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Extinction and Stationary Distribution of a Stochastic $SEII_aI_qHR$ Epidemic Model with Intervention

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Abstract

Mathematical and statistical models serve as valuable tools for the analysis and simulation of infectious disease transmission. This study explores the dynamics of Covid-19 through the utilization of a deterministic epidemic model denoted as $SEII_aI_qHR$, incorporating interventions. The investigation focuses on essential aspects such as the positivity, boundedness, existence of various equilibria based on the basic reproduction number (R_0), and asymptotic behavior of solutions around these equilibria in the deterministic model. Recognizing the significance of environmental noise and the involvement of random factors in real-world disease propagation systems, we also develop a stochastic version of the $SEII_aI_qHR$ model to account for the impact of noise. We establish the necessary conditions for the existence and uniqueness of solutions for the system and discuss the ergodic stationary distribution as well as the conditions for system extinction. To validate our analytical findings, we conduct numerical studies. Our results indicate that the rate of intervention and the fraction of the population in quarantine actively influence disease control efforts.

Key words: Stochastic model; Disease intervention; Extinction; Stationary distribution; Sieve bootstrap test.

AMS Subject Classifications: 37H30, 37A50, 60G17, 34A34, 37N25

1. Introduction

Infectious diseases are the leading cause of deaths in the low-income countries (W.H.O., 2020). As of 2019, all communicable diseases together accounted for 36% of all deaths worldwide (W.H.O., 2020). Some example of communicable diseases are SARS, MERS-CoV, COVID-19, Dengue, Malaria, *etc.* Severe acute respiratory syndrome (SARS) is a viral respiratory disease caused by a SARS-associated coronavirus. Burden of SARS outbreak in 2003 in Asian countries is around USD $60 billion (Ding and Zhang, 2022). Middle East Respiratory Syndrome (MERS) is viral respiratory illness and it was first occurred in 2012 in Saudi Arabia. Approximately 35% of MERS cases reported to WHO have died (W.H.O., 2022). Recent outbreak of COVID-19 infection causes around 7 million deaths worldwide (W.H.O.,

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2023b). Dengue is a viral infection caused by the bite of infected mosquitoes. Around half of the world population are at risk of dengue infection with 100–400 million infections occurring each year (Bhatt *et al.*, 2013). Along with the dengue, as of 2021, around half of the world population at risk of Malaria with around 247 million cases and approximately 0.61 million deaths currently occurring each year (W.H.O., 2023a).

In epidemiology, compartmental SIR type models can provide an overall understanding of the dynamics of infectious diseases. Information like spread dynamics, incidence peak timing, transmission severity, effect of disease control strategies *etc.* can be obtained by studying mathematical models (Cai *et al.*, 2017; Ding and Zhang, 2022; Tang *et al.*, 2020; Li *et al.*, 2020). Classical epidemiological models of communicable diseases are mainly deterministic compartmental systems (Choisy *et al.*, 2007; Wearing *et al.*, 2005). However, disease incidence growth in general random in nature since uncertainty in contact rates (Cai *et al.*, 2013; Allen, 2017). Furthermore, disease incidence also depend on population demographic rates which in-general follows Markovian process therefore, it is related to environmental noise (Cai *et al.*, 2013; Allen, 2017). Thus, stochastic differential equation (SDE) based models can provide more realistic information on disease spread at initial stage of infection (Allen, 2008, 2017; Cai *et al.*, 2013; Mao, 2007; Oksendal, 2013).

Recently, there are few works on infectious diseases can be found in literature based on stochastic differential equations (Cai *et al.*, 2013; Lahrouz and Omari, 2013; Ding and Zhang, 2022; Cai *et al.*, 2017; Rao *et al.*, 2012; Din *et al.*, 2021; Sun *et al.*, 2022; Din *et al.*, 2020; Tuckwell and Williams, 2007). Randomness in these models are incorporated either by adding random noise in the state equations or by considering environmental fluctuations in some model parameters (Allen, 2008, 2017). Cai *et al.* (2013) found that random fluctuations can suppress the disease outbreak that leads some insight on disease control strategies. Lahrouz and Omari (2013) considered a SIRS epidemic model with general incidence rate in a population of varying size. They analytically determined the sufficient conditions for the extinction and the existence of a unique stationary distribution. Ding and Zhang (2022) developed a stochastic SIRS epidemic model with information intervention. Author's determined that the average in time of the second moment of the solutions of the stochastic system is bounded for a relatively small noise. Furthermore, they found that information interaction response rate have a vital role in reducing disease incidence, and as the intensity of the response increases, the number of infected population decreases, which is beneficial for disease control (Ding and Zhang, 2022). Cai *et al.* (2017) considered a stochastic version of SIRS epidemic model with ratio-dependent incidence rate. Author's mathematically derived some results on permanence and extinction of the proposed stochastic epidemic model. Rao *et al.* (2012) determined stability of an epidemic model with diffusion and stochastic perturbation. Din *et al.* (2021) use a stochastic Markovian dynamics approach to describe the spreading of dengue and the threshold of the disease. Some mathematical properties of the stochastic epidemic model are determined.

In this paper, we first develop a deterministic $SEII_aI_qHR$ epidemic model with frequency dependent incidence rate based on the assumption that a susceptible individual may get infection either by contacting a symptomatic or an asymptomatic or an exposed individual. This deterministic model also considered the transmission variability among different transmission rates from symptomatic, asymptomatic and exposed individuals. Furthermore, model also considered the awareness effect (for example spreading awareness program through media, proper hand sanitization, social distancing, wearing mask, *etc*), and infection (exposed population) quarantine effect. Main objective of this work is to study the effect of stochastic perturbations in the developed deterministic $SEII_aI_qHR$ epidemic model. In particular, we focused on answering the following questions:

- A detailed study of the $SEII_aI_qHR$ epidemic model and its stochastic version. Then comparison between their dynamics based on various factors.
- How the effect of intervention and quarantine effect influenced the dynamics of a disease in presence of environmental fluctuations.

The rest of the paper is presented as follows: In section 2, detailed $SEII_aI_qHR$ model is formulated. In section 3, some basic properties (example: positivity of solution, global stability of the disease-free equilibrium, local stability of the endemic equilibrium, etc) of the deterministic $SEII_aI_qHR$ model are studied. Detailed formulation of the stochastic $SEII_aI_qHR$ model is shown in section 4. We also discussed Euler Maruyama scheme to determine the numerical solution of the stochastic differential equation. Next, we analytically studied the existence and uniqueness of the solution for the SDE model in section 5. Moreover, long term disease extinction, ergodicity of the solution is studied analytically through various mathematical as well statistical concept. In section 6, we numerically studied the deterministic system to support its analytical findings. We further studied the stochastic system and generated various sample paths, average density paths, histograms of densities, stochastic extinction scenario, *etc.* We have replicated the system very large times to adress the role of quarantine population in the trend of infection. Finally, we discuss and conclude our study.

2. The mathematical model

We start with a deterministic compartmental SIR-type model where population is subdivided into seven mutually exclusive sub-classes namely susceptible (S), exposed (E), symptomatic (I), asymptomatic (I_a), quarantined (I_q), hospitalized (H) and recovered (R), respectively. We considered frequency dependent force of infection with the assumption that susceptible can get infection in contact with the symptomatic (I), asymptomatic (I_a), and exposed (E) cases, respectively. However, we also assumed that the probability of infection form the exposed and asymptomatic cases are lesser compared to the symptomatic cases with transmission modification parameters $\eta_1(0 \leq \eta_1 \leq 1)$, and $\eta_2(0 \leq \eta_2 \leq 1)$, respectively. Furthermore, we also considered the effect of some intervention that reduce the transmission rate β by a factor $(1-k)$, where $0 \leq k \leq 1$. In epidemiological point of view, this intervention represents some awareness effect among the susceptible population that reduce the contact with the infected populations (exposed, symptomatic and asymptomatic). The intervention strategies includes the preventive measures such as lock-down, spreading awareness program through media, proper hand sanitization, social distancing, wearing mask, *etc.* which results in slowing down the disease transmission process.

We assume variable human population with recruitment rate Π. The susceptible compartment reduced due to new infection and natural deaths at rate μ_d. Exposed population increased due to new infection coming from the susceptible compartment and reduced due to natural deaths at a rate μ_d. After the incubation period $\frac{1}{\sigma}$, a fractions ρ_1 and ρ_2 of the exposed population become symptomatic and asymptomatic infected and the remaining fraction $(1 - \rho_1 - \rho_2)$ of the exposed population become quarantined. Symptomatic infected compartment (I) is increased due to inflow of infected population coming from the exposed class (E) and reduced due to natural death at a rate μ_d and a fraction α become hospitalized. Asymptomatic infected compartment (I_a) increased due to inflow of a fraction ρ_2 of the exposed population after completion of the incubation period $\frac{1}{\sigma}$. This compartment is decreased due to natural recovery and death at rates γ_a and μ_d, respectively. Quarantined compartment (I_q) increased due to those exposed individuals who are quarantined and this compartment is reduced due to hospitalization of symptomatic cases, natural death and recovery at rates α_q, γ_q and μ_d, respectively. Hospitalized compartment (H) is increased by the patient coming from the symptomatic class and quarantined compartments at rates α, and α_q, respectively. This compartment is decreased due to recovery, disease related death, and natural death at rates γ, δ, and μ_d, respectively. Recovered compartment increased due to inflow of individuals coming from asymptomatic, quarantined, and hospitalized compartments, respectively. This population is reduced by natural death at a rate μ_d. Based on all the assumptions our deterministic the epidemic model that represents the rate of change of different disease classes are provided below:

$$
\begin{aligned}
\frac{dS}{dt} &= \Pi - (1-k)\frac{\beta S}{N}(I + \eta_1 I_a + \eta_2 E) - \mu_d S, \\
\frac{dE}{dt} &= (1-k)\frac{\beta S}{N}(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E, \\
\frac{dI}{dt} &= \rho_1 \sigma E - \alpha I - \mu_d I, \\
\frac{dI_a}{dt} &= \rho_2 \sigma E - \gamma_a I_a - \mu_d I_a, \\
\frac{dI_q}{dt} &= (1-\rho_1-\rho_2)\sigma E - (\alpha_q + \gamma_q) I_q - \mu_d I_q, \\
\frac{dH}{dt} &= \alpha I + \alpha_q I_q - (\gamma + \delta) H - \mu_d H, \\
\frac{dR}{dt} &= \gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R,
\end{aligned} \tag{1}
$$

The schematic diagram and the description of the parameters used in the model (1) is presented in Fig. 1 and Table 1 respectively.

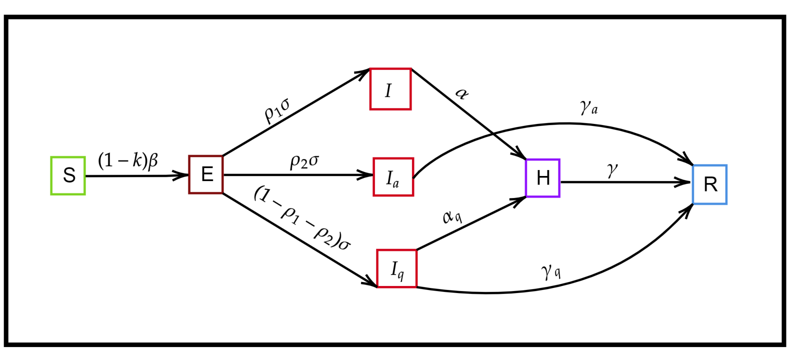

Figure 1: A Flow diagram of the model (1).

Table 1: Description of various parameters used in the model (1).

Parameter	Definitions	Value	Reference
Π	Recruitment rate	10	Din *et al.* (2021)
μ_d	Death rate	0.2	Din *et al.* (2020)
η_1	Modification parameter	0.1002	Senapati *et al.* (2021)
η_2	Modification parameter	(0.1,0.4)	Assumed
k	Strength of intervention	$(0, 0.6544)$	Senapati *et al.* (2021)
β	Rate of disease transmission	1.7399	Senapati *et al.* (2021)
σ	Rate of transition from E to I	0.1923	Li *et al.* (2020)
ρ_1	Fraction of the E move to I	0.3362	Senapati *et al.* (2021)
ρ_2	Fraction of the E move to I_a	0.4204	Senapati *et al.* (2021)
α	Rate of transition from I to H	0.2174	Li *et al.* (2020)
α_q	Rate of transition from I_q to H	0.1429	Senapati *et al.* (2021)
γ_a	Recovery rate of I_a	0.13978	Tang *et al.* (2020)
γ_q	Recovery rate of I_q	0.11624	Tang *et al.* (2020)
γ	Recovery rate of H	0.0701	Senapati *et al.* (2021)
δ	Rate of disease induced death	0.0175	Senapati *et al.* (2021)

3. Analysis

3.1. Model positivity

Theorem 1: The solution to the system (1) remains positive for all time t (≥ 0) given a non-negative initial condition.

Proof: From (1) we can write
$\frac{dS}{dt}\Big|_{S=0} = \Pi \geq 0,\ \frac{dE}{dt}\Big|_{E=0} = \frac{(1-k)}{N}\beta S(I+\eta_1 I_a) \geq 0,\ \frac{dI}{dt}\Big|_{I=0} = \rho_1\sigma E \geq 0,\ \frac{dI_a}{dt}\Big|_{I_a=0} = \rho_2\sigma E \geq 0,$

$\frac{dI_q}{dt}\Big|_{I_q=0} = (1-\rho_1-\rho_2)\sigma E \geq 0, \frac{dH}{dt}\Big|_{H=0} = \alpha I + \alpha_q I_q \geq 0, \frac{dR}{dt}\Big|_{R=0} = \gamma_a I_a + \gamma_q I_q + \gamma H \geq 0.$
Consequently, the system (1) is positive at all times when positive initial conditions are given. □

3.2. Boundness

Theorem 2: The system (1) is bounded in the feasible region $\{(S, E, I, I_a, I_q, H, R) \in R^7_+ : N(t) \leq \frac{\Pi}{\mu_d};\ S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t) \geq 0,\ \text{at any time } t \geq 0\}$.

Proof: We begin by considering the total population density $N(t)$ and utilize the model (1) in the following manner:

$$N(t) = S(t) + E(t) + I(t) + I_a(t) + I_q(t) + H(t) + R(t),$$

$$\frac{dN}{dt} = \Pi - \mu_d N,$$

By using Gronwall's inequality,

$$\begin{aligned} N(t) &= N(0)e^{-\Pi t} + \tfrac{\Pi}{\mu_d}, \quad t \geq 0, \\ \Rightarrow \lim_{n\to\infty} Sup N(t) &\leq \tfrac{\Pi}{\mu_d}. \end{aligned} \tag{2}$$

So we can say that the system (1) is bounded in the region $\{(S, E, I, I_a, I_q, H, R) \in R^7_+ : N(t) \leq \frac{\Pi}{\mu_d};\ S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t) \geq 0, \text{ at any time } t \geq 0\}$. □

3.3. Local stability of disease-free equilibrium (DFE)

The DFE of the model (1) is given by $E_0(\frac{\Pi}{\mu_d}, 0, 0, 0, 0, 0, 0)$.
The local stability of E_0 in the system (1) can be established using the next generation operator method. Following the notation in Driessche and Watmough (2002), we denote the matrices F and V for the new infection and transition terms, respectively, as follows:

$$F = \begin{bmatrix} \eta_2(1-k)\beta & (1-k)\beta & (1-k)\beta\eta_1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} (\sigma+\mu_d) & 0 & 0 & 0 \\ -\rho_1\sigma & (\alpha+\mu_d) & 0 & 0 \\ -\rho_2\sigma & 0 & (\gamma_a+\mu_d) & 0 \\ -(1-\rho_1-\rho_2)\sigma & 0 & 0 & (\alpha_q+\gamma_q+\mu_d) \end{bmatrix}$$

Therefore, the basic reproduction number, denoted by R_0 (Hethcote, 2000) and calculated as $\rho(FV^{-1})$ where ρ represents the spectral radius, can be expressed as $R_0 = \frac{\eta_2(1-k)\beta}{(\sigma+\mu_d)} + \frac{\rho_1\sigma(1-k)\beta}{(\sigma+\mu_d)(\alpha+\mu_d)} + \frac{(1-k)\beta\eta_1\rho_2\sigma}{(\sigma+\mu_d)(\gamma_a+\mu_d)}$.

