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Heterogeneous Auto-Regressive Modeling based Realised Volatility Forecasting

G. Avinash[1], Ramasubramanian V.[2] and Badri Narayanan Gopalakrishnan[3]
[1] *The Graduate School, ICAR-Indian Agricultural Research Institute, New Delhi*
[2] *ICAR-Indian Agricultural Statistics Research Institute, New Delhi*
[3] *Lead Adviser & Head, Trade and Commerce, NITI Aayog, GoI, New Delhi*

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Abstract

Volatility forecasting is a critical task in the financial markets. It exhibits persistence, which is implicit in option prices. In this study, estimation of Realised Volatility (RV) through high frequency data on the basis of realised variance measures by Heterogeneous Auto-Regressive (HAR) modeling termed as HAR-RV is discussed. This volatility cascade leads to a simple AR-type model in the realised volatility with the feature of considering different volatility components realised over different time horizons successfully capturing the main characteristics of finance data. The HAR model can be extended by adding different decompositions of volatility components into the benchmark model. Thus HAR-RV and its extensions namely, HAR with the simple jump measure (HAR-J), HAR augmented with Quarticity component (HAR-Q), Bi-power variation (BPV) to separate the continuous and jump component named as HAR with continuous and jump component (HAR-CJ), HAR with Quarticity and Jump component (HAR-QJ), Without Jump component (CHAR) and along with Quarticity component (CHAR-Q) models were studied. HAR models have been widely used to forecast crude oil futures volatility, agricultural commodities, stock returns *etc.* An attempt has been done on real dataset relating to Standard and Poor's 500 (S&P 500) stock market high frequency data and its volatility was estimated by using HAR models and its extensions and were compared on different horizons with their volatility studied. The results revealed that CHAR-Q models perform well in the estimation period compared to all other models.

Key words: Bi-power variation; Continuous component; Jump component; High frequency data; Quarticity; Standard and Poor's 500.

1. Introduction

Volatility modeling and forecasting are integral to finance, and are used in a variety of financial applications such as risk management and hedging, because volatility plays an

Corresponding Author: Ramasubramanian V.
Email: ram.vaidhyanathan@gmail.com

important role in asset pricing, portfolio construction, risk management, and trading strategy creation. Researchers and practitioners continuously strive for improving the forecast accuracy of asset return volatility. Engle (1982), Bollerslev (1986), Nelson (1991) and others have done extensive and in-depth research on the measurement and modeling of the volatility of asset price, and they believe that the volatility of financial markets has a particular time-varying nature. Later, they introduced the Auto-Regressive Conditional Heteroscedasticity (ARCH) or Generalized ARCH (GARCH) model to capture the aggregation effect on market volatility and achieved better results. Taylor (1994) worked on Stochastic Volatility (SV) model, which is more elastic than the ARCH type for representing the time-varying character of market volatility. The classic GARCH model, SV model, and other research outcomes based on low frequency financial data on asset price fluctuations have been immensely recognised by domestic and international research institutions. Almost all GARCH models are associated with daily, close-to-close returns, or with even lower-frequency data requirements. Though these models perform well in predicting volatility, they fail to capture the intraday activity patterns. Once high-frequency data available, researchers recognized that these data are even more informative regarding volatility, and the concept of realised volatility emerged (Barndorff-Nielsen and Shephard, 2002). However, daily squared returns are a noisy proxy for true volatility (Molnar, 2012). Realised volatility quickly found its way into the volatility modeling and forecasting literature (Andersen *et al.*, 2003) and became popular, not only in volatility models but also in price forecasting (Degiannakis and Filis, 2017).

With the availability and broad application of high-frequency financial data, the Realised Volatility (RV) and the realised double power variation based on high-frequency data measurement contain more market information than the low-frequency model volatility. An attempt has been made by combining their research to model high-frequency volatility from different perspectives. Based on the theory of heterogeneous markets, Corsi (2009) presented an article that discussed the HAR-RV model. The first order autoregressive volatility process is implemented, which represents the market's heterogeneous trading behaviour. Also, constructed a new HAR model (HAR-RV-CJ) based on the original one, decomposing realised volatility into continuous sample path variance and jump variance to study the impact of volatility. Specifically, the current applications of the HAR models follow the (1, 5, 22) time horizon structure originally proposed for developed markets, using daily (1 day), weekly (5 days), and monthly (22 days) periods to represent the short-term, medium-term, and long-term investors trading frequencies, respectively. However, investors cultural backgrounds and investment habits, as well as the alternative investment choices, differ largely across markets, which will probably result in different heterogeneous structures across markets. Furthermore, investors trading frequencies may be affected by financial and economic policies as well as market conditions, which will probably lead to a market's heterogeneous structure varying over time.

It is well known that stock market prices fluctuate the most during and in the early moments of bubbles and crashes due to uncertainty in the markets. Volatility forecasting, therefore, plays a crucial role in determining the distress of an asset or a market and the research in this area has grown over time. Even till date, forecasting volatility still "remains very much an art rather than a science" quoted two decades earlier by Figelwski (2004).

Traditionally, the multivariate volatility models include the multivariate GARCH and the multivariate SV models. Despite the numerous modifications to multivariate volatility models, such models consider covariance as a latent variable and suffer from intraday information loss due to the use of low frequency data. However, this resulted in a considerable loss of information on inter-day trading data and also caused bias in estimating and forecasting the conditional volatility. Hence, with the availability of reliable high-frequency intraday asset prices, researchers were motivated to conduct further research aiming to primarily produce short-run volatility forecasts better. In this study, the existing HAR-type models and its extensions have been studied empirically to infer about their predictive power for forecasting realised volatilities by taking the case of S&P 500 futures. These findings add to the concepts of financial risk management and volatility forecasting. When faced with high equity market uncertainty, the findings will assist market participants to choose appropriate strategies to limit risk and maximize returns.

The structure of paper is as follows. Section 2 deals with genesis of HAR modelling. Some preliminaries and methodology are given in Section 3. A case study on real data of S&P 500 market is given in Section 4 followed by concluding remarks in Section 5.

2. Genesis of HAR modeling

Volatility is arguably referred to as a quantitative measure of risk where the higher the volatility, higher the risk of a specific asset and therefore it's forecast becomes crucial in areas such as portfolio management and asset allocation. Most available studies apply models based on low-frequency transaction data, such as GARCH, SV, and ARMA to forecast the volatility of crude oil futures (Chang *et al.*, 2010). Although these models perform well in predicting volatility in crude oil futures markets, they fail to capture the intraday activity patterns, the macroeconomic announcements and the volatility persistence that are separately quantified and have been shown to account for a substantial fraction of return variability, both at the intraday and daily level.

Giot and Laurent (2003) have employed GARCH-type models to create estimates for cocoa, coffee, and sugar futures price volatility. Tian *et al.* (2017) and Yang *et al.* (2017) on the other hand, used high-frequency data and enhanced Corsi's (2009) HAR model to create short-run volatility projections (up to 20 days ahead) motivated by the 'Heterogeneous Market Hypothesis' and the measure of RV.

