+k\sigma)/\beta}}\left[\begin{array}{l} e^{-\mu/\beta}\int_{-\infty}^{k}\frac{1}{\sqrt{2\pi}}e^{-\sigma w/\beta-w^2/2}dw \\ -e^{-(\mu+k\sigma)\mu/\beta}\int_{-\infty}^{k}\frac{1}{\sqrt{2\pi}}e^{-w^2/2}dw \end{array}\right]$$

$$= \frac{1}{1-e^{-(\mu+k\sigma)/\beta}} \left[\begin{array}{l} e^{-\mu/\beta} \left\{ \frac{\sigma^2}{2\beta^2} - -e^{-(\mu+k\sigma)/\beta} \int_k^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\sigma w/\beta - w^2/2} dw \right\} \\ -e^{-(\mu+k\sigma)/\beta} \Phi(k) \end{array} \right] \quad (3.2)$$

Here the probability of disaster, α = P($X > \theta$) has to be small which implies k must be positive. For k >0, using the result 3.322(1), page:333, of Gradshteyn and Ryzhik (2006), we have

$$\int_u^{\infty} e^{-\gamma u - u^2/4\beta} du = \sqrt{\pi\beta} e^{\beta\gamma^2} \left[1 - \Phi\left(\gamma\sqrt{\beta} + \frac{u}{2\sqrt{\beta}} \right) \right]; \mathrm{Re}\,\beta > 0, u > 0.$$

Thus for μ = 0, σ = 1, we obtained R from (3.2) for selected values of β and k. The values of R are shown in Table 3.

Table 3: Values of R for different choice of β and k.

β	k							
	1.2	1.5	1.8	2.1	2.4	2.7	3.0	3.3
0.75	0.8451	0.8499	0.8632	0.8731	0.8793	0.8831	0.8855	0.8869
0.80	0.6927	0.7199	0.7404	0.7549	0.7646	0.7706	0.7746	0.7769
0.90	0.4565	0.5069	0.5419	0.5662	0.5827	0.5939	0.6013	0.6064
0.95	0.3558	0.4162	0.4592	0.4887	0.5091	0.5229	0.5324	0.5388
1.00	0.2640	0.3348	0.3845	0.4193	0.4435	0.4603	0.4719	0.4798
1.10	0.0971	0.1893	0.2543	0.2995	0.3315	0.3539	0.3699	0.3814
1.15	0.0213	0.1248	0.1961	0.2466	0.2825	0.3080	0.3263	0.3395

4 An Illustrative example

While manufacturing an item with its strength following power function or right truncated exponential distribution, it is likely that the possible values of θ may have an upper limit, for example say 3.3. Without loss of generality, let us assume that μ =0 and σ = 1 in case of normal stress as discussed in the section 2. For tolerance level $\alpha \leq 0.02$, from table 1 we find that $k = \theta \geq 2.576$. Since θ can not exceed 4 we have the option of fixing the item in such a way that $2.576 \leq \theta \leq 3.3$. That is $2.576 \leq k \leq 3.3$ and corresponding value of α or β from Table 2 or 3 respectively can be decide to achieve maximum strength reliability.

5 Conclusions

Table 2 and 3 reveal that even for large tolerance level , very high strength reliability can be achieved in case of normal stress rather than exponential stress, and also for a given tolerance level normal stress gives higher strength reliability for some values of the parameter a of a power function strength distribution rather than the exponential stress considered by Alam and Roohi (2003)

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