By utilizing Theorem 2 from Driessche and Watmough (2002), we can establish the following result.

Lemma 1: The DFE, E_0 of the model (1) is locally-asymptotically stable (LAS) if $R_0 < 1$, and unstable if $R_0 > 1$.

3.4. Global stability of DFE

In order to demonstrate the global stability of E_0 in the model (1), we can rewrite the system as follows:

$$\begin{aligned} \frac{dX}{dt} &= T(X, I') \\ \frac{dI'}{dt} &= G(X, I'), \quad G(X, 0) = 0, \end{aligned} \tag{3}$$

where $X = (S, H, R) \in \mathbb{R}^3_+$ represents the components denoting the number of uninfected individuals, and $I' = (E, I, I_a, I_q) \in \mathbb{R}^4_+$ represents the components denoting the number of infected individuals, including latent, infectious, and other categories. $E_0 = (X^*, 0)$

represents the disease-free equilibrium of the system eqrefEQ:eqn 2.3. For the system (1), the expressions for $T(X, I')$ and $G(X, I')$ are in the Annexure.

From the expression of $G(X, I')$, it is evident that $G(X, 0) = 0$.

To demonstrate the global stability of $\varepsilon_0 = (X^*, 0)$, the following two conditions must be satisfied:

(H1) For $\frac{dX}{dt} = T(X, 0)$, X^* is globally asymptotically stable.

(H2) $G(X, I') = AI - \widehat{G}(X, I)$, $\widehat{G}(X, I) \geq 0$ for $(X, I') \in \Omega$,

Here, $A = D_{I'}G(X^*, 0)$ represents an M-matrix, where the off-diagonal elements are non-negative. Additionally, Ω denotes the region in which the model (1) holds biological significance.

Now, we can express the system defined in **(H1)** as follows:

$$\begin{aligned} \frac{dS}{dt} &= \Pi - \mu_d S, \\ \frac{dR}{dt} &= -\mu_d R. \end{aligned} \tag{4}$$

By solving this system of equations analytically, we obtain the following solution: $S(t) = \frac{\Pi}{\mu_d} + e^{-\mu_d t}(S(0) - \frac{\Pi}{\mu_d})$, $R(t) = e^{-\mu_d t}R(0)$. As $t \to \infty$, $S(t) = \frac{\Pi}{\mu_d}$, $R(t) \to 0$. Hence, X^* is globally asymptotically stable for $\frac{dX}{dt} = T(X, 0)$.

Therefore, we can conclude that **(H1)** holds for the system (1). Now, the matrices A and $\widehat{G}(X, I)$ for the system (1) are in the Annexure.

It is evident that A is an M-matrix, and since $S(t) \leq N(t)$ holds in Ω, we can conclude that $\widehat{G}(X, I) \geq 0$ for $(X, I) \in \Omega$. Based on the findings presented in Castillo-Chavez *et al.* (2002), the following result can be stated:

Theorem 3: The DFE of the model (1) is globally asymptotically stable in Ω whenever $R_0 < 1$.

3.5. Existence of endemic equilibria

In this section, we establish the existence of the endemic equilibrium for the model (1). Let us denote $k_1 = \sigma + \mu_d$, $k_2 = \alpha + \mu_d$, $k_3 = \gamma_a + \mu_d$, $k_4 = \alpha_q + \gamma_q + \mu_d$, $k_5 = \gamma + \delta + \mu_d$. Let $E_*(S^*, E^*, I^*, I_a^*, I_q^*, H^*, R^*)$ represents any arbitrary endemic equilibrium point (EEP) of the model (1). Further, define $\lambda^* = \frac{(1-k)\beta I^*}{N^*} + \frac{(1-k)\beta\eta_1 I_a^*}{N^*} + \frac{(1-k)\beta\eta_2 E^*}{N^*}$.
So we have E_* in terms of λ^* by solving the equations in (1) at steady-state (see Annexure).

By substituting the E_* expressions into λ^*, we can observe that the non-zero equilibrium of the model (1) satisfies the following linear equation in terms of λ^*: $a_0\lambda^* + a_1 = 0$, where, $a_0 = k_2k_3k_4k_5\mu_d + k_3k_4k_5\mu_d\rho_1\sigma + k_2k_4k_5\mu_d\rho_2\sigma + k_2k_3k_5\mu_d(1-\rho_1-\rho_2)\sigma + k_3k_4\mu_d\alpha\rho_1\sigma +$

$k_2k_3\mu_d\alpha_q(1-\rho_1-\rho_2)\sigma+k_2k_4k_5\gamma_a\rho_2\sigma+k_2k_3k_5\gamma_a(1-\rho_1-\rho_2)\sigma+k_3k_4\gamma\alpha\rho_1\sigma+k_2k_3\alpha_q(1-\rho_1-\rho_2)\sigma$ $a_1 = k_1k_2k_3k_4k_5\mu_d(1-R_0)$. Since $a_0 > 0,\ k_1 > 0,\ k_2 > 0,\ k_3 > 0,\ k_4 > 0,\ k_5 > 0$ and $\mu_d > 0$, it becomes evident that the model (1) possesses a unique endemic equilibrium point (EEP) when $R_0 > 1$. On the other hand, when $R_0 < 1$, there is no positive endemic equilibrium point in the model. Based on the analysis, we can conclude that there is no existence of equilibrium other than the disease-free equilibrium (DFE) when $R_0 < 1$. Additionally, it can be demonstrated that the DFE E_0 of the model (1) is globally asymptotically stable (GAS) when $R_0 < 1$.

From the above discussion we have concluded that,

Theorem 4: The model (1) possesses a unique endemic (positive) equilibrium, denoted as E^*, whenever the basic reproduction number $R_0 > 1$. However, for $R_0 \leq 1$, the model does not have any endemic equilibrium.

3.6. Local stability of endemic equilibrium point (EEP)

The EEP of the model (1) is given by $E_*(S^*, E^*, I^*, I_a^*, I_q^*, H^*, R^*)$ where the expressions are computed analytically in the Annexure.

3.7. Local stability

Theorem 5: The endemic equilibrium E_* exhibits local asymptotic stability if all the roots of the characteristic equation possess negative real parts.

Proof: The Jacobian matrix of the system at E_* is as follows:

$$J_{E_*} = \begin{pmatrix} -P_{11} & -P_{12} & -P_{13} & -P_{14} & P_{15} & P_{16} & P_{17} \\ P_{21} & P_{22} & P_{23} & P_{24} & -P_{25} & -P_{26} & -P_{27} \\ 0 & P_{32} & -P_{33} & 0 & 0 & 0 & 0 \\ 0 & P_{42} & 0 & -P_{44} & 0 & 0 & 0 \\ 0 & P_{52} & 0 & 0 & -P_{55} & 0 & 0 \\ 0 & 0 & P_{63} & 0 & P_{65} & -P_{66} & 0 \\ 0 & 0 & 0 & P_{74} & P_{75} & P_{76} & -P_{77} \end{pmatrix},$$

where, $P_{11} = \frac{\beta(1-k)(N-S^*)}{N^2}(I^* + \eta_1 I_a^* + \eta_2 E^*) + \mu_d$, $P_{12} = \frac{(1-k)\beta S^*}{N^2}(\eta_2 N - I^* - \eta_1 I_a^* - \eta_2 E^*)$, $P_{13} = \frac{(1-k)\beta S^*}{N^2}(N - I^* - \eta_1 I_a - \eta_2 E)$, $P_{14} = \frac{(1-k)\beta S^*}{N^2}(\eta_1 N - I^* - \eta_1 I_a^* - \eta_2 E^*)$, $P_{15} = P_{16} = P_{17} = \frac{(1-k)\beta S^*}{N^2}(I^* + \eta_1 I_a^* + \eta_2 E^*)$ $P_{21} = \frac{\beta(1-k)(N-S^*)}{N^2}(I^* + \eta_1 I_a^* + \eta_2 E^*)$, $P_{22} = \frac{(1-k)\beta S^*}{N^2}(\eta_2 N - I^* - \eta_1 I_a^* - \eta_2 E^*) - \sigma - \mu_d$, $P_{23} = \frac{(1-k)\beta S^*}{N^2}(N - I^* - \eta_1 I_a - \eta_2 E)$, $P_{24} = \frac{(1-k)\beta S^*}{N^2}(\eta_1 N - I^* - \eta_1 I_a^* - \eta_2 E^*)$, $P_{25} = P_{26} = P_{27} = \frac{(1-k)\beta S^*}{N^2}(I^* + \eta_1 I_a^* + \eta_2 E^*)$, $P_{32} = \rho_1\sigma$, $P_{33} = (\alpha + \mu_d)$, $P_{42} = \rho_2\sigma$, $P_{44} = (\gamma_a + \mu_d)$, $P_{52} = \rho_3\sigma$, $P_{55} = (\alpha_q + \gamma_q + \mu_d)$, $P_{63} = \alpha$, $P_{65} = \alpha_q$, $P_{66} = (\gamma + \delta + \mu_d)$, $P_{74} = \gamma_a$, $P_{75} = \gamma_q$, $P_{76} = \gamma$, $P_{77} = \mu_d$.

Here the stability of E_* is determined by the presence of negative real roots in the characteristic equation of J_{E_*}. □

Now, the corresponding characteristic equation is a polynomial of degree 7, and analytical computation becomes challenging. Therefore, we will validate Theorem 5 by performing numerical computations.

4. Stochastic model

The role of environmental change in shaping epidemic development has been widely recognized (Oksendal, 2006). The unpredictable nature of human contact introduces inherent randomness into the growth and spread of epidemics, leading to ongoing disruptions in population dynamics (Beddington and May, 1977; Chen *et al.*, 2023). In the study of epidemic dynamics, the utilization of SDE models is often necessary due to their ability to provide a more suitable framework in various scenarios. These models effectively capture the stochastic nature of population fluctuations and account for the dynamical changes resulting from subtle parameter variations. In a recent investigation, Hussain *et al.* (2023) explored a stochastic version of the MERS-CoV epidemic model, focusing on the ergodic stationary distribution and criteria for disease extinction. Concurrently, Shi and Jiang (2023) introduced a stochastic compartmental model for COVID-19, integrating an Ornstein-Uhlenbeck (OU) process into the contact rate. Their analysis included the criteria for stationary distribution and the derivation of the probability density function near quasi-equilibrium. Additionally, the impact of the OU process on the stochastic model's dynamic behavior was examined. Tan *et al.* (2023) delved into a stochastic SIS epidemic model enriched by media coverage. Through the consideration of two threshold quantities, they investigated the stochastic dynamics, illustrating scenarios where disease eradication is certain or persistent with a distinct stationary distribution. Their study also inferred insights based on the intensity of random disturbances. Furthermore, Ullah *et al.* (2023) explored a stochastic epidemic model incorporating vaccination programs. Extinction and persistence conditions were scrutinized, supported by graphical representations to validate analytical findings.

Many real-world stochastic epidemic models are formulated based on their deterministic counterparts, with the deterministic version serving as a foundation for their development (Jiang *et al.*, 2010; Mao *et al.*, 2002; Li *et al.*, 2020; Thomas and Shelemyahu, 1989). Under the assumption that the coefficients of model (1) are influenced by random noise, which can be accurately represented by Brownian motion, the resulting model (1) can be transformed into a SDE in the following manner:

$$
\begin{aligned}
dS &= \Big[\Pi - \mu_d S - \frac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E)\Big]dt + \theta_1 S\ dB_1,\\
dE &= \Big[\frac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E\Big]dt + \theta_2 E\ dB_2,\\
dI &= \Big[\rho_1 \sigma E - \alpha I - \mu_d I\Big]dt + \theta_3 I\ dB_3,\\
dI_a &= \Big[\rho_2 \sigma E - \gamma_a I_a - \mu_d I_a\Big]dt + \theta_4 I_a\ dB_4,\\
dI_q &= \Big[(1-\rho_1-\rho_2)\sigma E - (\alpha_q + \gamma_q) I_q - \mu_d I_q\Big]dt + \theta_5 I_q\ dB_5,
\end{aligned}
\tag{5}
$$

$$dH = \Big[\alpha I + \alpha_q I_q - (\gamma+\delta)H - \mu_d H\Big]dt + \theta_6 H\ dB_6,$$

$$dR = \Big[\gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R\Big]dt + \theta_7 R\ dB_7,$$

In the model (5), all parameters and state variables are assumed to be non-negative real numbers. The influence of noise is taken into account through the functions $B_i(t),\ i = 1(1)7$, which represent standard Brownian motions, and $\theta_i(>0),\ i = 1(1)7$, which represent the corresponding intensities of the white noise. Additionally, the Brownian motion satisfies the fundamental axiom $B_1(0) = B_2(0) = B_3(0) = B_4(0) = B_5(0) = B_6(0) = B_7(0)$.

Let's define the vector G for the system (5) as $G = [S, E, I, I_a, I_q, H, R]^T$. The transition probability is specified in Table 2. The expectation $E_x[\Delta G]$ and variance $E_x[\Delta G \Delta G^T]$ are defined as follows.

So the Expectation is $E_x[\Delta G] = \sum_{i=1}^{22} P_i(\Delta G)_i =$

$$\begin{bmatrix} \Pi - \mu_d S - \dfrac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) \\ \dfrac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E \\ \rho_1 \sigma E - \alpha I - \mu_d I \\ \rho_2 \sigma E - \gamma_a I_a - \mu_d I_a \\ (1-\rho_1-\rho_2)\sigma E - (\alpha_q + \gamma_q) I_q - \mu_d I_q \\ \alpha I + \alpha_q I_q - (\gamma+\delta)H - \mu_d H \\ \gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R \end{bmatrix} \Delta t.$$

Also the variance is given below:

$$E_x[\Delta G \Delta G^T] = \sum_{i=1}^{22} P_i[(\Delta G)_i][(\Delta G)_i]^T = \begin{bmatrix} M_{11} & M_{12} & 0 & 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & 0 & 0 \\ 0 & M_{32} & M_{33} & 0 & 0 & M_{36} & 0 \\ 0 & M_{42} & 0 & M_{44} & 0 & 0 & M_{47} \\ 0 & M_{52} & 0 & 0 & M_{55} & M_{56} & M_{57} \\ 0 & 0 & M_{63} & 0 & M_{65} & M_{66} & M_{67} \\ 0 & 0 & 0 & M_{74} & M_{75} & M_{76} & M_{77} \end{bmatrix} \Delta t,$$

Here,
$M_{11} = P_1 + P_2 + P_3 + P_4 + P_5 = \Pi + \mu_d S + \frac{(1-k)}{N}\beta S \eta_2 E + \frac{(1-k)}{N}\beta S I + \frac{(1-k)}{N}\beta S \eta_1 I_a$;
$M_{12} = M_{21} = -P_3 = -(\frac{(1-k)}{N}\beta S \eta_2 E)$;
$M_{22} = P_3 + P_4 + P_5 + P_6 + P_7 = \frac{(1-k)}{N}\beta S \eta_2 E + \frac{(1-k)}{N}\beta S I + \frac{(1-k)}{N}\beta S \eta_1 I_a + \sigma E + \mu_d E$;
$M_{23} = M_{32} = P_8 = \rho_1 \sigma E$;
$M_{24} = M_{42} = P_{11} = \rho_2 \sigma E$;
$M_{25} = M_{52} = P_{14} = (1-\rho_1-\rho_2)\sigma E$;
$M_{33} = P_8 + P_9 + P_{10} = \rho_1 \sigma E + \alpha I + \mu_d I$;
$M_{36} = M_{63} = -P_9 = -\alpha I$;