The HAR-RV model uses high-frequency transaction data to successfully capture the main characteristics of financial data. Hence, many scholars extend the HAR-RV model by adding different decompositions of volatility components into the benchmark model (Andersen *et al.*, 2007; Patton and Sheppard, 2015; Gong and Lin, 2018). HAR-type models have been widely used to forecast crude oil futures volatility, and have been proven to be better than the traditional models which are based on low-frequency transaction data (Haugom *et al.*, 2014 and Andersen *et al.*, 2007) further proposed the use of BPV to separate the realised volatilities into continuous and jump components termed as HAR-CJ model. Corsi and Reno (2012) extended HAR model by adding the jump component and termed it as HAR-J model and also introduced without jump component (CHAR) and Quarticity component (CHAR-Q). Bollerslev *et al.* (2016) introduced the HAR-Q models using realised Quarticity (RQ) as an estimator of Integrated Quarticity (IQ) to capture temporal variation in the measurement error.

Degiannakis *et al.* (2022) used variants of the HAR model, and forecasted the realised volatility of agricultural commodities. They obtained data from Chicago Mercantile Exchange (CME)/ Intercontinental Exchange (ICE) with tick-by-tick data on five widely traded agricultural commodities (corn, rough rice, soybean, sugar, and wheat) during the period January 01, 2010 to June 30, 2017. The data was divided as In-sample estimation period and Out-sample forecasting period. Their results revealed that HAR model performed well when the variations in volatility measurements were decomposed into their continuous path and jump components.

Although the HAR-type models discussed above offer good predictive capacity for volatility forecasting, higher the prediction accuracy, better for risk management, financial asset pricing and portfolio optimization. Hence, it will be of interest to fit various HAR models on a given dataset (here, S&P 500 prices) and ascertain about which model better represent the underlying pattern and also their forecasting performance.

3. Preliminaries and methodology

3.1. Terminologies

The terminologies relating to volatility are described briefly. Implied volatility represents the current market price for volatility, or the fair value of volatility based on the market's expectation for movement over a defined period of time. Realised volatility is nothing but the assessment of variation in returns for an investment product when its historical returns within a defined time period are analysed. Analysts make use of high-frequency intraday data to determine measures of volatility at hourly/ daily/ weekly/ monthly frequency. Hence, volatility traders obviously care not only about what is expected but also what actually transpired. Note that in econometrics, sum of squared returns is called as realised volatility (Barndorff-Nielsen and Sheppard, 2004). Stochastic volatility models are similar to GARCH models but introduce a stochastic innovation term to the equation that describes the evolution of the conditional variance σ_t^2. To ensure positiveness of the conditional variances, stochastic volatility models are defined in terms of $ln\sigma_t^2$ instead of σ_t^2. If the autocorrelation function ρ_k of stationary ARMA(p, q) process decreases rapidly as $k \rightarrow \infty$, processes then it is often referred to as short memory processes. Stationary processes with much more slowly decreasing autocorrelation function are known as long memory processes. High-frequency data are mostly used in financial analysis and in high frequency trading which basically contain intraday observations that can be used to understand market behavior, dynamics, and micro-structures. Tick-by-tick market data, in which each single 'event' (transaction, quote, price movement, *etc.*) is characterised by a "tick" was first used to create high frequency data collections. The quantity of daily data acquired in 30 years can be equalled by high frequency observations over one day of a liquid market.

3.2. Tests used for realised variance measures

The Ljung-Box statistic is computed under the null hypothesis that there is no autocorrelation in the residuals in order to see whether the best-fitted model residuals are white noise or not. Normality of residuals can be tested by employing Shapiro-Wilk's (W) test. Jarque-Bera test is a goodness-of-fit test to test whether sample data have the skewness and kurtosis matching a normal distribution. Augmented Dickey-Fuller (ADF) test is used for testing the presence of a unit root in a time series by under the assumption that the time series is non-stationary.

3.3. Realised volatility (RV)

RV is a model free measurement of financial market volatility and was proposed by Andersen *et al.* (2001, 2003) and Barndorff-Nielsen and Shephard (2002) by defining a continuous time diffusion process. Andersen *et al.* (2003) showed that, under suitable conditions, including the absence of serial correlation in the intraday returns, RV is a consistent estimator of Integrated Volatility (IV_t). Hence

$$RV_t = \sum_{i=1}^{m} r_{t,i}^2 \xrightarrow{P} \int_{t-1}^{t} \sigma_s^2 ds$$

at day $t = 1, 2, ..., M$ for $i = 2, 3, ..., m$ with the number of intraday observations as m and the total number of observation days as M. Following Andersen and Bollerslev (1998), discretizing the data by equidistant sampling, might introduce intraday price jumps which translate into higher realised variances. In order to obtain a more robust measure of the realised volatility, Barndorff-Nielsen and Sheppard (2004) introduced the concept of the BPV for separating the realised variance into a continuous part and a discontinuous (jump) part. Using the approach of Huang (2004), the jump component is identified. Hence RV provides an ex-post measure of the true total variation including the discontinuous jump part.

3.4. HAR models

With the widespread availability of high-frequency intraday data, the recent literature has focused on employing RV to build forecasting models for time-varying return volatility. Among these forecasting models, the HAR model proposed by Corsi (2009) has gained popularity due to its simplicity and consistent forecasting performance in applications. The formulation of the HAR model is based on a straightforward extension of the heterogeneous ARCH (HARCH) class of models dealt by Muller *et al.* (1997). Under this approach, the conditional variance of the discretely sampled returns is parameterized as a linear function of lagged squared returns over the same horizon together with the squared returns over longer and/or shorter horizons.

The original HAR model specifies RV as a linear function of daily, weekly and monthly realised variance components, and can be expressed as

$$RV_t = \beta_o + \beta_1 RV_{t-1}^d + \beta_2 RV_{t-1}^w + \beta_3 RV_{t-1}^m + \varepsilon_t$$

where β_j ($j = 0, 1, 2, 3$) are unknown parameters that need to be estimated, RV_t is the realised variance of day t, and $RV_{t-1}^d = RV_{t-1}, RV_{t-1}^w = \frac{1}{5}\sum_{i=1}^{5} RV_{t-i}, RV_{t-1}^m = \frac{1}{22}\sum_{i=1}^{22} RV_{t-i}$ denote the daily, weekly and monthly lagged realised variance, respectively. This specification of RV parsimoniously captures the high persistence observed in most realised variance series. The various types of HAR models are discussed subsequently.

3.4.1. Standard HAR model

$$RV_{t+h}^{(h)} = \beta_0^{(t)} + \beta_1^{(t)} RV_t + \beta_2^{(t)} RV_t^{(5)} + \beta_3^{(t)} RV_t^{(22)} + \varepsilon_{t+h}^{(h)}$$

where RV_t denotes the previous day's volatility $RV_t^{(5)}$ denotes the averaged volatility during the previous week, and $RV_t^{(22)}$ denotes the averaged volatility over the previous month, h denotes the forecasting horizon.

3.4.2. HAR-J model

Augmenting the above standard HAR with the simple jump measure forms HAR-J model.

$$RV_{t+h}^{(h)} = \beta_0^{(t)} + \beta_1^{(t)} RV_t + \beta_2^{(t)} RV_t^{(5)} + \beta_3^{(t)} RV_t^{(22)} + \varepsilon_{t+h}^{(h)} + \beta_4^{(t)} RJ_t + \varepsilon_{t+h}^{(h)}$$

where RJ_t is the daily discontinuous jump variation.

3.4.3. HAR-Q model

It is obtained by using Realised Quarticity (RQ) as an estimator of Integrated Quarticity (IQ) to capture temporal variation in the measurement error by Bollerslev *et al.* (2016).

$$RV_{t+h}^{(h)} = \beta_0 + \beta_1^{(t)} RV_t + Q^{(1)} \underline{RQ_t^{1/2}} + \beta_2^{(t)} RV_t^{(5)} + \beta_3^{(t)} RV_t^{(22)} + \varepsilon_{t+h}^{(h)}$$

where $\underline{RQ_t^{1/2}}$ is the daily lagged realised quarticity and it is useful as most of the attenuation bias in the forecasts (due to RV_t being less persistent than unobserved IV_t) is due to the estimation error in RV_{t-1} . In other words, RQ_t as an estimator of IQ_t to capture temporal variation in the measurement error with $\underline{RQ_t^{1/2}}$ as the de-meaned values of $RQ_t^{1/2}$ for easy interpretation.

3.4.4. HAR-CJ model

Andersen *et al.* (2007) further proposed the use of BPV to separate the realised volatilities into continuous and jump components, which model is resulted as HAR-CJ and defined as

$$RV_{t+h}^{(h)} = \beta_0^{(t)} + \beta_1^{(t)} C_t + \beta_2^{(t)} C_t^{(5)} + \beta_3^{(t)} C_t^{(22)} + J^{(1)} RJ_t + J^{(5)} RJ_t^{(5)} + J^{(22)} RJ_t^{(22)} + \varepsilon_{t+h}^{(h)}$$

where C_t and RJ_t are continuous and discontinuous jump components respectively.

3.4.5. HAR-QJ model

It is obtained by using standard HAR along with previous day's Quarticity and jump component respectively.

$$RV_{t+h}^{(h)} = \beta_0 + \beta_1^{(t)} RV_t + Q^{(1)} \underline{RQ_t^{1/2}} + J^{(1)} RJ_t + \beta_2^{(t)} RV_t^{(5)} + \beta_3^{(t)} RV_t^{(22)} + \varepsilon_{t+h}^{(h)}$$

where RQ as an estimator of IQ to capture temporal variation in the measurement error. Using $\underline{RQ_t^{1/2}}$ as the de-meaned values of $RQ_t^{1/2}$ and RJ_t is the daily discontinuous jump variation.

3.4.6. CHAR model

$$RV_{t+h}^{(h)} = \beta_0^{(t)} + \beta_1^{(t)} C_t + \beta_2^{(t)} C_t^{(5)} + \beta_3^{(t)} C_t^{(22)} + \varepsilon_{t+h}^{(h)}$$

where, C_t, $C_t^{(5)}$ and $C_t^{(22)}$ are respectively, the daily continuous path variation, the daily average over the past five days and daily average over the past 22 days at time t. Without jump component, it is better at capturing volatility persistence and long memory than RV in HAR model.

3.4.7. CHAR-Q model

$$RV_{t+h}^{(h)} = \beta_0^{(t)} + \beta_1^{(t)} C_t + \beta_2^{(t)} C_t^{(5)} + \beta_3^{(t)} C_t^{(22)} + \beta_4^{(t)} (TPQ)^{1/2} + \varepsilon_{t+h}^{(h)}$$

where, $TPQ^{1/2}$ is Tri-power quarticity, which is consistent for the integrated quarticity in the presence of jumps.

3.5. Forecasting and evaluation

To quantitatively evaluate the forecasting of each model, three popular accuracy measures, namely the Mean Squared Prediction Error (MSPE), the Mean Absolute Prediction Error (MAPE), and Quasi Likelihood (QLIKE) by Patton (2011) have been used (and multiplied by 100 to express in percentages):

$$\text{MSPE} = \sqrt{N^{-1} \sum_{t=1}^{N} \left(RV_t - \widehat{RV}_t \right)^2}$$

$$\text{MAPE} = N^{-1} \sum_{t=1}^{N} \frac{\left| RV_t - \widehat{RV}_t \right|}{RV_t}$$

$$\text{QLIKE} = N^{-1} \sum_{t=1}^{N} \left(\log \widehat{RV}_t + \frac{\left| \widehat{RV}_t \right|}{RV_t} \right)$$

where RV_t and $\widehat{RV}_t$ are the actual and the forecasted RV respectively at the different forecasting horizons, and N is the number of real out-of-sample forecasts.

4. Case study

The Standard and Poor's 500 (S&P 500) is usually referred as leading indicator of the stock market in the United States. The S&P 500 index is made up of 500 large-cap stocks that represent the most important industries in the US economy. Furthermore, because of their high liquidity, they can easily be bought or sold in the market without influencing the asset price. Forecasting of asset return volatility S&P 500 index futures prices from Tick Data Inc (http://public.econ.duke.edu/ ap172/code.html), during April 8, 1997 to August 30, 2013 (4096 trading days) has been considered in this case study. These data points

Table 1: Descriptive statistics for S&P 500 market realised volatility measures, relative jump component and bi-power variation

Descriptive Statistics	RV	RJ	BPV
Mean	1.17	0.09	1.11
Minimum	0.04	0.00	0.03
Maximum	60.56	10.25	50.31
Standard Deviation	2.31	0.36	2.21
Skewness	10.02	17.6	9.36
Kurtosis	166.92	398.99	134.74
Tests used for checking assumptions for required analysis			
Ljung-Box test Q (1)	1735.60**	24.68**	1852.90**
Ljung-Box test Q (5)	7117.20**	135.04**	7747.80**
Ljung-Box test Q (10)	11989.00**	531.26**	12821.00**
Ljung-Box test Q (22)	20042.00**	884.56**	21112.00**
Shapiro-Wilk's (W) test	0.35**	0.19**	0.36**
J-B test	4654549.00**	26974481.00**	3021571.00**
ADF test	-6.86**	-8.87**	-7.09**
PP test	-2042.90**	-4966.50**	-1864.00**