$M_{44} = P_{11} + P_{12} + P_{13} = \rho_2\sigma E + \gamma_a I_a + \mu_d I_a;$
$M_{47} = M_{74} = -P_{12} = -\gamma_a I_a;$
$M_{55} = P_{14} + P_{15} + P_{16} + P_{17} = (1-\rho_1-\rho_2)\sigma E + \alpha_q I_q + \gamma_q I_q + \mu_d I_q;$
$M_{56} = M_{65} = -P_{15} = -\alpha_q I_q;$
$M_{57} = M_{75} = -P_{16} = -\gamma_q I_q;$
$M_{66} = P_9 + P_{15} + P_{19} + P_{20} + P_{21} = \alpha I + \alpha_q I_q + \gamma H + \delta H + \mu_d H;$
$M_{67} = M_{76} = -P_{19} = -\gamma H;$
$M_{77} = P_{12} + P_{16} + P_{19} + P_{22} = \gamma_a I_a + \gamma_q I_q + \gamma H + \mu_d R;$

Now we define,

$$Drift = \mathcal{C}(\mathcal{G}, t) = \frac{E_x[\Delta G]}{\Delta t} = \begin{bmatrix} \Pi - \mu_d S - \dfrac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) \\ \dfrac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E \\ \rho_1\sigma E - \alpha I - \mu_d I \\ \rho_2\sigma E - \gamma_a I_a - \mu_d I_a \\ (1-\rho_1-\rho_2)\sigma E - (\alpha_q + \gamma_q) I_q - \mu_d I_q \\ \alpha I + \alpha_q I_q - (\gamma + \delta) H - \mu_d H \\ \gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R \end{bmatrix}.$$

Also the diffusion is defined as

$$Diffusion = \mathcal{D}(\mathcal{G}, t) = \sqrt{\frac{E_x[\Delta G \Delta G^T]}{\Delta t}} = \sqrt{\begin{bmatrix} M_{11} & M_{12} & 0 & 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & 0 & 0 \\ 0 & M_{32} & M_{33} & 0 & 0 & M_{36} & 0 \\ 0 & M_{42} & 0 & M_{44} & 0 & 0 & M_{47} \\ 0 & M_{52} & 0 & 0 & M_{55} & M_{56} & M_{57} \\ 0 & 0 & M_{63} & 0 & M_{65} & M_{66} & M_{67} \\ 0 & 0 & 0 & M_{74} & M_{75} & M_{76} & M_{77} \end{bmatrix}}.$$

By incorporating the drift and diffusion equations, the SDE for the system can be expressed as follows:

$$d\mathcal{G}(t) = \mathcal{C}(\mathcal{G}, t)\ dt + \mathcal{D}(\mathcal{G}, t)\ dB(t)$$

i.e.,

$$d\begin{bmatrix} S \\ E \\ I \\ I_a \\ I_q \\ H \\ R \end{bmatrix} = \begin{bmatrix} \Pi - \mu_d S - \dfrac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) \\ \dfrac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E \\ \rho_1\sigma E - \alpha I - \mu_d I \\ \rho_2\sigma E - \gamma_a I_a - \mu_d I_a \\ (1-\rho_1-\rho_2)\sigma E - (\alpha_q + \gamma_q) I_q - \mu_d I_q \\ \alpha I + \alpha_q I_q - (\gamma + \delta) H - \mu_d H \\ \gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R \end{bmatrix} dt + \sqrt{\begin{bmatrix} M_{11} & M_{12} & 0 & 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & 0 & 0 \\ 0 & M_{32} & M_{33} & 0 & 0 & M_{36} & 0 \\ 0 & M_{42} & 0 & M_{44} & 0 & 0 & M_{47} \\ 0 & M_{52} & 0 & 0 & M_{55} & M_{56} & M_{57} \\ 0 & 0 & M_{63} & 0 & M_{65} & M_{66} & M_{67} \\ 0 & 0 & 0 & M_{74} & M_{75} & M_{76} & M_{77} \end{bmatrix}}\, dB(t).$$

Table 2: Possible changes in the process of the model

Transition	Probability
$(\Delta G)_1 = [1\,0\,0\,0\,0\,0\,0]^T$	$P_1 = \Pi\ \Delta t$
$(\Delta G)_2 = [-1\,0\,0\,0\,0\,0\,0]^T$	$P_2 = \mu_d S\ \Delta t$
$(\Delta G)_3 = [-1\,1\,0\,0\,0\,0\,0]^T$	$P_3 = \frac{(1-k)}{N}\beta S\eta_2 E\ \Delta t$
$(\Delta G)_4 = [-1\,0\,1\,0\,0\,0\,0]^T$	$P_4 = \frac{(1-k)}{N}\beta SI\ \Delta t$
$(\Delta G)_5 = [-1\,0\,0\,1\,0\,0\,0]^T$	$P_5 = \frac{(1-k)}{N}\beta S\eta_1 I_a\ \Delta t$
$(\Delta G)_6 = [0\,-1\,0\,0\,0\,0\,0]^T$	$P_6 = \sigma E\ \Delta t$
$(\Delta G)_7 = [0\,-1\,0\,0\,0\,0\,0]^T$	$P_7 = \mu_d E\ \Delta t$
$(\Delta G)_8 = [0\,1\,1\,0\,0\,0\,0]^T$	$P_8 = \rho_1\sigma E\ \Delta t$
$(\Delta G)_9 = [0\,0\,-1\,0\,0\,0\,0]^T$	$P_9 = \alpha I\ \Delta t$
$(\Delta G)_{10} = [0\,0\,-1\,0\,0\,0\,0]^T$	$P_{10} = \mu_d I\ \Delta t$
$(\Delta G)_{11} = [0\,1\,0\,1\,0\,0\,0]^T$	$P_{11} = \rho_2\sigma E\ \Delta t$
$(\Delta G)_{12} = [0\,0\,0\,1\,0\,0\,0]^T$	$P_{12} = \gamma_a I_a\ \Delta t$
$(\Delta G)_{13} = [0\,0\,0\,-1\,0\,0\,0]^T$	$P_{13} = \mu_d I_a\ \Delta t$
$(\Delta G)_{14} = [0\,0\,0\,-1\,0\,0\,0]^T$	$P_{14} = (1-\rho_1-\rho_2)\sigma E\ \Delta t$
$(\Delta G)_{15} = [0\,0\,0\,0\,-1\,0\,0]^T$	$P_{15} = \alpha_q I_q\ \Delta t$
$(\Delta G)_{16} = [0\,0\,0\,0\,-1\,0\,0]^T$	$P_{16} = \gamma_q I_q\ \Delta t$
$(\Delta G)_{17} = [0\,0\,0\,0\,-1\,0\,0]^T$	$P_{17} = \mu_d I_q\ \Delta t$
$(\Delta G)_{18} = [0\,0\,1\,0\,0\,0\,0]^T$	$P_{18} = \alpha I\ \Delta t$
$(\Delta G)_{19} = [0\,0\,0\,0\,0\,-1\,0]^T$	$P_{19} = \gamma H\Delta t$
$(\Delta G)_{20} = [0\,0\,0\,0\,0\,-1\,0]^T$	$P_{20} = \delta H\Delta t$
$(\Delta G)_{21} = [0\,0\,0\,0\,0\,-1\,0]^T$	$P_{21} = \mu_d H\Delta t$
$(\Delta G)_{22} = [0\,0\,0\,0\,0\,0\,-1]^T$	$P_{22} = \mu_d R\Delta t$

4.1. Euler Maruyama scheme

In this section, we employ the Euler-Maruyama scheme to obtain the numerical solution of the stochastic differential equation. The model parameters used in the computations are listed in Table 1. The following computational procedure is followed:

$$d\mathcal{G}_n(t) = \mathcal{C}(\mathcal{G}_n, t)\ dt + \mathcal{D}(\mathcal{G}_n, t) dB(t)$$

$$\begin{bmatrix} S^{n+1} \\ E^{n+1} \\ I^{n+1} \\ I_a^{n+1} \\ I_q^{n+1} \\ H^{n+1} \\ R^{n+1} \end{bmatrix} = \begin{bmatrix} S^{n} \\ E^{n} \\ I^{n} \\ I_a^{n} \\ I_q^{n} \\ H^{n} \\ R^{n} \end{bmatrix} + \begin{bmatrix} \Pi - \mu_d S - \frac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) \\ \frac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E \\ \rho_1 \sigma E - \alpha I - \mu_d I \\ \rho_2 \sigma E - \gamma_a I_a - \mu_d I_a \\ (1 - \rho_1 - \rho_2)\sigma E - (\alpha_q + \gamma_q) I_q - \mu_d I_q \\ \alpha I + \alpha_q I_q - (\gamma + \delta) H - \mu_d H \\ \gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R \end{bmatrix} dt$$

$$+ \sqrt{\begin{bmatrix} M_{11} & M_{12} & 0 & 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} & 0 & 0 \\ 0 & M_{32} & M_{33} & 0 & 0 & M_{36} & 0 \\ 0 & M_{42} & 0 & M_{44} & 0 & 0 & M_{47} \\ 0 & M_{52} & 0 & 0 & M_{55} & M_{56} & M_{57} \\ 0 & 0 & M_{63} & 0 & M_{65} & M_{66} & M_{67} \\ 0 & 0 & 0 & M_{74} & M_{75} & M_{76} & M_{77} \end{bmatrix}} \delta B_n$$

5. Parametric perturbation of the model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{P})$ be a complete probability space equipped with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. The filtration is assumed to be increasing and right-continuous, and $\mathcal{F}_0$ contains all $\mathcal{P}$-null sets. Throughout the paper, we denote $a \wedge b$ as the minimum of a and b, $a \vee b$ as the maximum of a and b, and $\langle y(t) \rangle$ as the time average of $y(t)$ defined as $\frac{1}{t}\int_0^t y(s)\ ds$.

5.1. Existence and uniqueness of the global solutions

In order to investigate the dynamic characteristics of the system described by equation (5), the initial step involves verifying the presence of a unique positive solution for this system. This section aims to provide a comprehensive explanation regarding the existence of a unique positive solution to the SDE model represented by equation (5).

Theorem 6: For any initial value $(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) \in \mathbb{R}_+^7$, there is a positive solution $(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t))$ of the stochastic model (5) for $t \geq 0$ and the solution will maintain in $\mathbb{R}^7_+$ with probability one.

Proof: The constants involved in the equations are locally Lipschitz continuous for the given initial population sizes $(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) \in \mathbb{R}^7+$ when $t \in [0, \tau e]$, where τ_e is the explosion time (Yanan and Daqing, 2014; Ji and Jiang, 2014). To establish the global nature of the solution, it is necessary to prove that $\tau_e = \infty$ almost surely (a.s.). We select $k_0 \geq 0$ to be sufficiently large such that $S(0), E(0), I(0), I_a(0), I_q(0), H(0)$, and $R(0)$ all fall within the interval $[\frac{1}{k_0}, k_0]$. For each integer $k \geq k_0$, we define the stopping time $\tau_k = \inf\{t \in [0, \tau_e] : \min(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t)) \leq \frac{1}{k}\ or,\ \max(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t)) \geq k\}$.

We define $\inf(\phi) = \infty$ for the empty set ϕ according to the given notation. By definition, as k approaches infinity, τ_k increases. We set τ_∞ as the limit of τ_k as k tends to infinity, with $0 \leq \tau_\infty \leq \tau_e$ almost surely (a.s.). By proving that $\tau_\infty = \infty$ almost surely, we can demonstrate that $\tau_e = \infty$, and it follows that $(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t)) \in \mathbb{R}^7_+$ a.s. for all $t \geq 0$.

Now, we define a C^2 function $V : \mathbb{R}^7_+ \to \mathbb{R}_+$ such that
$V = V(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t)) = S(t) - 1 - \log S(t) + E(t) - 1 - \log E(t) + I(t) - 1 - \log I(t) + I_a(t) - 1 - \log I_a(t) + I_q(t) - 1 - \log I_q(t) + H(t) - 1 - \log H(t) + R(t) - 1 - \log R(t)$.

Here the function V is non negative as $y - 1 - \log y \geq 0, \forall\ y \geq 0$. For arbitrary values of $k \geq k_0$ and $T \geq 0$, applying the Itô formula to equation (5) yields the following result.

$$dV(S, E, I, I_a, I_q, H, R) = (1 - \tfrac{1}{S})dS + \theta_1(S-1)dB_1(t) + (1 - \tfrac{1}{E})dE + \theta_2(E-1)dB_2(t) + (1 - \tfrac{1}{I})dI + \theta_3(I-1)dB_3(t) + (1 - \tfrac{1}{I_a})dI_a + \theta_4(I_a - 1)dB_4(t) + (1 - \tfrac{1}{I_q})dI_q + \theta_5(I_q - 1)dB_5(t) + (1 - \tfrac{1}{H})dH + \theta_6(H-1)dB_6(t) + (1 - \tfrac{1}{R})dR + \theta_7(R-1)dB_7(t)$$

$$= LV(S, E, I, I_a, I_q, H, R)dt + \theta_1(S-1)dB_1(t) + \theta_2(E-1)dB_2(t) + \theta_3(I-1)dB_3(t) + \theta_4(I_a - 1)dB_4(t) + \theta_5(I_q - 1)dB_5(t) + \theta_6(H-1)dB_6(t) + \theta_7(R-1)dB_7(t).$$

In equation (5), $LH : \mathbb{R}^7_+ \to \mathbb{R}_+$ is defined by the following equation

$$LV(S, E, I, I_a, I_q, H, R) = (1 - \tfrac{1}{S})[\Pi - \mu_d S - \frac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E)] + \frac{\theta_1^2}{2} + (1 - \frac{1}{E})\Big[\frac{(1-k)}{N}\beta S(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E\Big] + \frac{\theta_2^2}{2} + (1 - \frac{1}{I})(\rho_1 \sigma E - \alpha I - \mu_d I) + \frac{\theta_3^2}{2} + (1 - \frac{1}{I_a})(\rho_2 \sigma E - \gamma_a I_a - \mu_d I_a) + \frac{\theta_4^2}{2} + (1 - \frac{1}{I_q})((1 - \rho_1 - \rho_2)\sigma E - (\alpha_q + \gamma_q)I_q - \mu_d I_q) + \frac{\theta_5^2}{2} + (1 - \frac{1}{H})(\alpha I + \alpha_q I_q - (\gamma + \delta)H - \mu_d H) + \frac{\theta_6^2}{2} + (1 - \frac{1}{R})(\gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R) + \frac{\theta_7^2}{2}$$

$$\leq \Pi(1 - \tfrac{1}{S}) + 7\mu_d + \tfrac{\beta I}{N} + \eta_1 \beta \tfrac{I_a}{N} + \eta_2 \beta \tfrac{E}{N} - k\beta \tfrac{I}{N} - k\eta_1 \beta \tfrac{I_a}{N} - k\eta_2 \beta \tfrac{E}{N} + \sigma + \alpha + \gamma_a + \rho_1 \sigma \tfrac{E}{I_q} + \rho_2 \sigma \tfrac{E}{I_q} - \alpha_q - \gamma_q - \alpha \tfrac{I}{H} - \alpha_q \tfrac{I_q}{H} + \gamma + \delta - \gamma_a \tfrac{I_a}{R} - \gamma_q \tfrac{I_q}{R} - \gamma \tfrac{H}{R} + \sum_{i=1}^{7} \tfrac{\theta_i^2}{2}$$

$$\leq \Pi + 7\mu_d + \sigma + \alpha + \gamma_a + \gamma + \delta - \alpha_q - \gamma_q + \sum_{i=1}^{7} \tfrac{\theta_i^2}{2} = K \text{ (say)}$$