Note: Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

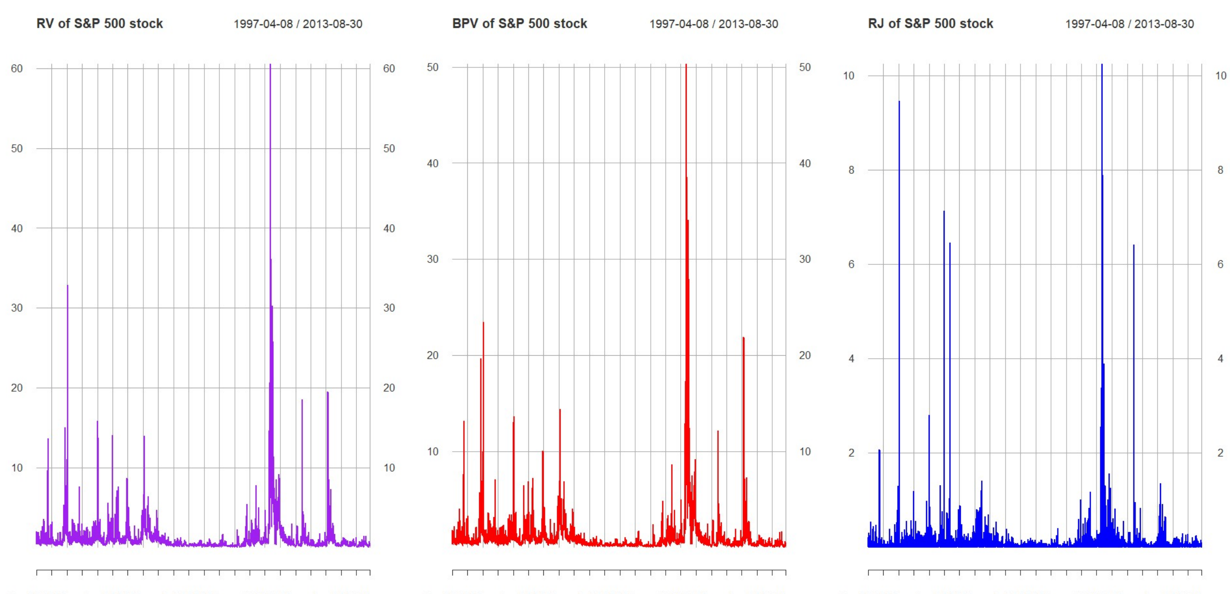

Figure 1: Realised volatility (RV), bi-power variation (BPV) and realised jump (RJ) components in S&P 500 stock

available as tick-by-tick data prices, which were 23,400 data points for each trading day resulted in 9,58,46,400 data points. In order to avoid microstructure noise, aggregating the data to 5 min prices led to 3,19,488 data points. For in-sample analysis, April 8, 1997 to January 06, 2012 (3686 trading days) period was considered which is 90 per cent of 4096 trading days for estimation period, whereas, the remaining period till August 30, 2013 were used as real out-of-sample forecasts based on rolling window approach. The use of rolling window approach works best to capture changes in the market conditions as suggested by Degiannakis and Filis (2017), Degiannakis *et al.* (2018), and Engle *et al.* (1990). The same data which has been used in this study has been used as a default data in the R package for fitting HAR models, but, in this study, the data has been aggregated to 5 min prices and also more variants of HAR models have been tried for comparison purposes.

4.1. Empirical results

Table 1 provides an overview of descriptive statistics and test statistics of the Ljung–test for one, five, ten and 22 lags (trading days). The descriptive statistics like skewness and kurtosis indicate the data considered were very much erratic. All the data series had positive skewness and were highly leptokurtic in nature. W and J-B tests for normality showed that all the series deviate from normality. Phillip-Perron (PP) test employed to test a unit root in a time series indicated presence of stationarity in the RV, RJ and BPV time series.

Table 2: In-sample HAR results for S&P 500 with RV

HAR	$h = 1$	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
β_0	0.12** (3.23)	0.18** (7.64)	0.24** (10.46)	0.37** (15.53)	0.59** (22.89)	0.72** (27.90)
$\beta_1^{(1)}$	0.22** (3.24)	0.18** (13.11)	0.13** (10.05)	0.10** (7.49)	0.08* (5.59)	0.06** (4.24)
$\beta_2^{(5)}$	0.49** (3.33)	0.39** (16.55)	0.37* (16.51)	0.33* (14.22)	0.30** (12.31)	0.21** (8.68)
$\beta_3^{(22)}$	0.18** (3.04)	0.26** (12.50)	0.28** (14.03)	0.26** (12.55)	0.14** (6.44)	0.15** (7.15)
$Adj R^2$	0.51	0.63	0.62	0.54	0.38	0.30
AIC	3852.83	1444.90	1134.06	1301.31	1809.72	1676.70
BIC	3883.86	1475.92	1165.07	1332.30	1809.74	1676.70
RMSE	1.69	1.22	1.17	1.19	1.28	1.26
Q-LIKE	0.15	0.12	0.13	0.17	0.22	0.25

Note: Parenthesis in the above table indicates test statistic value. Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

The measures for realised volatilities for S&P 500 stock index have shown significant autocorrelations at 1, 5, 10, 22 lags, tested with Ljung-Box chi-square test. This motivated further for the application of autoregressive models such as HAR and its extensions. Astonishingly, even the jump components (J_t) showed autoregressive behavior of jumps indicating that because of the impact of major economic events, there were structural breaks in the volatility of returns of financial assets, which feature may help in improving the predictive ability of the HAR-type models. As the continuous component refers to the realised volatility that remained after discarding jumps, the Ljung Box test statistics, ADF, W, J-B test and PP test were naturally much higher and had similar patterns like realised volatility components. Figure 1 depicts the Realised Volatilities RV, BPV and RJ components in S&P 500 stock price index considered. It can be seen from Figure 1 that RV plot subsumes BPV along with other components whereas BPV consists of both continuous and jump components. In

Figure 1, the third plot relating to jump component arises due to the intra-day variations in the data which occur on a daily basis whose magnitudes and ranges are much smaller than the other two components as can be seen in the plot.

Huge spike of realised volatility in 2009 can be observed in Figure 1 through Figure 8. This is so because S&P 500 market price bottomed out during 2008-2009 owing to financial crisis that resulted in great U.S market recession. S&P 500 lost approximately 50% of its value due to market crash and took two years to recover from it. As a result, squared returns increased irrespective of direction of their original values leading to sudden increase in realised volatility.

4.2. In-sample parameter estimation results

In-sample analysis results are presented in Tables 2 through 8 for RV of S&P 500 market with Figures 2 through 8 depicting these results. In Tables 2 through 8, the estimation results and model performance accuracy measures have been reported for the seven models considered *viz.*, HAR, HAR-J, HAR-Q, HAR-CJ, HAR-QJ, CHAR and CHAR-Q at six prediction horizons (h = 1, 5, 10, 22, 44 and 66 days). The analysis was done using R software. For fitting the models, Ordinary Least Squares (OLS) estimation was employed. It can be seen that most of the parameters were significant at the 1% level, suggesting strong persistence in the realised volatility. The Adjusted R^2 (higher values), AIC, BIC, RMSE and Q-LIKE (comparatively smaller) measures at 5 days and 10 days ahead prediction horizons revealed that the fitted models performed well for these days as compared to 1, 22, 44, 66 days prediction horizons. In HAR and HARJ models, all the parameters were significant at all horizons. Most of the parameters of the HAR-Q, HAR-CJ, HAR-QJ models were significant at short and medium horizons, but for long memory horizons some of the components like Quarticity (especially for HAR-QJ) and jump components showed non-significance. CHAR model showed significant contribution by all continuous components.

When all the seven HAR and its extension models fitted were compared based on their prediction performances, CHAR-Q type of HAR model came out to be the best model at horizons $h = 5$ and $h = 10$ and hence can be considered superior with regard to model fit. This shows that the continuous component along with the Quarticity component work better as compared to all other models for S&P 500 stock market price index data.

In-sample analysis results showed that as the h-day-ahead horizon increases, the HAR and its extension models fail to estimate well compared to the short and medium memory realised volatility.

4.3. Out-of-sample forecasting results

The out-of-sample predictive performance of seven models were compared by using a rolling window prediction method for forecasting the volatility of S&P 500 stock price returns over the multi-period horizons (1, 5, 10, 22, 44 and 66 days). For this, firstly, the whole sample was divided into two sub-samples called "estimation sample" and "prediction sample". Estimation sample is the estimation window containing the 3686 days at any given time starting from the first day (with rolling window method, the period shifts by one day every time, but the sample size will remain 3686), the prediction sample contained days from

Table 3: In-sample HAR-J results for S&P 500 with RV

HAR-J	$h=1$	$h=5$	$h=10$	$h=22$	$h=44$	$h=66$
β_0	0.13** (3.86)	0.19* (7.92)	0.25* (10.66)	0.38** (15.63)	0.59** (22.95)	0.72** (27.95)
$\beta_1^{(1)}$	0.35** (15.71)	0.27** (16.44)	0.19** (12.32)	0.13** (8.40)	0.10** (6.10)	0.08** (4.74)
$\beta_2^{(5)}$	0.43** (13.14)	0.35** (15.04)	0.35** (15.31)	0.31** (13.47)	0.29** (11.76)	0.20** (8.24)
$\beta_3^{(22)}$	-0.18** (6.27)	0.26** (12.65)	0.28** (14.11)	0.26** (12.57)	0.14** (6.44)	0.15** (7.15)
$J^{(1)}$	-1.00** (3.39)	-0.64** (4.98)	-0.45** (4.86)	-0.25** (1.90)	-0.18** (1.55)	-0.15** (1.55)
$Adj\,R^2$	0.53	0.64	0.63	0.55	0.39	0.30
AIC	3733.44	1351.50	1086.37	1288.54	1731.86	1674.01
BIC	3770.66	1388.71	1123.58	1325.72	1769.03	1711.12
RMSE	1.66	1.20	1.16	1.19	1.27	1.26
Q-LIKE	0.16	0.12	0.13	0.17	0.22	0.25

Note: Parenthesis in the above table indicates test statistic value. Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

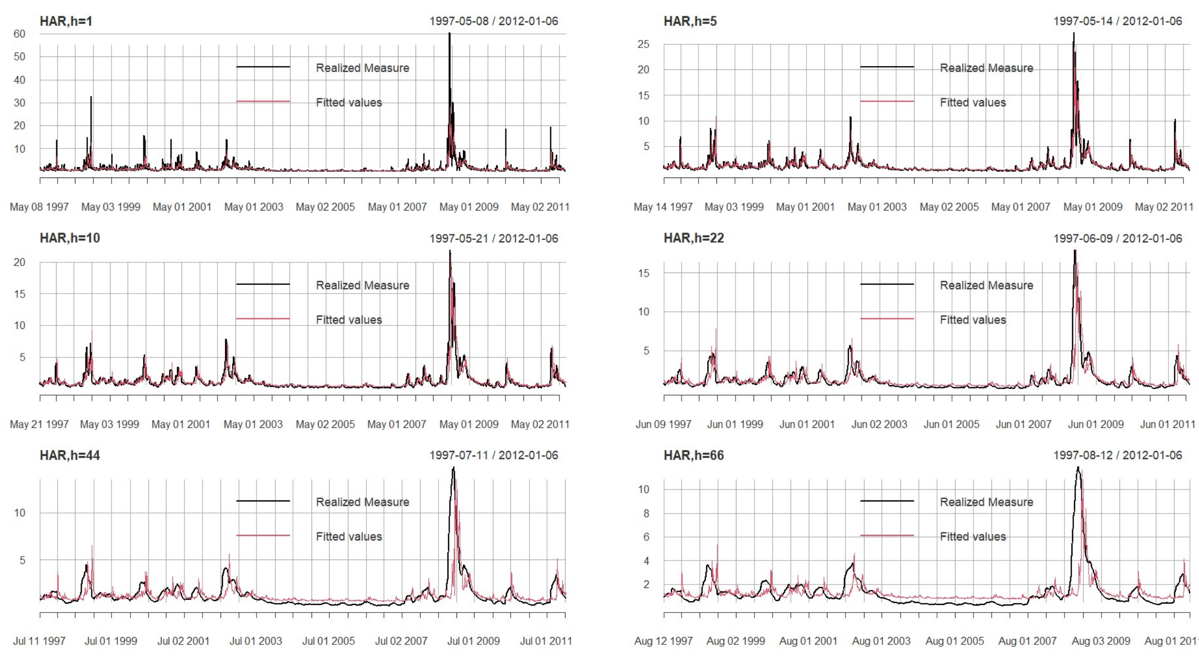

Figure 2: Plots for the fitted HAR model at different horizons

the 3687$^{\text{th}}$ day till the end of data period. In this way, on estimation samples, models were fitted to compute the predicted values for each of the subsequent 1-day, 5-days, 10-days, 22-days, 44-days and 66-days periods for the given samples.

It is noted here that the estimation sample was moved forward every time by one

Table 4: In-sample HAR-Q results for S&P 500 with RV

HAR-Q	$h = 1$	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
β_0	-0.01 (0.25)	0.02 (0.82)	0.18* (1.77)	0.32** (2.86)	0.55** (3.71)	0.68** (3.00)
$\beta_1^{(1)}$	0.59** (21.55)	0.18** (13.73)	0.30** (5.35)	0.24** (2.55)	0.19** (2.03)	0.16** (1.95)
$\beta_2^{(5)}$	0.35** (11.00)	0.68** (22.30)	0.31* (1.84)	0.27* (1.88)	0.26** (2.08)	0.17** (2.02)
$\beta_3^{(22)}$	0.09** (3.35)	0.15** (7.21)	0.24 (1.12)	0.22 (1.16)	0.11 (0.78)	0.13 (1.13)
$Q^{(1)}$	-0.36** (18.28)	-0.56** (14.41)	-0.16** (5.04)	-0.14** (2.12)	-0.11** (1.70)	-0.10* (1.68)
$Adj\,R^2$	0.56	0.65	0.64	0.56	0.40	0.30
AIC	3519.61	1224.52	991.53	1201.99	1638.03	1635.26
BIC	3556.85	1261.76	1028.75	1239.19	1720.20	1672.38
RMSE	1.61	1.18	1.14	1.18	1.26	1.25
Q-LIKE	0.14	0.11	0.11	0.14	0.20	0.24

Note: Parenthesis in the above table indicates test statistic value. Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