Here, K is a positive constant that is independent of the variables S, E, I, I_a, I_q, H, R, and the time t. Therefore, $dV(S, E, I, I_a, I_q, H, R) \leq Kdt + \theta_1(S-1)dB_1(t) + \theta_2(E-1)dB_2(t) + \theta_3(I-1)dB_3(t) + \theta_4(I_a-1)dB_4(t) + \theta_5(I_q-1)dB_5(t) + \theta_7(R-1)dB_7(t)$

Integration both sides of above equation from 0 to $\tau_k \wedge T$

$E[V(S(\tau_k \wedge T), E(\tau_k \wedge T), I(\tau_k \wedge T), I_a(\tau_k \wedge T), I_q(\tau_k \wedge T), H(\tau_k \wedge T), R(\tau_k \wedge T))] \leq V(S(0), E(0), I(0), I_a(0),$
$I_q(0), H(0), R(0)) + K(\tau_k \wedge T) + E[\int_0^{\tau_k \wedge T} \theta_1(S-1)dB_1(t) + \theta_2(E-1)dB_2(t) + \theta_3(I-1)dB_3(t) + \theta_4(I_a-1)dB_4(t) + \theta_5(I_q-1)dB_5(t) + \theta_7(R-1)dB_7(t)] \leq V(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) + KT$

Setting $\Omega_k = \tau_k \leq T$ for $k \geq k_1$ and by $P(\tau_\infty \leq T) > \epsilon,\ P(\Omega_k) \geq \epsilon$.
It is worth noting that for every $w \in \Omega_k$, there exists at least one combination of $S(\tau_k, w)$, $E(\tau_k, w)$, $I(\tau_k, w)$, $I_a(\tau_k, w)$, $I_q(\tau_k, w)$, $H(\tau_k, w)$,$R(\tau_k, w)$ that is equal to either k or $\frac{1}{k}$ and hence $V(S(\tau_k), E(\tau_k), I(\tau_k), I_a(\tau_k), I_q(\tau_k), H(\tau_k),$
$R(\tau_k))$ is not less than $(k - 1 - \log k)$ or $(\frac{1}{k} - 1 + \log k)$.
Consequently, $V(S(\tau_k), E(\tau_k), I(\tau_k), I_a(\tau_k), I_q(\tau_k), H(\tau_k), R(\tau_k)) \geq E[(k - 1 - \log k) \wedge (\frac{1}{k} - 1 + \log k)]$.
Thus, it follows from $P(\tau_\infty \leq T) > \epsilon$ and equation (5) that
$V(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) + KT \geq E[1_{\Omega_{(w)}} V(S(\tau_k), E(\tau_k), I(\tau_k), I_a(\tau_k),$
$I_q(\tau_k), H(\tau_k), R(\tau_k))] \geq \epsilon[(k - 1 - \log k) \wedge (\frac{1}{k} - 1 + \log k)]$
Here, $1_{\Omega_{(w)}}$ denotes the indicator function of Ω. By letting $k \to \infty$, we arrive at the contradiction $\infty > V(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) + KT = \infty$. This implies that $\tau_\infty = \infty$ a.s., thereby completing the proof. □

5.2. Extinction of the disease

Next, we will investigate the dynamic behavior of the epidemic model to determine the conditions for long-term disease elimination. We aim to derive the conditions under which the disease will become extinct within the community. This leads us to the following lemma.

Lemma 2 (Strong Law of Large Number, (Lahrouz and Omari, 2013; Din *et al.*, 2020))**:** Let $M = \{M\}_{t \geq 0}$ be continuous and real-valued local martingale, which vanish as $t \to 0$, then $\lim_{t\to\infty} \langle M, M \rangle_t = \infty$, a.s., $\Rightarrow \lim_{t\to\infty} \frac{M_t}{\langle M, M \rangle_t} = 0$, a.s. and also, $\lim_{t\to\infty} sup \frac{\langle M, M \rangle_t}{t} < 0$ a.s., $\Rightarrow \lim_{t\to\infty} \frac{M_t}{t} = 0$, a.s.

Theorem 7: Let $(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t))$ represent the solution of system (5) for any initial value $(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) \in \mathbb{R}^7_+$. If $R^0_E < 1$, then the solution $(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t))$ of system (5) satisfies $\lim_{t\to\infty} sup \frac{\ln E(t)}{t} \leq (\sigma + \mu_d + \frac{\theta_2^2}{2})(R^0_E - 1) < 0$ a.s., where $R^0_E = \frac{(1-k)\beta(1+\eta_1+\eta_2)}{(\sigma+\mu_d+\frac{\theta_2^2}{2})}$. So for $R^0_E < 1$ the disease will be eradicated in the long term.

Proof: Applying the *Itô* formula to the second equation of model (5), we obtain
$d\ln E(t) = \frac{dE(t)}{E(t)} = [\frac{(1-k)}{N}\beta\frac{S}{E}(I+\eta_1 I_a+\eta_2 E)-\sigma-\mu_d-\frac{\theta_2^2}{2}]dt+\theta_2 dB_2(t)$

$$\leq [(1-k)\beta+(1-k)\beta\eta_1+(1-k)\beta\eta_2-\sigma-\mu_d-\frac{\theta_2^2}{2}]dt+\theta_2 dB_2(t)$$

$$\leq [(1-k)\beta(1+\eta_1+\eta_2)-(\sigma+\mu_d)-\frac{\theta_2^2}{2}]dt+\theta_2 dB_2(t)$$

Integrating the above formula from 0 to t on both sides, we obtain
$\ln E(t)-\ln E(0) \leq \int_0^t [(1-k)\beta(1+\eta_1+\eta_2)-(\sigma+\mu_d)-\frac{\theta_2^2}{2}]ds+\int_0^t \theta_2 dB_2(t)$.

According to the strong law of large numbers (Lahrouz and Omari, 2013; Khasminskii, 2011), we have, $\limsup_{t\to\infty}\frac{\theta_2}{t}\int_0^t dB_2(t)=0$, a.s.
So, $\lim_{t\to\infty} sup\frac{\ln E(t)}{t} \leq \frac{1}{t}\int_0^t [(1-k)\beta(1+\eta_1+\eta_2)-(\sigma+\mu_d)-\frac{\theta_2^2}{2}]ds$

$$\leq \left[(1-k)\beta(1+\eta_1+\eta_2)-(\sigma+\mu_d)-\frac{\theta_2^2}{2}\right]$$

$$\leq (\sigma+\mu_d+\frac{\theta_2^2}{2})\left[\frac{(1-k)\beta(1+\eta_1+\eta_2)}{(\sigma+\mu_d+\frac{\theta_2^2}{2})}-1\right]$$

If we choose $R_E^0 = \frac{(1-k)\beta(1+\eta_1+\eta_2)}{(\sigma+\mu_d+\frac{\theta_2^2}{2})}$, it implies $\lim_{t\to\infty} sup\frac{\ln E(t)}{t} \leq (\sigma+\mu_d+\frac{\theta_2^2}{2})[R_E^0-1] < 0$ if $R_E^0 < 1$.
Therefore, the above result indicates that

$$\lim_{t\to\infty} E(t) = 0 \text{ a.s.},$$

which implies that the disease will be eradicated. This completes the proof. □

5.3. Ergodic stationary distribution

When a disease spreads rapidly within a population, understanding its long-term dynamics becomes a significant concern for health officials. In order to study and address this issue mathematically, stability analysis tools are commonly utilized. Deterministic models, under certain conditions, can show the existence of an endemic equilibrium and its global asymptotic stability. However, in the context of stochastic systems, the presence of an endemic equilibrium is not guaranteed, posing challenges in predicting the persistence of the disease within the population (Din *et al.*, 2020). In our study, inspired by the work of Khasminskii (2011), we aim to investigate the existence of an ergodic stationary distribution for system (5). This analysis provides insights into the long-term persistence of the disease. The deterministic version of the system (5) can be easily obtained by setting $\theta_i = 0$ for $i = 1$ to 7, resulting in a straightforward conversion. However, it is important to note that the original stochastic model and its deterministic counterpart exhibit significant differences. Moreover, empirical evidence suggests the absence of an endemic disease state in the stochastic system, challenging the applicability of traditional linear stability analysis to assess the disease's sustained presence. Consequently, our research focuses on investigating the stationary distribution of the proposed system (5), specifically exploring the existence of ergodic stationary components.

Let's consider the assumption that $X(t)$ is a regular time-homogeneous Markov process in $\mathbb{R}^n_+$. Mathematically, it can be represented as $dX(t) = b(X)dt + \sum_{r=1}^{k} \sigma_r dB_r(t)$, where $b(X)$ represents the drift term.
The diffusion matrix is defined as $A(X) = [a_{ij}(x)],\ a_{ij}(x) = \sum_{r=1}^{k} \sigma_r^i(x)\sigma_j^r(x)$ a.s.

Lemma 3: (Din *et al.*, 2020) The Markov process $X(t)$ has a unique stationary distribution $m(\cdot)$ if there exists a bounded domain $U \subseteq \mathbb{R}^d$ with a regular boundary such that the closure $U \in \mathcal{R}^d$ satisfies the following properties:

1. In the open domain U and some of its neighbors, the smallest eigenvalue of the diffusion matrix $A(t)$ is set far from zero.

2. If $x \in R^d U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $sup_{x\in k} E\tau^x < \infty$ for every compact subset. Moreover, if $f(.)$ is a function integrable with respect to the measure π, then $P\left[\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X_x(t))dt = \int_{R^d} f(x)\pi dx\right] = 1.$

For future reference, let us define another threshold value $R_0^* = \left[\frac{\mu_d(1-k)\beta\rho_1\sigma}{\left(\mu_d+\frac{\theta_1^2}{2}\right)\left(\sigma+\mu_d+\frac{\theta_2^2}{2}\right)\left(\alpha+\mu_d+\frac{\theta_3^2}{2}\right)}\right].$

Theorem 8: If $R_0^* > 1$, then a solution $(S(t), E(t), I(t), I_a(t), I_q(t), H(t), R(t))$, of system (5) is ergodic. Moreover, $\exists$ a unique stationary distribution $\pi(.)$.

Proof: First, we will demonstrate that the second condition of Lemma 3 is satisfied. To accomplish this, we will construct a non-negative C^2 function $\overline{V} : \mathbb{R}^7_+ \to \mathbb{R}_+$ such that it satisfies the following properties:
$\overline{V} = N(t) - c_1 \ln S(t) - c_2 \ln E(t) - c_3 \ln I(t)$, with $c_i \geq 0,\ i = 1(1)3$. Applying $It\hat{o}'s$ formula (Mao, 1997), we obtain

$$L\overline{V} = (\Pi - \mu_d N - \delta H) - c_1\left[\frac{\Pi}{S} - \mu_d - \frac{(1-k)}{N}\beta(I + \eta_1 I_a + \eta_2 E) - \frac{\theta_1^2}{2}\right] - c_2\left[\frac{(1-k)}{N}\beta\frac{S}{E}(I + \eta_1 I_a + \eta_2 E) - \sigma - \mu_d - \frac{\theta_2^2}{2}\right] - c_3\left[\rho_1\sigma\frac{E}{I} - \alpha - \mu_d - \frac{\theta_3^2}{2}\right]$$

$$= \Pi - \mu_d N - \delta H - c_1\frac{\Pi}{S} + c_1\mu_d + c_1(1-k)\beta\frac{I}{N} + c_1(1-k)\beta\eta_1\frac{I_a}{N} + c_1(1-k)\beta\eta_2\frac{E}{N} + c_1\frac{\theta_1^2}{2} - c_2(1-k)\beta\frac{SI}{NE} - c_2(1-k)\beta\eta_1\frac{SI_a}{NE} - c_2(1-k)\beta\eta_2\frac{SE}{NE} + c_2(\sigma+\mu_d) + c_2\frac{\theta_2^2}{2} - c_3\rho_1\sigma\frac{E}{I} + c_3(\alpha+\mu_d) + c_3\frac{\theta_3^2}{2}$$

$$\leq -\left[\mu_d N + c_1\frac{\Pi}{S} + c_2(1-k)\beta\frac{SI}{NE} + c_3\rho_1\sigma\frac{E}{I}\right] + \Pi + c_1(\mu_d + \frac{\theta_1^2}{2}) + c_2(\sigma + \mu_d + \frac{\theta_2^2}{2}) + c_3(\alpha + \mu_d + \frac{\theta_3^2}{2}) + c_1\left[(1-k)\beta\frac{I}{N} + (1-k)\beta\eta_1\frac{I_a}{N} + (1-k)\beta\eta_2\frac{E}{N}\right]$$

$$= -4\left[\mu_d N c_1\frac{\Pi}{S} c_2(1-k)\beta\frac{SI}{NE} c_3\rho_1\sigma\frac{E}{I}\right]^{\frac{1}{4}} + \Pi + c_1(\mu_d + \frac{\theta_1^2}{2}) + c_2(\sigma+\mu_d+\frac{\theta_2^2}{2}) + c_3(\alpha+\mu_d+\frac{\theta_3^2}{2}) + c_1\left[(1-k)\beta\frac{I}{N} + (1-k)\beta\eta_1\frac{I_a}{N} + (1-k)\beta\eta_2\frac{E}{N}\right]$$

$$= -4\left[\mu_d(1-k)\beta\rho_1\sigma\Pi c_1c_2c_3\right]^{\frac{1}{4}} + \Pi + c_1(\mu_d + \frac{\theta_1^2}{2}) + c_2(\sigma+\mu_d+\frac{\theta_2^2}{2}) + c_3(\alpha+\mu_d+\frac{\theta_3^2}{2}) + c_1\left[(1-k)\beta\frac{I}{N} + (1-k)\beta\eta_1\frac{I_a}{N} + (1-k)\beta\eta_2\frac{E}{N}\right].$$

Now we assume that, $\Pi = c_1(\mu_d + \frac{\theta_1^2}{2}) = c_2(\sigma + \mu_d + \frac{\theta_2^2}{2}) = c_3(\alpha + \mu_d + \frac{\theta_3^2}{2})$ where, $c_1 = \frac{\Pi}{\left(\mu_d+\frac{\theta_1^2}{2}\right)}, c_2 = \frac{\Pi}{\left(\sigma+\mu_d+\frac{\theta_2^2}{2}\right)}$ and $c_3 = \frac{\Pi}{\left(\alpha+\mu_d+\frac{\theta_3^2}{2}\right)}$.

So, $L\overline{V} \leq -4\Big[\frac{\mu_d(1-k)\beta\rho_1\sigma\Pi^4}{\left(\mu_d+\frac{\theta_1^2}{2}\right)\left(\sigma+\mu_d+\frac{\theta_2^2}{2}\right)\left(\alpha+\mu_d+\frac{\theta_3^2}{2}\right)}\Big]^{\frac{1}{4}} + 4\Pi + c_1\Big[(1-k)\beta\frac{I}{N} + (1-k)\beta\eta_1\frac{I_a}{N} + (1-k)\beta\eta_2\frac{E}{N}\Big] \leq -4\Pi\Big[(R_0^*)^{\frac{1}{4}} - 1\Big] + c_1\Big[(1-k)\beta\frac{I}{N} + (1-k)\beta\eta_1\frac{I_a}{N} + (1-k)\beta\eta_2\frac{E}{N}\Big]$
where, $R_0^* = \Big[\frac{\mu_d(1-k)\beta\rho_1\sigma}{\left(\mu_d+\frac{\theta_1^2}{2}\right)\left(\sigma+\mu_d+\frac{\theta_2^2}{2}\right)\left(\alpha+\mu_d+\frac{\theta_3^2}{2}\right)}\Big]$.

We define another function of the form:

$V = c_4\Big[N(t) - c_1\ln S(t) - c_2\ln E(t) - c_3\ln I(t)\Big] - \ln S(t) - \ln E(t) - \ln I(t) - \ln I_a(t) - \ln I_q(t) - \ln H(t) - \ln R(t) + N(t)$, where, $c_4 > 0$ represents a constant that will be determined later.
Therefore, $V = c_4\overline{V} - \ln S(t) - \ln E(t) - \ln I(t) - \ln I_a(t) - \ln I_q(t) - \ln H(t) - \ln R(t) + N(t)$.