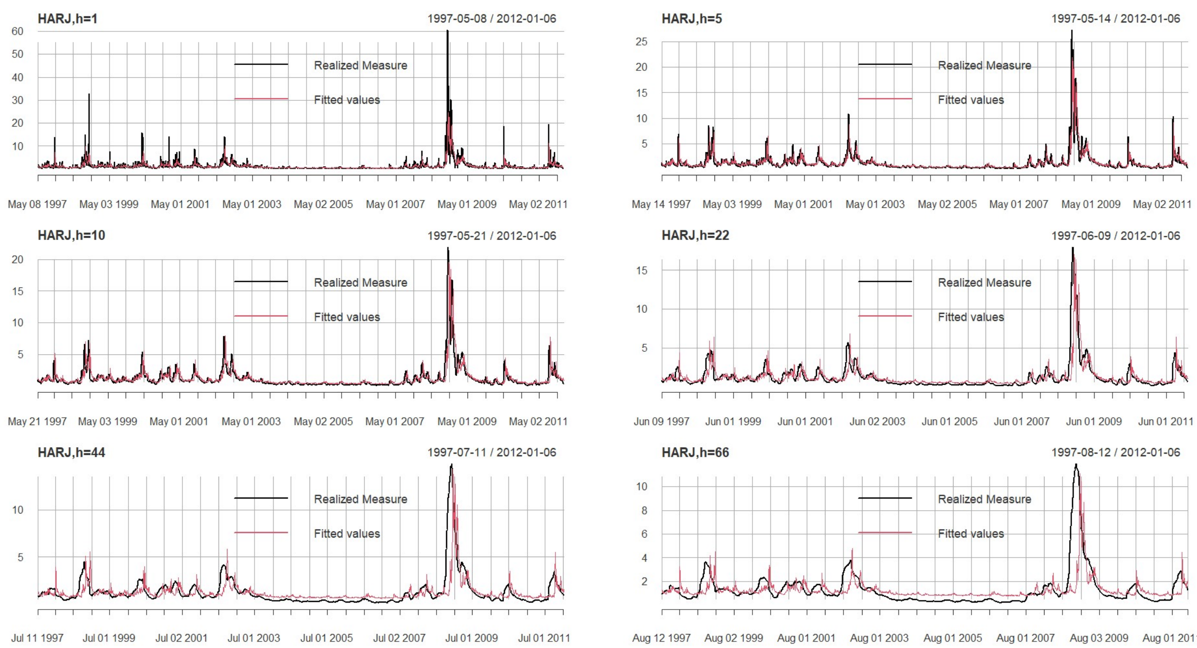

Figure 3: Plots for the fitted HAR-J model at different horizons

day. The estimation sample still contained 3686 observations, the last estimation sample with same number of observations but with the last observation in it belonging to the 4095$^{\text{th}}$ day. The predicted values of the 1, 5, 10, 22, 44 and 66 days were obtained from the fitted models on each of these 410 estimation samples. The forecasting accuracies of each model were measured using MSPE, MAPE and Q-LIKE functions to evaluate the deviation between

Table 5: In-sample HAR-CJ results for S&P 500 with RV

HAR-CJ	$h = 1$	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
β_0	0.07 (1.30)	0.17** (2.15)	0.27** (2.44)	0.39** (2.41)	0.57** (2.49)	0.66** (2.10)
$C_1^{(1)}$	0.33** (3.58)	0.22** (4.20)	0.18** (3.02)	0.13** (1.95)	0.10 (1.49)	0.08 (1.29)
$C_2^{(5)}$	0.58** (2.19)	0.56** (3.26)	0.43** (2.49)	0.35** (2.26)	0.31** (2.15)	0.22* (1.92)
$C_3^{(22)}$	0.05 (0.32)	0.06 (0.35)	0.14 (0.61)	0.19 (0.77)	0.13 (0.66)	0.21* (1.83)
$J_1^{(1)}$	-0.56* (1.67)	-0.14 (0.45)	-0.20 (0.94)	-0.12 (0.77)	-0.09 (0.67)	-0.05 (0.61)
$J_2^{(5)}$	-1.11 (0.90)	-1.83 (1.57)	-0.59 (0.85)	-0.04 (0.06)	0.06 (0.10)	0.09 (0.16)
$J_3^{(22)}$	1.66 (1.07)	2.64 (1.19)	2.18 (0.96)	1.21 (0.62)	0.24 (0.20)	-0.67 (0.53)
$Adj\,R^2$	0.54	0.66	0.63	0.55	0.39	0.30
AIC	3692.10	1288.01	1055.27	1283.86	1731.00	1667.71
BIC	3714.75	1277.65	1104.90	1333.47	1780.56	1717.22
RMSE	1.65	1.18	1.15	1.19	1.27	1.26
Q-LIKE	0.15	0.10	0.11	0.16	0.21	0.24

Note: Parenthesis in the above table indicates test statistic value. Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

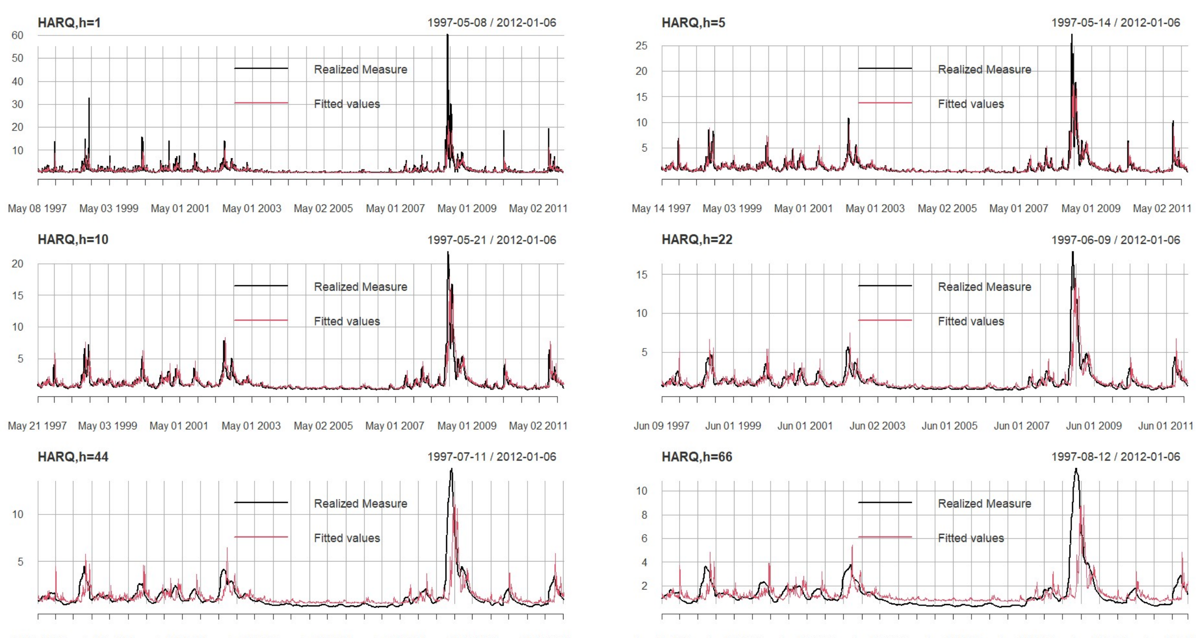

Figure 4: Plots for the fitted HAR-Q model at different horizons

Table 6: In-sample HAR-QJ results for S&P 500 with RV

HAR-QJ	$h = 1$	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
β_0	0.00 (0.17)	0.12* (1.72)	0.19* (1.95)	0.32** (2.88)	0.54** (3.88)	0.68** (3.15)
$\beta_1^{(1)}$	0.60** (21.83)	0.40** (5.26)	0.30** (5.17)	0.24** (2.63)	0.19** (2.16)	0.15** (1.92)
$\beta_2^{(5)}$	0.35** (10.80)	0.31** (2.94)	0.31* (1.87)	0.28* (1.88)	0.26** (2.09)	0.17** (1.13)
$\beta_3^{(22)}$	0.10** (3.62)	0.22* (1.79)	0.25 (1.15)	0.22 (1.11)	0.11 (0.77)	0.13 (1.13)
$Q^{(1)}$	-0.33 ** (14.83)	-0.25 (0.89)	-0.13 (0.82)	0.05 (0.25)	0.06 (0.33)	0.06 (0.41)
$J^{(1)}$	-0.33** (3.39)	-0.19** (4.98)	-0.15** (4.86)	-0.15* (1.90)	-0.12 (1.55)	-0.10 (1.55)
$Adj\,R^2$	0.56	0.66	0.64	0.56	0.40	0.30
AIC	3508.58	1214.08	989.88	1203.43	1684.29	1636.49
BIC	3552.03	1257.52	1033.31	1246.84	1727.66	1679.81
RMSE	1.61	1.18	1.14	1.18	1.26	1.25
Q-LIKE	0.13	1.1	0.12	0.15	0.21	0.23

Note: Parenthesis in the above table indicates test statistic value. Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

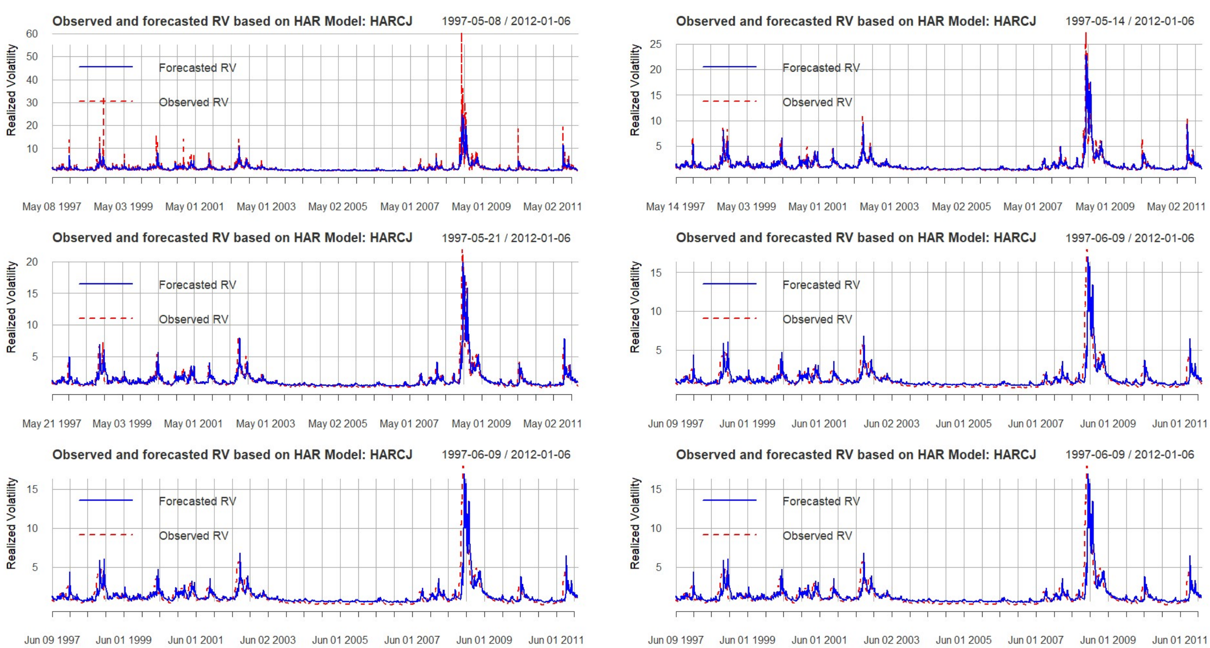

Figure 5: Plots for the fitted HAR-CJ model at different horizons

the predicted values and the true values of realised volatilities.

Table 9 and Figure 9 report the values of forecasting performance measures of all the

Table 7: In-sample CHAR results for S&P 500 with RV

CHAR	$h = 1$	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
β_0	0.14** (4.36)	0.21** (8.85)	0.27** (11.71)	0.40** (16.61)	0.61** (23.76)	0.73** (28.69)
$\beta_1^{(1)}$	0.26** (12.62)	0.20** (13.91)	0.16** (11.05)	0.12** (8.03)	0.09** (5.94)	0.07** (4.49)
$\beta_2^{(5)}$	0.49** (14.54)	0.42** (17.15)	0.37** (15.92)	0.32** (13.43)	0.30** (11.66)	0.21** (8.18)
$\beta_3^{(22)}$	0.17** (5.75)	0.24** (11.58)	0.28** (13.78)	0.27** (12.68)	0.15** (6.56)	0.16** (7.32)
$Adj R^2$	0.53	0.65	0.63	0.54	0.39	0.29
AIC	3757.68	1315.29	1099.14	1309.77	1746.68	1682.51
BIC	3788.71	1346.29	1130.16	1340.77	1777.63	1713.43
RMSE	1.67	1.2	1.16	1.2	1.27	1.26
Q-LIKE	0.15	0.12	0.14	0.17	0.22	0.24

Note: Parenthesis in the above table indicates test statistic value. Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

Table 8: In-sample CHAR-Q results for S&P 500 with RV

CHAR-Q	$h = 1$	$h = 5$	$h = 10$	$h = 22$	$h = 44$	$h = 66$
β_0	0.03 (0.90)	0.14** (0.03)	0.22** (2.40)	0.35** (3.36)	0.57** (3.63)	0.70** (3.82)
$\beta_1^{(1)}$	0.55** (20.24)	0.37 ** (4.89)	0.28** (4.69)	0.22** (2.23)	0.17* (1.77)	0.14** (1.23)
$\beta_2^{(5)}$	0.40** (12.16)	0.36** (3.28)	0.33* (1.91)	0.29* (1.86)	0.27** (1.99)	0.18** (1.72)
$\beta_3^{(22)}$	0.10 (3.60)	0.21 (1.53)	0.25 (1.16)	0.24 (1.21)	0.13 (0.82)	0.14 (1.16)
$\beta_4^{(1)}$	-0.35** (15.72)	-0.20** (4.43)	-0.16** (4.06)	-0.13* (1.72)	-0.10 (1.37)	-0.09 (1.40)
$Adj R^2$	0.56	0.66	0.64	0.55	0.39	0.30
AIC	3510.40	1159.47	997.35	1244.92	1712.05	1653.44
BIC	3547.64	1196.68	1034.55	1282.12	1749.22	1690.57
RMSE	1.61	1.17	1.14	1.18	1.26	1.26
Q-LIKE	0.13	0.10	0.12	0.17	0.22	0.23

Note: Parenthesis in the above table indicates test statistic value. Asterisks (** and *) indicate statistical significance at 1% and 5% respectively.

models for forecasting realised volatilities at 1, 5, 10, 22, 44 and 66 days. These results showed that extensions of HAR-type models using BPV, jump and quarticity components tend to have the good prediction accuracies. Moreover, it can be seen that the forecasting accuracy decreases with increase in prediction horizon, which indicates that HAR-type models are more accurate in predicting realised volatilities in the short and medium runs. For forecasting horizon h = 1, 5, 10 and 66-days, as per the Q-LIKE function, CHAR-Q performed better