According to Lemma 3 and the continuity of $\overline{V}(S,E,I,I_a,I_q,H,R)$, we can conclude that $\overline{V}(S,E,I,I_a,I_q,H,R)$ has a unique minimum value around $(S_0,E_0,I_0,I_{a_0},I_{q_0},H_0,R_0)$ in the interior of $\mathbb{R}_+^7$. Now we define a non-negative C^2 function $V : \mathbb{R}_+^7 \to \mathbb{R}_+$ as $V = \overline{V}(S,E,I,I_a,I_q,H,R) - \overline{V}(S_0,E_0,I_0,I_{a_0},I_{q_0},H_0,R_0)$.

Applying $It\hat{o}'s$ formula to V, we obtain

$$\begin{aligned}
LV &= c_4L\overline{V} - L\ln S(t) - L\ln E(t) - L\ln I(t) - L\ln I_a(t) - L\ln H(t) - L\ln R(t) + LN(t)\\
&\leq c_4\Big\{-4\Pi\Big[(R_0^*)^{\frac{1}{4}} - 1\Big] + c_1\Big[(1-k)\beta\tfrac{I}{N} + (1-k)\beta\eta_1\tfrac{I_a}{N} + (1-k)\beta\eta_2\tfrac{E}{N}\Big]\Big\} - \Big[\frac{\Pi}{S} - \mu_d\\
&\quad - \frac{(1-k)}{N}\beta(I + \eta_1 I_a + \eta_2 E) - \frac{\theta_1^2}{2}\Big] - \Big[\frac{(1-k)}{N}\beta\frac{S}{E}(I + \eta_1 I_a + \eta_2 E) - \sigma - \mu_d - \frac{\theta_2^2}{2}\Big] -\\
&\quad \Big[\rho_1\sigma\frac{E}{I} - \alpha - \mu_d - \frac{\theta_3^2}{2}\Big] - \Big[\rho_2\sigma\frac{E}{I_a} - \gamma_a - \mu_d - \frac{\theta_4^2}{2}\Big] - \Big[\alpha\frac{I}{H} + \alpha_q\frac{I_q}{H} - \gamma - \delta - \mu_d - \frac{\theta_6^2}{2}\Big] -\\
&\quad \Big[\gamma_a\frac{I_a}{R} + \gamma_q\frac{I_q}{R} + \gamma\frac{H}{R} - \mu_d - \frac{\theta_7^2}{2}\Big] + \Pi - \mu_d N - \delta H
\end{aligned}$$

$\leq -c_4c_5 + c_1c_4(1-k)\beta\frac{I}{N} + c_1c_4(1-k)\beta\eta_1\frac{I_a}{N} + c_1c_4(1-k)\beta\eta_2\frac{E}{N} - \frac{\Pi}{S} + \mu_d + (1-k)\beta\frac{I}{N} + (1-k)\beta\eta_1\frac{I_a}{N} + (1-k)\beta\eta_2\frac{E}{N} + \frac{\theta_1^2}{2} - (1-k)\beta\frac{SI}{NE} - (1-k)\beta\eta_1\frac{SI_a}{NE} - (1-k)\beta\eta_2\frac{S}{N} + \sigma + \mu_d + \frac{\theta_2^2}{2} - \rho_1\sigma\frac{E}{I} + \alpha + \mu_d + \frac{\theta_3^2}{2} - \rho_2\sigma\frac{E}{I_a} + \gamma_a + \mu_d + \frac{\theta_4^2}{2} - \alpha\frac{I}{H} - \alpha_q\frac{I_q}{H} + \gamma + \delta + \mu_d + \frac{\theta_6^2}{2} - \gamma_a\frac{I_a}{R} - \gamma_q\frac{I_q}{R} - \gamma\frac{H}{R} + \mu_d + \frac{\theta_7^2}{2} + \Pi - \mu_d N - \delta H$
where, $c_5 = \Pi\Big[\Big(R_0^*\Big)^{\frac{1}{4}} - 1\Big] > 0$.

So, $LV \leq -c_4c_5 + (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2) - \frac{\Pi}{S} - (1-k)\beta\frac{SI}{NE} - (1-k)\beta\eta_1\frac{SI_a}{NE} + 6\mu_d + \sigma + \alpha + \gamma_a + \gamma + \delta - \rho_1\sigma\frac{E}{I} - \rho_2\sigma\frac{E}{I_a} - \alpha\frac{I}{H} - \alpha_q\frac{I_q}{H} - \gamma_a\frac{I_a}{R} - \gamma_q\frac{I_q}{R} - \gamma\frac{H}{R} + \Pi - \mu_d N - \delta H + \frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$.

We define a set as follow:

$D = \Big\{\epsilon_1 \leq S \leq \frac{1}{\epsilon_2},\ \epsilon_1 \leq E \leq \frac{1}{\epsilon_2},\ \epsilon_1 \leq I \leq \frac{1}{\epsilon_2},\ \epsilon_1 \leq I_a \leq \frac{1}{\epsilon_2},\ \epsilon_1 \leq I_q \leq \frac{1}{\epsilon_2},\ \epsilon_1 \leq H \leq \frac{1}{\epsilon_2},\ \epsilon_1 \leq R \leq \frac{1}{\epsilon_2}\Big\}$

where $\epsilon_i > 0,\ i = 1, 2$ are constants, which are very small and will be determined later. We can divide $\mathbb{R}^7_+ \setminus D$ into the following sixteen domains:

$D_1 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, 0 < S < \epsilon_1\}$,

$D_2 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, 0 < E < \epsilon_2, S > \epsilon_1\}$;

$D_3 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, E > \epsilon_1, I < \epsilon_2\}$,

$D_4 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, E > \epsilon_1, I_a < \epsilon_2\}$,

$D_5 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, I > \epsilon_1, 0 < H < \epsilon_2\}$,

$D_6 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, I_q > \epsilon_1, 0 < H < \epsilon_2\}$,

$D_7 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, I_a > \epsilon_1, 0 < R < \epsilon_2\}$,

$D_8 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, I_q > \epsilon_1, 0 < R < \epsilon_2\}$,

$D_9 = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, H > \epsilon_1, 0 < R < \epsilon_2\}$,

$D_{10} = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, S > \frac{1}{\epsilon_2}\}$,

$D_{11} = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, E > \frac{1}{\epsilon_2}\}$,

$D_{12} = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, I > \frac{1}{\epsilon_2}\}$,

$D_{13} = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, I_a > \frac{1}{\epsilon_2}\}$,

$D_{14} = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, I_q > \frac{1}{\epsilon_2}\}$,

$D_{15} = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, H > \frac{1}{\epsilon_2}\}$,

$D_{16} = \{(S, E, I, I_a, I_q, H, R) \in R^7_+, R > \frac{1}{\epsilon_2}\}$.

For all the above cases, it can be observed that there exists a positive constant $c > 0$ such that

$LV(S, E, I, I_a, I_q, H, R) < -c, \forall\ (S, E, I, I_a, I_q, H, R) \in R^7_+ \setminus D$. (see Annexure for detail)

Let $(S, E, I, I_a, I_q, H, R) = x \in R^7_+ \setminus D$, the time τ^x at which a trajectory starting from x reaches to the set D, $\tau^n = inf\{t : |(X(t)| = n\}$ and $\tau^n(t) = min\{\tau^x, t, \tau^n\}$.

By integrating LV from 0 to $\tau^n(t)$ and using expectations, as well as applying Dynkin's formula, we have reached the conclusion that

$EV(S(\tau^n(t)), E(\tau^n(t)), I(\tau^n(t)), I_a(\tau^n(t)), I_q(\tau^n(t)), H(\tau^n(t)), R(\tau^n(t))) - V(x)$
$= E\int_0^{\tau^n(t)} LV(S(u), E(u), I(u), I_a(u), I_q(u), H(u), R(u))du$
$\leq E\int_0^{\tau^n(t)} -cdu = -cE\tau^n(t)$. By utilizing the fact that the function $V(x)$ is non-negative, we can deduce that $E\tau^n(t) \leq \frac{V(x)}{c}$.

Thus, $P(\tau_e = \infty) = 1$, which implies that the model (5) is regular. Applying the well-known Fatou's lemma, we obtain $E\tau^n(t) \leq \frac{V(x)}{c} < \infty$.

Obviously, $sup_{x\in K} E\tau^x < \infty$ where $K \subset R^7_+$. So the second condition of Lemma 3 is satisfied. Moreover, the diffusion matrix for system (5) takes the form

$$B = \begin{bmatrix} \theta_1^2 S^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_2^2 E^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_3^2 I^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_4^2 I_a^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta_5^2 I_q^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta_6^2 H^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_7^2 R^2 \end{bmatrix}$$

$M = min_{(S,E,I,I_a,I_q,H,R)\in D}\{\theta_1^2 S^2, \theta_2^2 E^2, \theta_3^2 I^2, \theta_4^2 I_a^2, \theta_5^2 I_q^2, \theta_6^2 H^2, \theta_7^2 R^2\}$, we can obtain

$\sum_{i,j=1}^{7} a_{ij}(S, E, I, I_a, I_q, H, R)\xi_i\xi_j = \theta_1^2 S^2\xi_1^2 + \theta_2^2 E^2\xi_2^2 + \theta_3^2 I^2\xi_3^2 + \theta_4^2 I_a^2\xi_4^2 + \theta_5^2 I_q^2\xi_5^2 + \theta_6^2 H^2\xi_6^2 + \theta_7^2 R^2\xi_7^2 > M|\xi^2|$
where $\xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7) \in R_+^7$. Thus, the first condition of Lemma 3 is satisfied. It follows from Lemma 3 that the proposed stochastic model is ergodic with a unique stationary distribution.

□

6. Numerical simulations

In this section, we perform numerical simulations using R programming to support our analytical findings. We have taken most of the parameter values from Table 1 and demonstrated the system dynamics for both R_0 greater and less than 1. For the parameter $k = 0$ *i.e.* under no intervention, it is observed that $R_0 = 1.68(> 1)$ which implies the disease persist in the deterministic system (1). Similarly, for the parameter $k = 0.6544$ *i.e.* with intervention effect, it is observed that $R_0 = 0.5805(< 1)$ which implies the disease will die out from the deterministic system (1).

First, we have plotted the relationship $F = \frac{\beta S(I+\eta_1 I_a+\eta_2 E)}{N}$ with respect to a) S, I, b) S, H and c) I, H respectively in Figure 2(a),(b) and (c). It is observed that curve (a) exhibits a quadratic shape, curve (b) follows a sigmoidal pattern, and curve (c) shows a linear relationship. Figure 2(a) illustrates the significant dependence of F on the infection I. The three-dimensional representation reveals that for a fixed I, the shape remains relatively stable concerning S. However, altering I while keeping S constant leads to a rapid increase or decrease in the shape of F, consequently resulting in a swift change in disease propagation within the system. Moving to Figure 2(b), an initial rapid increase in F is observed due to sudden changes in S, albeit with less intensity compared to the previous scenario. However, gradual increments in S result in a slower evolution of F, leading to an initial rapid disease propagation that gradually diminishes as the susceptible population increases. Finally, Figure 2(c) depicts a gradual yet consistent rise in disease propagation as the infection rate increases within the system. This indicates that the different compartments have varying and complex impacts on the spread of new infections. For the above-mentioned parameter values together with $\eta_2 = 0.2$, we have drawn a time series diagram to visualize these two scenarios in Figure 3(a), and (b) for two different values of control parameters $k = 0$ and $k = 0.6544$ respectively. Here it is clear that all the compartments go towards a stable equilibrium. So in Figure 3(a), the susceptible population S (green) goes to stable equilibrium density approximately 29.67, the exposed E (purple), infected I (red), asymptomatic I_a (black), quarantine I_q (pink), hospitalised H (yellow) and recovery population R (light blue) goes to stable equilibrium density approximately $(10.36, 1.6, 2.47, 1.06, 1.74, 2.95)$ respectively. It also supports Theorem 5, as $R_0 > 1$. Similarly, in Figure 3(b), the susceptible population S (green) goes to stable equilibrium density at 50, rest of the compartment dies out as time goes. It also supports Theorem 3, as $R_0 < 1$ and the DFE is $E_0(50, 0, 0, 0, 0, 0, 0)$.

Next, we have simulated the stochastic version of the model (5) through the Euler Maruyama method. To simulate the path of $S(t),\ E(t),\ I(t),\ I_a(t),\ I_q(t),\ H(t)$ and $R(t)$ for the model (5), we fixed the initial values $(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) = (40, 30, 10, 30, 12, 15, 8)$ throughout the stochastic simulation unless it stated in the figure caption. The parameter values are taken from Table 1 with $k = 0$ and intensity parameters $\theta_1 = 0.3, \theta_5 = 0.2, \theta_7 = 0.1$. In Figure 4(a), we consider the other intensity parameters $\theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_6 = 0.2$ and generated the stochastic densities for $S(t)$ (green), $E(t)$ (purple), $I(t)$ (red), $I_a(t)$ (blue), $I_q(t)$ (black), $H(t)$ (cyan) and $R(t)$ (violet). We further generated the stochastic densities corresponding to $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.3, \theta_6 = 0.4$ in Figure 4(b) and $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.6, \theta_6 = 0.4$ in Figure 4(c). In a similar way, we have also simulated the scenario in the presence of and high ($k = 0.6544$) and moderate interventions ($k = 0.4$). For high intervention we have generated the stochastic densities corresponding to $\theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_6 = 0.2$ in Figure 5(a); $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.3, \theta_6 = 0.4$ in Figure 5(b) and $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.6, \theta_6 = 0.4$ in Figure 5(c). Similarly, for low intervention we have generated the stochastic densities in Figure 6(a)-(c). We observed that all the Figures 4(a)-(c), Figures 5(a)-(c) and Figures 6(a)-(c) are stochastically bounded and have positive, unique solution converges in probability (Theorem 6). Figures 7(a)-(f) represents four different sample path and their average path of $S(t),\ E(t),\ I(t),\ I_a(t),\ I_q(t),\ H(t)$ and $R(t)$ respectively for the stochastic model (5). The parameters are taken from Figure 4 with $\sigma_1 = 0.3, \sigma_2 = 0.2, \sigma_3 = 0.1, \sigma_4 = 0.3, \sigma_5 = 0.2, \sigma_6 = 0.2, \sigma_7 = 0.1$ *i.e.* without the presence of intervention. In Figure 7(a) (*i.e.* stochastic densities with respect to S), we observed that the one sample path have decreasing flow, others and the average density path (black) shows stable trend. Similarly, in Figure 7(b) (*i.e.*, stochastic densities with respect to E) and Figure 7(c) (*i.e.*, stochastic densities with respect to I), we observed that almost all the sample path shows a stable type of path, as does the average path (black). Figure 7(d) (*i.e.*, stochastic densities with respect to I_a) and Figure 7(e) (*i.e.*, stochastic densities with respect to I_q) shows mixed types of sample path with a larger variation and the average path (black) also reveals a stable type scenario. Various stochastic densities with respect to H and its average path also shows a stable scenario (not shown here). Although, in Figure 7(f) (*i.e.* stochastic densities with respect to R), we observed that the one sample path goes to extinction, others and the average density path (black) shows stochastic oscillating, implies the complex dynamical behavior of the system. Here it reveals there is no extinction scenario on the average run (see Figure 7(a) -(f)), although some downward trend in sample paths is observed in $S(t), I(t), I_q(t)$ and $R(t)$. Next, we generate the figures of average sample path in the presence of intervention (*i.e.* $k = 0.6544$). Following the ideas of Figure 7, we have generated Figure 8 when $R_0 < 1$. In Figure 8(a)-(b), we observed stable scenario in the sample paths as well as average path. However, a downward trend is observed in the average path (see Figure 8(c)-(f)) and in certain extent the result shows a similar behavior like the deterministic system in long run.

To get more detail on the distribution of the densities of various compartments, we have drawn histograms (see Figure 9(a-f)) of the densities at the time point 150 for 5000 runs of the system (5). The parameters are taken from Fig. 4. Here, we have observed that some sample path shows extinction due to stochastic fluctuation in the I_a, I_q, R population. The average densities lies in the approximate range $(30, 100)$, $(20, 60)$, $(15, 26)$, $(20, 60)$, $(10, 32)$ and $(10, 28)$ for S, E, I, I_a, I_q and R respectively. Similarly, various histograms of the densities (see Figure 10(a)-(f)) at the time point 150 shows I_a, I_1, R compartments have the chance to extinct in the present scenario, although it is possible to have more probability of extinction for a large time point instead of 150 as we have already observed a sharp downtrend in the various compartments in the average run. Histograms were also calculated at time point 100 to provide enhanced understanding of the temporal dynamics (not shown here). An extinction scenario may occur for I_a at a frequency lower than that of I_q, H, and R. The S compartment exhibits a distribution with a long right tail. Furthermore, the distributions of E, I, and I_a are leptokurtic, while that of I_q is positively skewed. Moreover, we have studied the stochastic extinction of the exposed compartment (see Theorem 7) and plot R_E^0 with respect to the parameters k and θ_2. Other parameters are taken from Table 1 with $\eta_2 = 0.1$. We have drawn two heat map diagrams by varying disease transmission rate (β). In Fig. 11(a), we consider a low value of $\beta = 0.74$ and observed that moderate value of control (k) leads to $R_E^0 < 1$. Consequently its easy to control the disease in a long time. Similarly, Fig. 11(b), we consider a moderate value of $\beta = 1.74$ and observed that large value of control (k) needed to make $R_E^0 < 1$. Consequently its no so easy to control the disease in a long time as more area has R_E^0 value greater than one. Two different sample path are drawn (see Fig. 12(a),(b)) for the parameter set same as Fig. 11(a) with $k = 0.6544$ and $\theta_1 = 0.3,\ \theta_2 = 0.7,\ \theta_3 = 0.4,\ \theta_4 = 0.6,\ \theta_5 = 0.2,\ \theta_6 = 0.4,\ \theta_7 = 0.1$. We have computed the value of R_E^0 (< 1) and observed that both sample path leads to extinction.

6.1. Role of quarantine proportion to the trend of infection

Here we have numerically studied the impact of the fraction of quarantine population $\rho_3 = 1 - \rho_1 - \rho_2$ to the model (5) in terms of disease propagation. We defined a new infection term $I_{dis} = I + I_a + I_q$ and studied its long term behaviour with respect to the parameter ρ_3. We simulate the model (5) for two different values at $\rho_3 = 0.25$, $\rho_3 = 0.5$ and find the time series of I, I_a, I_q. We repeat the process for 5000 times and compute the average values *i.e.* $I^{av}, I_a^{av}, I_q^{av}$. After that we compute $I_{dis} = I^{av} + I_a^{av} + I_q^{av}$ to observe the flow of infection in the system. The quantity I_{dis} is simulated for $\rho_3 = 0.25, 0.25$ and plotted in Fig. 13(a). The time series plot $I_{dis}(t)$ for a lower value of $\rho = 0.25$ is presented in green colour and for a relatively higher value of $\rho = 0.5$ is presented in red colour. Now following Noguchi *et al.* (2011) we have performed robust sieve bootstrap approaches for linear trend detection for the generated $I_{dis}(t)$ data. As we found the p-value is very small (< 0.01) in both the case, we tried to fit linear regression models to check the slope of the trend line. The slope of green line is 0.002578 whereas for the red line its 0.003069. So comparing the slope we can say that in long term on average the disease for stochastic system with high value of ρ_3 leads to rapid fall of disease compare to the low one. In this context, it is to be noted that the first difference of $I_{dis}(t)$ *i.e.* $D(I_{dis}(t))$ is stationary (see Fig. 13(b)) in both the case due to Augmented Dickey-Fuller (ADF) test with p-value less than 0.01. Although $I_{dis}(t)$ is not stationary for both the case due to ADF test with p-values 0.8812 and 0.3716 respectively.

7. Discussion and conclusion

The World Health Organization (W.H.O., 2020) states that infectious diseases are the main reason for death in nations with low incomes. Furthermore, according to a recent report, 36% of all deaths worldwide in 2019 were attributable to communicable diseases (W.H.O., 2020). COVID-19 is a rapidly spreading infectious disease that could pose a worldwide threat. Mathematical and statistical models are useful tools for forecasting the pattern, duration, effects of different interventions, and other aspects of disease outbreaks. The present study aimed to develop an intervention-based, deterministic $SEII_aI_qHR$ epidemic model to study the dynamics of the most recent COVID-19 outbreak. Moreover, the model includes the intervention parameter k, which takes into consideration factors like vaccinations, social distancing policies, lockdowns, and other intervention tactics. Symptomatic, asymptomatic, and exposed compartments contribute to the spread of new infections. The disease circulates among the symptomatic, asymptomatic, and quarantine populations in proportions represented by the variables ρ_1, ρ_2, and $(1 - \rho_1 - \rho_2)$, respectively. We explored the positive invariance and boundedness of every forward solution of the model. Furthermore, using the basic reproduction number (R_0), we explore the local and global stability of the unique disease-free equilibrium of the model. In addition, we also studied the existence and local stability of the endemic equilibrium of the model. The deterministic model offers a general understanding of the spread of disease, but it ignores uncertain variables like immigration, human behavior, the effects of the climate, and other random factors. Therefore, we developed a stochastic version of the $SEII_aI_qHR$ model with a frequency-dependent force of infection and intervention to study the dynamics of the disease transmission in the context of changing environmental and population factors. Moreover, we calculated the transition probabilities to investigate the drift and diffusion components of the SDE while developing the stochastic $SEII_aI_qHR$ model. We then discussed some fundamental properties of the model, including the existence of a unique positive global solution with probability one, which shows that the problem is well-behaved. We also analytically found that the criteria $R_E^0 < 1$ leads to disease extinction in the long term. Additionally, we found the ergodic stationary distribution and the extinction conditions of the disease by constructing an appropriate Lyapunov function and using the I$\hat{t}$o formula. Finally, we validated the theoretical findings by generating several numerical solutions to the models. Furthermore, we numerically determined the relationship between the disease transfer function F and various disease compartments of the model (5). Our findings suggest the possibility of three different types of scenarios, e.g., linear, sigmoidal, and quadratic. Furthermore, for two different scenarios, $R_0 < 1$ (stability of the DFE) and $R_0 > 1$ (stability of the EE), we generated time-series diagrams of densities by varying the control parameter k. In addition, to visualize different sample paths, we simulated the SDE model by varying the intervention strength and intensity parameters. The results of our study indicate that the disease does not extinct in the majority of cases. However, the average density of the sample path in the presence of intervention shows a decline in average for the disease compartments compare to without intervention scenario. We have drawn multiple histograms and compared those in two distinct scenarios to see how the densities of various compartments are distributed at a given time. In order to observe the extension scenario, we additionally display the R_E^0 heat map in the (k, θ_2) plane. To calculate R_E^0, two distinct values of disease transmission—low and high—are used. It is noted that the values roughly fall between $(0, 2.5)$ and $(0, 5.5)$, respectively. Lastly, our numerical analysis has demonstrated the positive impact of quarantine proportions on the infection trend.

In conclusion, The study has mainly two aspects: (1) To study the deterministic aspects of the model and observe the disease propagation and impact of intervention. (2) To formulate the stochastic version of the model and observe the impact of noise, intervention and quarantine proportion in the disease propagation, extinction and ergodic stationary distribution. Here we found that as the intensity of intervention increases, the number of infected patients decreases. This means that intervention plays important roles in the outbreak of sudden infectious diseases. For example, media reports can be used to provide the public with information about the current situation of the epidemic and the effective prevention and control measures proposed by experts. Outbreaks of infectious diseases have led to a dramatic increase in interventions like media, self protection, containment zone, *etc.*, which in turn can help raise awareness and change their behaviors for better implementation of mitigation measures. People will adopt relatively conservative behaviors to reduce the possibility of infection, and individual behavior can effectively delay the peak period of infectious disease outbreaks and reduce the severity of infectious disease outbreaks. However, a part of this study only focuses on the qualitative analysis of the stochastic models. The estimation of some key parameters and studying the distribution of intervention scenario will be an interesting study for the future work.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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ANNEXURE

Expression of $T(X, I')$, $G(X, I')$, A and $\hat{G}(X, I)$ used in section 3.4.

$$T(X, I') = \begin{bmatrix} \Pi - (1-k)\frac{\beta S}{N}(I + \eta_1 I_a + \eta_2 E) - \mu_d S \\ \gamma_a I_a + \gamma_q I_q + \gamma H - \mu_d R \end{bmatrix},$$

$$G(X, I') = \begin{bmatrix} (1-k)\frac{\beta S}{N}(I + \eta_1 I_a + \eta_2 E) - \sigma E - \mu_d E \\ \rho_1 \sigma E - \alpha I - \mu_d I \\ \rho_2 \sigma E - \gamma_a I_a - \mu_d I_a \\ (1 - \rho_1 - \rho_2)\sigma E - (\alpha_q + \gamma_q) I_q - \mu_d I_q \\ \alpha I + \alpha_q I_q - (\gamma + \delta) H - \mu_d H \end{bmatrix}.$$

$$A = \begin{bmatrix} -(\mu_d + \sigma) + (1-k)\beta\eta_2 & (1-k)\beta & (1-k)\beta\eta_1 & 0 & 0 \\ \rho_1 \sigma & -(\alpha + \mu_d) & 0 & 0 & 0 \\ \rho_2 \sigma & 0 & -(\gamma_a + \mu_d) & 0 & 0 \\ (1 - \rho_1 - \rho_2)\sigma & 0 & 0 & -(\alpha_q + \gamma_q + \mu_d) & 0 \\ 0 & \alpha & 0 & \alpha_q & -(\gamma + \delta + \mu_d) \end{bmatrix},$$

$$\hat{G}(X, I) = \begin{bmatrix} (1-k)\beta\eta_2 E(1 - \frac{S}{N}) + (1-k)\beta I(1 - \frac{S}{N}) + (1-k)\beta\eta_1 I_a(1 - \frac{S}{N}) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Calculations used in section 3.5.

$$S^* = \frac{\Pi}{\lambda^* + \mu_d},\ E^* = \frac{\lambda^* S^*}{k_1},\ I^* = \frac{\rho_1 \sigma \lambda^* S^*}{k_1 k_2},\ I_a^* = \frac{\rho_2 \sigma \lambda^* S^*}{k_1 k_3},\ I_q^* = \frac{(1-\rho_1-\rho_2)\sigma\lambda^* S^*}{k_1 k_4},\ H^* = \frac{\alpha\rho_1\sigma\lambda^* S^*}{k_1 k_2 k_5} + \frac{\alpha_q(1-\rho_1-\rho_2)\sigma\lambda^* S^*}{k_1 k_4 k_5},\ R^* = \frac{\gamma_a \rho_2 \sigma \lambda^* S^*}{\mu_d k_1 k_3} + \frac{\gamma_q(1-\rho_1-\rho_2)\sigma\lambda^* S^*}{\mu_d k_1 k_4} + \frac{\gamma\alpha\rho_1\sigma\lambda^* S^*}{\mu_d k_1 k_2 k_5} + \frac{\gamma\alpha_q(1-\rho_1-\rho_2)\sigma\lambda^* S^*}{\mu_d k_1 k_4 k_5}.$$

Calculations used in section 3.6. From the model (1), we have

$$I^* = \frac{\rho_1\sigma E^*}{\alpha+\mu_d},\ I_a^* = \frac{\rho_2\sigma E^*}{\gamma_a+\mu_d},\ I_q^* = \frac{(1-\rho_1-\rho_2)\sigma E^*}{(\alpha_q+\gamma_q+\mu_d)},\ H^* = \frac{1}{\gamma+\delta+\mu_d}\Big(\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}\Big)E^*,$$

$$R^* = \Big[\frac{\gamma_a\rho_2\sigma}{\mu_d(\gamma_a+\mu_d)} + \frac{\gamma_q\sigma(1-\rho_1-\rho_2)}{\mu_d(\alpha_q+\gamma_q+\mu_d)} + \frac{\gamma}{\mu_d(\gamma+\delta+\mu_d)}\Big(\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}\Big)\Big]E^*.$$

$$I^* = \frac{\rho_1\sigma E^*}{\alpha+\mu_d},\ I_a^* = \frac{\rho_2\sigma E^*}{\gamma_a+\mu_d},\ I_q^* = \frac{(1-\rho_1-\rho_2)\sigma E^*}{(\alpha_q+\gamma_q+\mu_d)},\ H^* = \Big(\frac{\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}}{\gamma+\delta+\mu_d}\Big)E^*,$$

$$R^* = \frac{\frac{\gamma_a\rho_2\sigma}{\gamma_a+\mu_d} + \frac{\gamma_q\sigma(1-\rho_1-\rho_2)}{(\alpha_q+\gamma_q+\mu_d)} + \gamma\Big(\frac{\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}}{\gamma+\delta+\mu_d}\Big)}{\mu_d}E^*$$

$$N = S + E + I + I_a + I_q + H + R$$

$$N = S + E\Big(1 + \frac{\rho_1\sigma}{\alpha+\mu_d} + \frac{\rho_2\sigma}{\gamma_a+\mu_d} + \frac{(1-\rho_1-\rho_2)\sigma}{(\alpha_q+\gamma_q+\mu_d)} + \frac{\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}}{\gamma+\delta+\mu_d} + \frac{\frac{\gamma_a\rho_2\sigma}{\gamma_a+\mu_d} + \frac{\gamma_q\sigma(1-\rho_1-\rho_2)}{(\alpha_q+\gamma_q+\mu_d)} + \gamma\Big(\frac{\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}}{\gamma+\delta+\mu_d}\Big)}{\mu_d}\Big)$$

$N = S + m_1 E$,

where, $m_1 = \Big(1 + \frac{\rho_1\sigma}{\alpha+\mu_d} + \frac{\rho_2\sigma}{\gamma_a+\mu_d} + \frac{(1-\rho_1-\rho_2)\sigma}{(\alpha_q+\gamma_q+\mu_d)} + \frac{\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}}{\gamma+\delta+\mu_d} + \frac{\frac{\gamma_a\rho_2\sigma}{\gamma_a+\mu_d} + \frac{\gamma_q\sigma(1-\rho_1-\rho_2)}{(\alpha_q+\gamma_q+\mu_d)} + \gamma\Big(\frac{\frac{\alpha\rho_1\sigma}{\alpha+\mu_d} + \frac{\alpha_q\rho_2\sigma}{\gamma_a+\mu_d}}{\gamma+\delta+\mu_d}\Big)}{\mu_d}\Big)$

$$\Rightarrow \frac{(1-k)\beta S}{S+m_1 E}(I + \eta_1 I_a + \eta_2 E) = E(\sigma+\mu_d)$$
$$\Rightarrow \frac{(1-k)\beta S}{S+m_1 E}\Big(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2\Big) = (\sigma+\mu_d)$$
$$\Rightarrow (1-k)\beta S\Big(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2\Big) = (S+m_1E)(\sigma+\mu_d)$$
$$\Rightarrow S\Big[(1-k)\beta\Big(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2\Big) - (\sigma+\mu_d)\Big] = m_1(\sigma+\mu_d)E$$
$$\Rightarrow S^* = \frac{m_1(\sigma+\mu_d)}{[(1-k)\beta(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2) - (\sigma+\mu_d)]}E^*$$
$$\text{Now, } N = \Big(\frac{m_1(\sigma+\mu_d)}{[(1-k)\beta(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2) - (\sigma+\mu_d)]} + m_1\Big)E$$
$$\Rightarrow N = m_2E; \text{where, } m_2 = \Big(\frac{m_1(\sigma+\mu_d)}{[(1-k)\beta(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2) - (\sigma+\mu_d)]} + m_1\Big)$$
$$\text{Again, } \Pi = S\Big[\mu_d + \frac{(1-k)\beta}{m_2}\Big(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2\Big)\Big]$$
$$\Rightarrow E^* = \frac{\Pi}{[\mu_d + \frac{(1-k)\beta}{m_2}(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2)]}\frac{[(1-k)\beta(\frac{\rho_1\sigma}{\alpha+\mu_d} + \eta_1\frac{\rho_2\sigma}{\gamma_a+\mu_d} + \eta_2) - (\sigma+\mu_d)]}{m_1(\sigma+\mu_d)}$$

Proof of $LV < 0$ for $(S, E, I, I_a, I_q, H, R) \in D_i,\ i = 1(1)16$ used in Theorem 8.

Case I: $(S, E, I, I_a, I_q, H, R) \in D_1$

$$LV \leq -c_4c_5 + (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2) - \frac{\Pi}{S} - (1-k)\beta\frac{SI}{NE} - (1-k)\beta\eta_1\frac{SI_a}{NE} + 6\mu_d + \sigma + \alpha + \gamma_a + \gamma + \delta - \rho_1\sigma\frac{E}{I} - \rho_2\sigma\frac{E}{I_a} - \alpha\frac{I}{H} - \alpha_q\frac{I_q}{H} - \gamma_a\frac{I_a}{R} - \gamma_q\frac{I_q}{R} - \gamma\frac{H}{R} + \Pi - \mu_d N - \delta H + \frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$
$$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2) + 6\mu_d + \sigma + \alpha + \gamma_a + \gamma + \delta + \Pi + \frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2} - \frac{\Pi}{S}$$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\Pi}{\epsilon_1}$

Let $\epsilon_1>0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\Pi}{\epsilon_1}<0$.

In such case, we have $LV<0$.

Case II: $(S,E,I,I_a,I_q,H,R)\in D_2$

$LV\leq -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-(1-k)\beta\frac{S}{E}-(1-k)\beta\eta_1\frac{S}{E}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-(1-k)\beta\frac{\epsilon_1}{\epsilon_2}-(1-k)\beta\eta_1\frac{\epsilon_1}{\epsilon_2}$

Let $\epsilon_1>\epsilon_2^2$, very small, such that $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-(1-k)\beta\frac{\epsilon_1}{\epsilon_2}-(1-k)\beta\eta_1\frac{\epsilon_1}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case III: $(S,E,I,I_a,I_q,H,R)\in D_3$

$LV\leq -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\rho_1\sigma\frac{E}{I}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\rho_1\sigma\frac{\epsilon_1}{\epsilon_2}$

Let $\epsilon_1>\epsilon_2^2$, very small, such that $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\rho_1\sigma\frac{\epsilon_1}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case IV: $(S,E,I,I_a,I_q,H,R)\in D_4$

$LV\leq -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\rho_2\sigma\frac{E}{I_a}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\rho_2\sigma\frac{\epsilon_1}{\epsilon_2}$

Let $\epsilon_1=\epsilon_2^2$, choose$\epsilon_1>0$ small enough such that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\rho_2\sigma\frac{\epsilon_1}{\epsilon_2}<0$.

For this case, we get $LV<0$.

Case V: $(S,E,I,I_a,I_q,H,R)\in D_5$

$LV\leq -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\alpha\frac{I}{H}$

$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\alpha\frac{\epsilon_1}{\epsilon_2}$

Let $\epsilon_1 = \epsilon_2^2$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\alpha\frac{\epsilon_1}{\epsilon_2}<0$. Here we get $LV<0$.

Case VI: $(S,E,I,I_a,I_q,H,R)\in D_6$

$$\begin{aligned}LV \le& -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\\&\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\alpha_q\frac{I_q}{H}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\alpha_q\frac{\epsilon_1}{\epsilon_2}\end{aligned}$$

Let $\epsilon_1 = \epsilon_2^2$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\alpha_q\frac{\epsilon_1}{\epsilon_2}<0$.

Therefore, we have $LV<0$.

Case VII: $(S,E,I,I_a,I_q,H,R)\in D_7$

$$\begin{aligned}LV \le& -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\\&\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma_a\frac{I_a}{R}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma_a\frac{\epsilon_1}{\epsilon_2}\end{aligned}$$

Let $\epsilon_1 = \epsilon_2^2$, choose$\epsilon_1>0$ small enough such that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma_a\frac{\epsilon_1}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case VIII: $(S,E,I,I_a,I_q,H,R)\in D_8$

$$\begin{aligned}LV \le& -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\\&\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma_q\frac{I_q}{R}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma_q\frac{\epsilon_1}{\epsilon_2}\end{aligned}$$

Let $\epsilon_1 > \epsilon_2^2$, very small, such that $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma_q\frac{\epsilon_1}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case IX: $(S,E,I,I_a,I_q,H,R)\in D_9$

$$\begin{aligned}LV \le& -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\\&\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma\frac{H}{R}\\ \le& (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma\frac{\epsilon_1}{\epsilon_2}\end{aligned}$$

Let $\epsilon_1 = \epsilon_2^2$, choose $\epsilon_1>0$ small enough such that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\gamma\frac{\epsilon_1}{\epsilon_2}<0$.

For this case, we have $LV<0$.

Case X: $(S,E,I,I_a,I_q,H,R)\in D_{10}$

$$LV \le -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\mu_dN$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}$$

Let $\epsilon_2>0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case XI: $(S,E,I,I_a,I_q,H,R)\in D_{11}$

$$LV \le -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\mu_dN$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}$$

Again choosing $\epsilon_2>0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case XII: $(S,E,I,I_a,I_q,H,R)\in D_{12}$

$$LV \le -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\mu_dN$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}$$

Again choosing $\epsilon_2>0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case XIII: $(S,E,I,I_a,I_q,H,R)\in D_{13}$

$$LV \le -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\mu_dN$$
$$\le (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}$$

Again choosing $\epsilon_2>0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}<0$.

In such case, we have $LV<0$.

Case XIV: $(S,E,I,I_a,I_q,H,R)\in D_{14}$

$$LV \le -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_dN-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$

$$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\mu_d N$$
$$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}$$

Again choosing $\epsilon_2 > 0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2} < 0$.

Here we get $LV < 0$.

Case XV: $(S, E, I, I_a, I_q, H, R) \in D_{15}$

$$LV \leq -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_d N-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$
$$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\mu_d N-\delta H$$
$$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}-\frac{\delta}{\epsilon_2}$$

Again choosing $\epsilon_2 > 0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}-\frac{\delta}{\epsilon_2} < 0$.

In such case, we have $LV < 0$.

Case XVI: $(S, E, I, I_a, I_q, H, R) \in D_{16}$

$$LV \leq -c_4c_5+(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)-\frac{\Pi}{S}-(1-k)\beta\frac{SI}{NE}-(1-k)\beta\eta_1\frac{SI_a}{NE}+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta-\rho_1\sigma\frac{E}{I}-\rho_2\sigma\frac{E}{I_a}-\alpha\frac{I}{H}-\alpha_q\frac{I_q}{H}-\gamma_a\frac{I_a}{R}-\gamma_q\frac{I_q}{R}-\gamma\frac{H}{R}+\Pi-\mu_d N-\delta H+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}$$
$$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\mu_d N$$
$$\leq (c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2}$$

Again choosing $\epsilon_2 > 0$ be as sufficiently small so that, $(c_1c_4+1)(1-k)\beta(1+\eta_1+\eta_2)+6\mu_d+\sigma+\alpha+\gamma_a+\gamma+\delta+\Pi+\frac{\theta_1^2+\theta_2^2+\theta_3^2+\theta_4^2+\theta_6^2+\theta_7^2}{2}-\frac{\mu_d}{\epsilon_2} < 0$.

In such case, we have $LV < 0$.

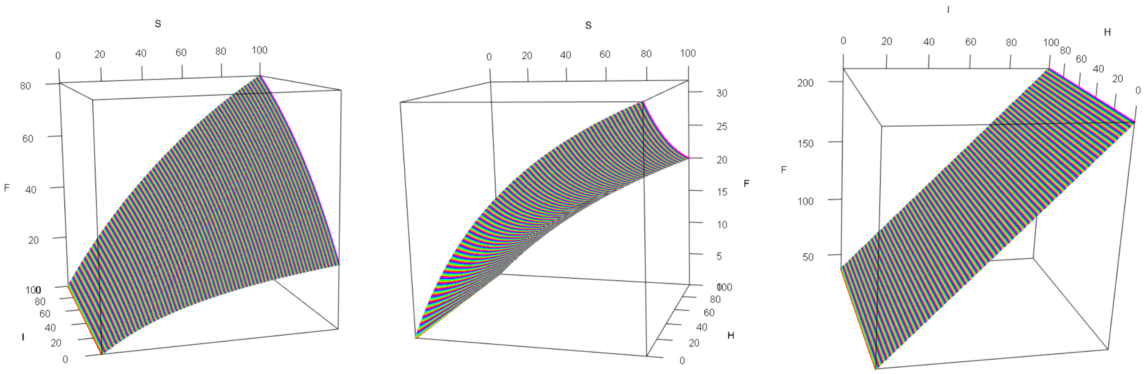

Figure 2: The relationship between $F = \frac{\beta S(I+\eta_1 I_a+\eta_2 E)}{N}$ **and (a)** S, I **[upper left panel] (b)** S, H **[upper right panel] and (c)** I, H **[lower panel]. Figure (a) depicts a quadratic shape, while Figure (b) illustrates a sigmoidal form, and Figure (c) exhibits a linear shape. The other parameters are** $\eta_2 = 0.4$ **and the same from Table 1.**

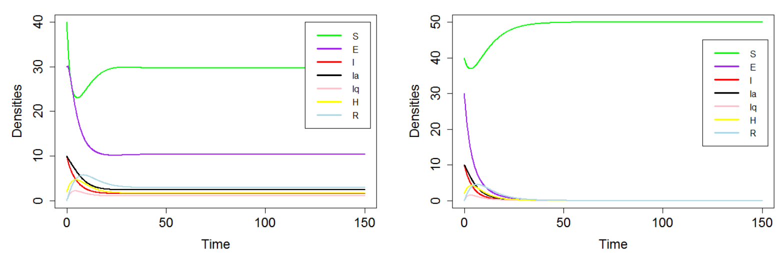

Figure 3: The time series plot of the model (1) **for (a)** $k = 0$ **and (b)** $k = 0.6544$**. The other parameters are same as in Table 1 with** $\eta_2 = 0.2$**.**

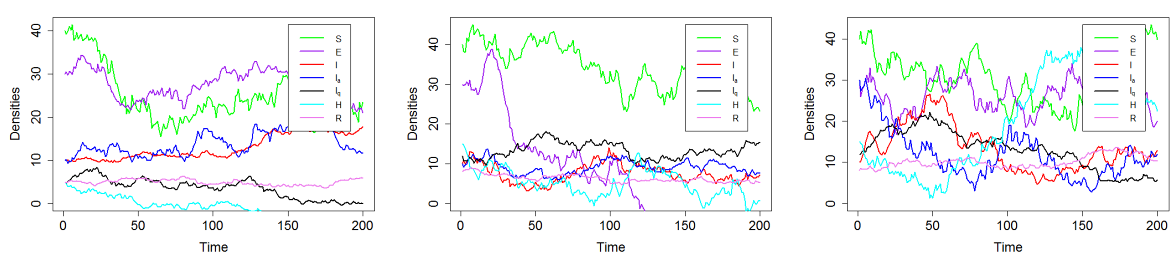

Figure 4: The path $S(t)$, $E(t)$, $I(t)$, $I_a(t)$, $I_q(t)$, $H(t)$ **and** $R(t)$ **for the stochastic model** (5) **with initial values** $(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) = (40, 30, 10, 30, 12, 15, 8)$**. The parameters are taken from Table 1,** $\theta_1 = 0.3, \theta_5 = 0.2, \theta_7 = 0.1$**,** $k = 0$ **with a)** $\theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_6 = 0.2$**; b)** $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.3, \theta_6 = 0.4$ **and c)** $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.6, \theta_6 = 0.4$**.**

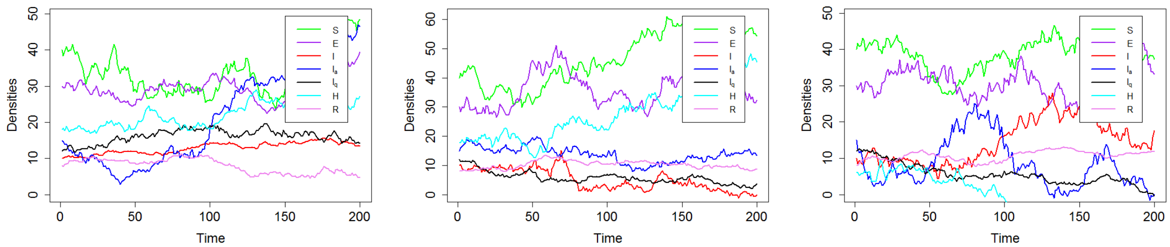

Figure 5: The path $S(t)$, $E(t)$, $I(t)$, $I_a(t)$, $I_q(t)$, $H(t)$ **and** $R(t)$ **for the stochastic model** (5) **with initial values** $(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) = (40, 30, 10, 15, 12, 18, 8)$**. The parameters are taken from Table 1,** $\theta_1 = 0.3, \theta_5 = 0.2, \theta_7 = 0.1$ **and** $k = 0.6544$ **with a)**$\theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_6 = 0.2$**; b)** $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.3, \theta_6 = 0.4$ **and c)** $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.6, \theta_6 = 0.4$**.**

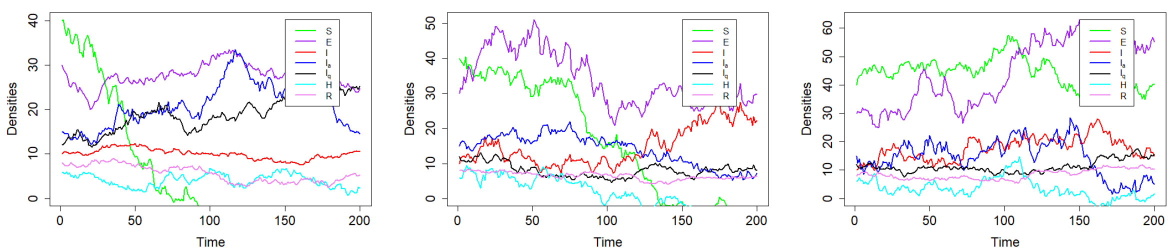

Figure 6: The path $S(t)$, $E(t)$, $I(t)$, $I_a(t)$, $I_q(t)$, $H(t)$ **and** $R(t)$ **for the stochastic model** (5) **with initial values** $(S(0), E(0), I(0), I_a(0), I_q(0), H(0), R(0)) = (40, 30, 10, 15, 12, 18, 8)$**. The parameters are taken from Table 1,** $\theta_1 = 0.3, \theta_5 = 0.2, \theta_7 = 0.1$ **and** $k = 0.4$ **with a)**$\theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_6 = 0.2$**; b)** $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.3, \theta_6 = 0.4$ **and c)** $\theta_2 = 0.4, \theta_3 = 0.4, \theta_4 = 0.6, \theta_6 = 0.4$**.**

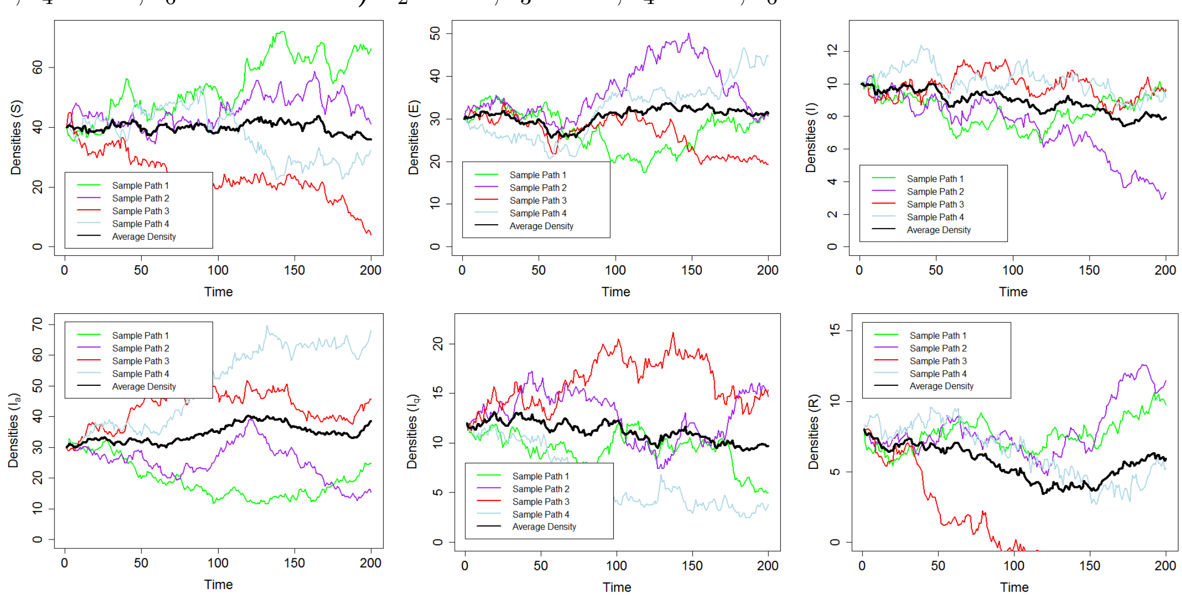

Figure 7: The four different sample paths and their average path of $S(t)$, $E(t)$, $I(t)$, $I_a(t)$, $I_q(t)$, $H(t)$ **and** $R(t)$ **for the stochastic model** (5)**. The parameters are taken from Fig. 4 with** $\theta_1 = 0.3, \theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_5 = 0.2, \theta_6 = 0.2, \theta_7 = 0.1$ **and** $k = 0$**.**

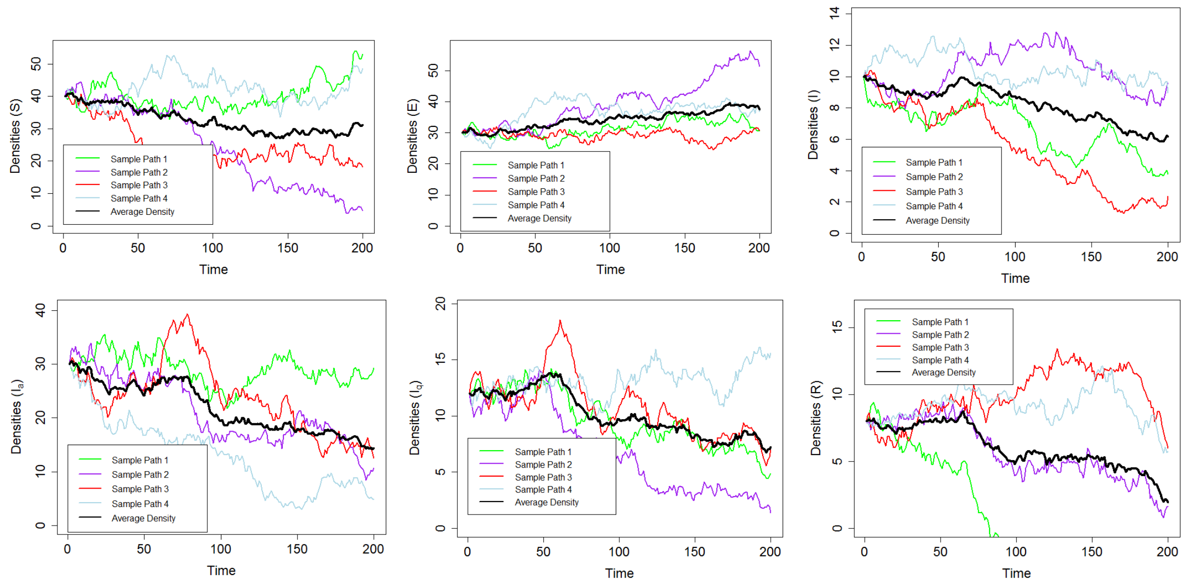

Figure 8: The four different sample paths and their average path of $S(t)$, $E(t)$, $I(t)$, $I_a(t)$, $I_q(t)$, $H(t)$ and $R(t)$ for the stochastic model (5). The parameters are taken from Fig. 5 with $\theta_1 = 0.3, \theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_5 = 0.2, \theta_6 = 0.2, \theta_7 = 0.1$ and $k = 0.6544$.

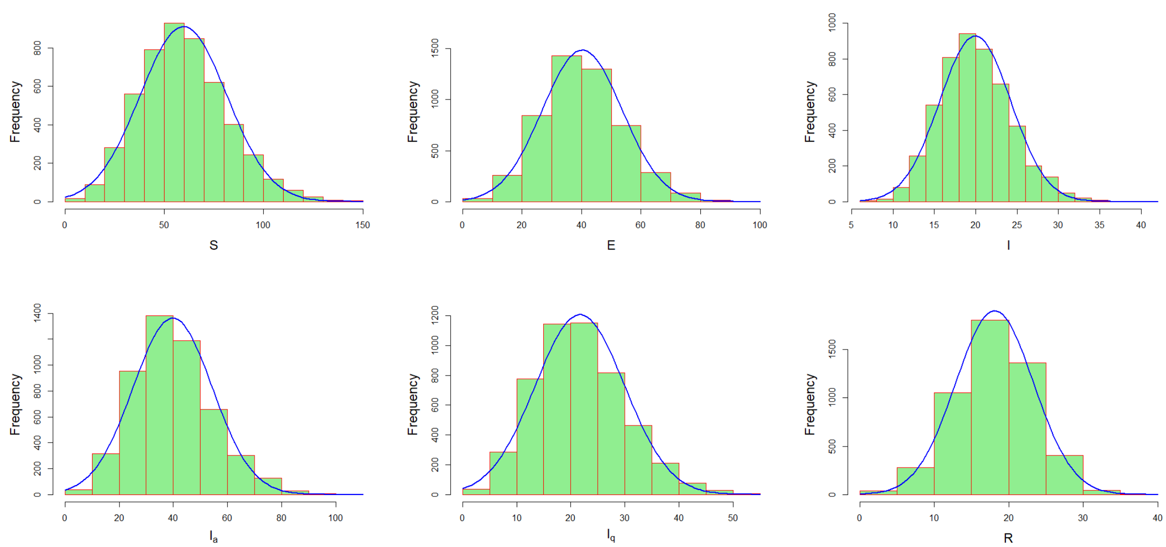

Figure 9: Histogram of the densities at the time point 150 of the system (5). The parameters are taken from Fig. 4 with $\theta_1 = 0.3, \theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_5 = 0.2, \theta_6 = 0.2, \theta_7 = 0.1$.

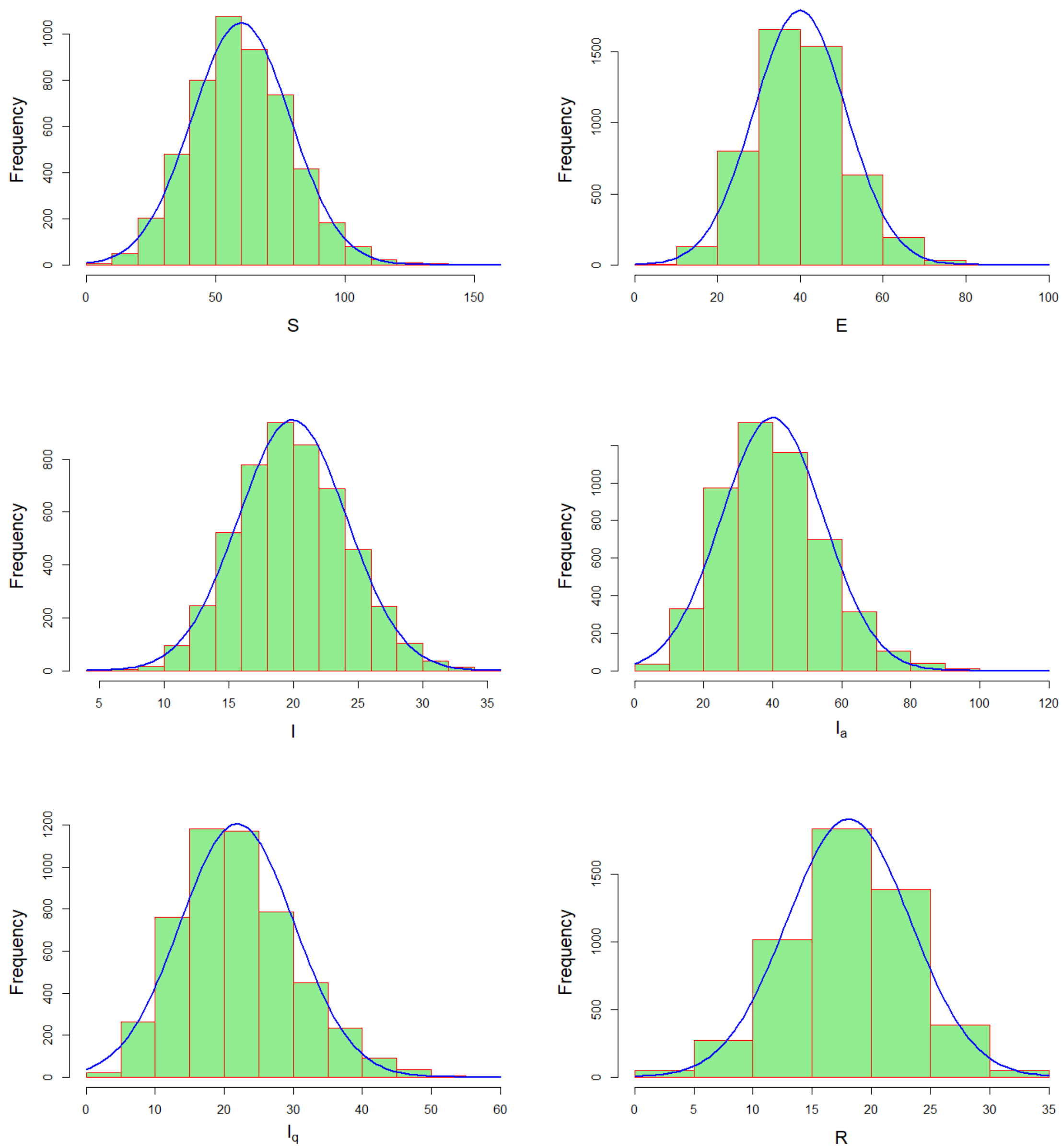

Figure 10: Histogram of the densities at the time point 150 of the system (5). The parameters are taken from Fig. 5 with $\theta_1 = 0.3, \theta_2 = 0.2, \theta_3 = 0.1, \theta_4 = 0.3, \theta_5 = 0.2, \theta_6 = 0.2, \theta_7 = 0.1$.

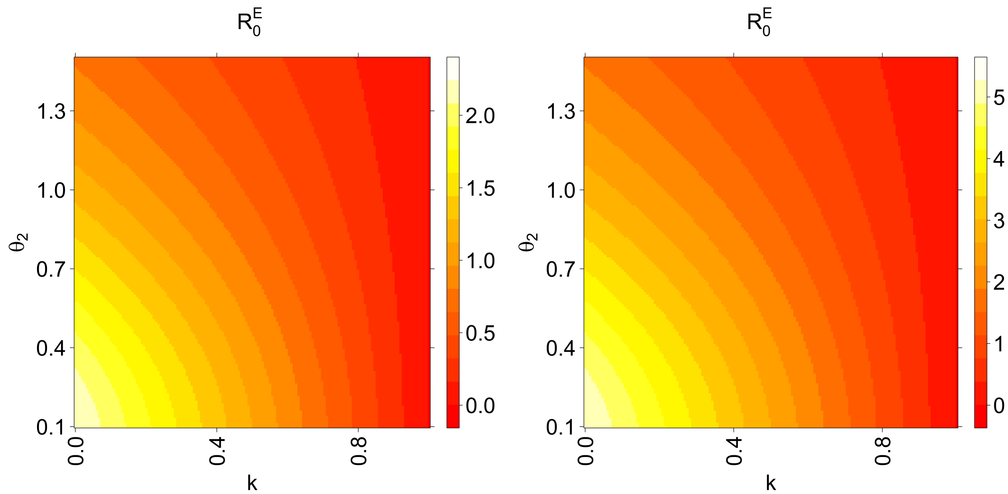

Figure 11: Heat map diagram of R_E^0 with respect to k and θ_2 for the system (5). The parameters are taken from Table 1 with $\eta_2 = 0.1$. The left figure corresponding to $\beta = 0.74$ and right figure corresponding to $\beta = 1.74$ respectively.

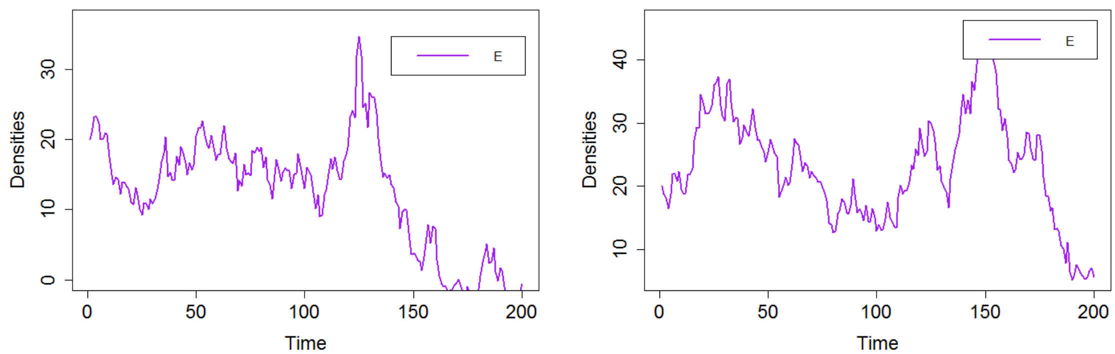

Figure 12: Two different sample paths are drawn for the parameter set same as Fig. 11(a) with $k = 0.6544$.

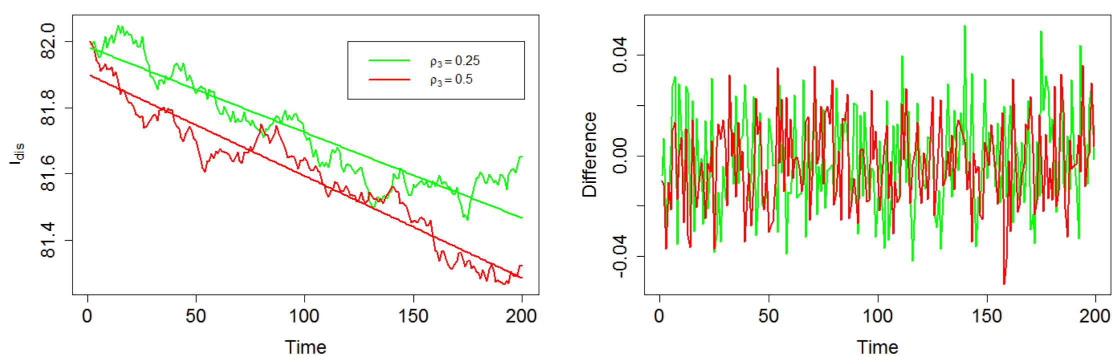

Figure 13: a) Average paths for I_{dis} are drawn for two different values of parameter ρ_3, the other parameters are same as Fig. 8. b) The difference plot corresponding to a).