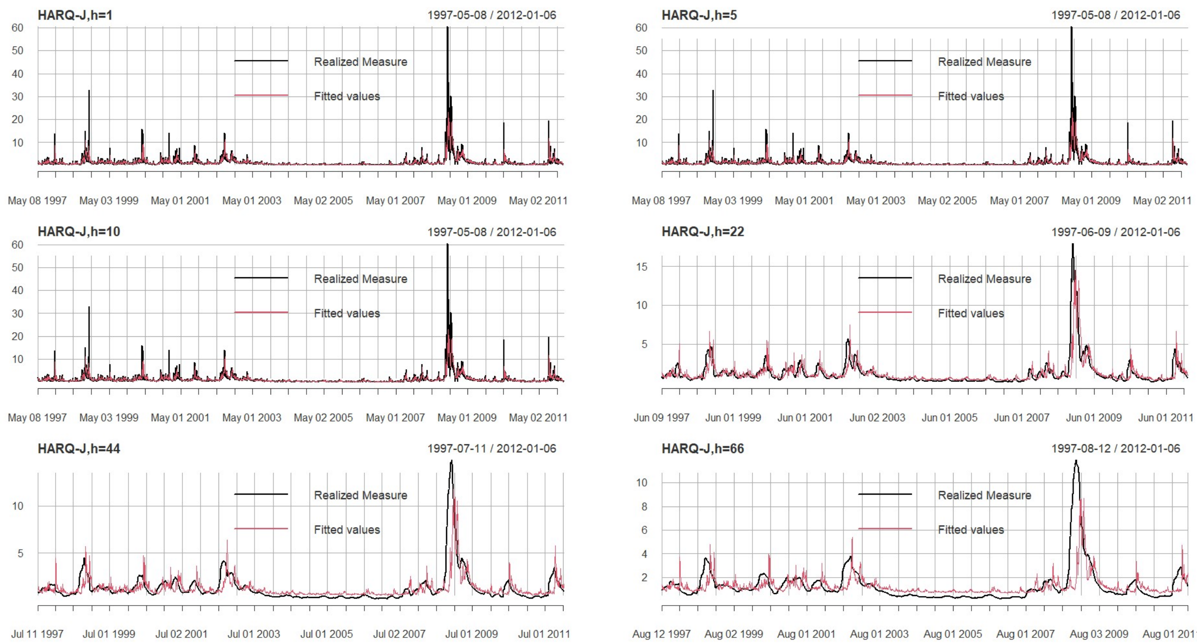

Figure 6: Plots for the fitted HAR-QJ model at different horizons

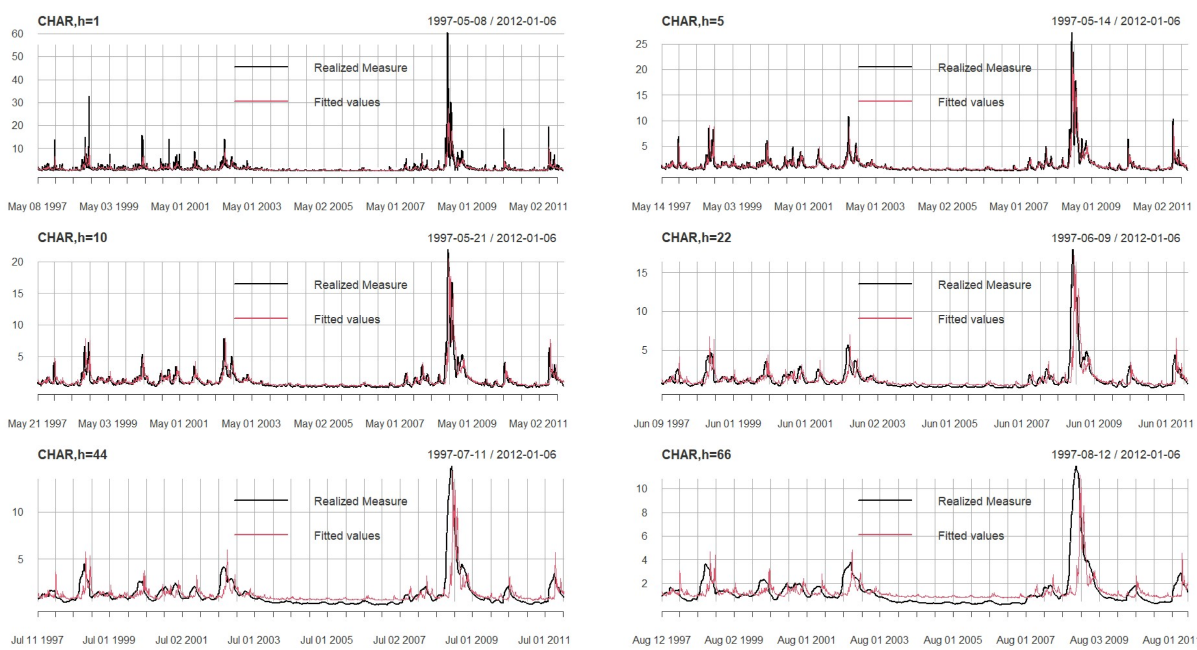

Figure 7: Plots for the fitted CHAR model at different horizons

than all other HAR model types while only for $h = 22$ and 44 horizons HARQJ-model performed well. When MSPE is considered for $h = 1$, 5, CHAR-Q performed well whereas at $h = 22$ and 66 days, CHAR performed better and for $h = 10$, HAR-QJ model performed better. At $h = 44$, HARQ model performed well for forecasting realised volatility. Overall, while considering all these measures, CHARQ, HAR-QJ, CHAR and HAR-Q performed well as compared to all other HAR and its extensions.

Table 9: Forecasting evaluation for S&P market with RV

Accuracy measures	h	HAR	HAR-J	HAR-Q	HAR-QJ	HARCJ	CHAR	CHAR-Q
MSPE	1	5.17	5.10	4.99	4.89	4.42	4.4	4.29
	5	3.92	3.84	3.15	2.89	5.64	5.62	2.21
	10	4.96	4.82	3.88	3.79	5.60	6.31	5.47
	22	9.44	9.36	8.12	7.26	6.88	6.85	7.27
	44	18.96	18.82	15.4	16.24	17.04	16.83	18.63
	66	27.09	27.07	24.06	24.62	25.23	17.23	20.63
MAPE	1	17.96	17.52	15.38	15.37	14.98	14.99	14.77
	5	22.55	22.1	19.58	18.84	17.27	17.25	17.11
	10	26.39	25.94	23.86	23.59	17.38	18.43	18.37
	22	34.85	34.72	32.96	31.63	19.75	19.67	20.28
	44	47.83	47.67	43.85	44.79	30.33	19.92	42.38
	66	57.49	57.42	53.66	54.53	42.13	20.61	44.29
Q-LIKE	1	14.87	14.55	13.98	13.19	14.73	14.88	12.00
	5	11.49	11.15	8.50	8.47	21.17	21.07	8.21
	10	12.65	12.37	9.90	9.74	21.52	25.21	9.79
	22	16.95	16.81	14.51	13.209	26.23	26.19	13.80
	44	23.32	23.21	19.35	20.29	28.79	28.15	27.69
	66	26.19	26.12	22.4	23.35	35.38	29.35	23.15

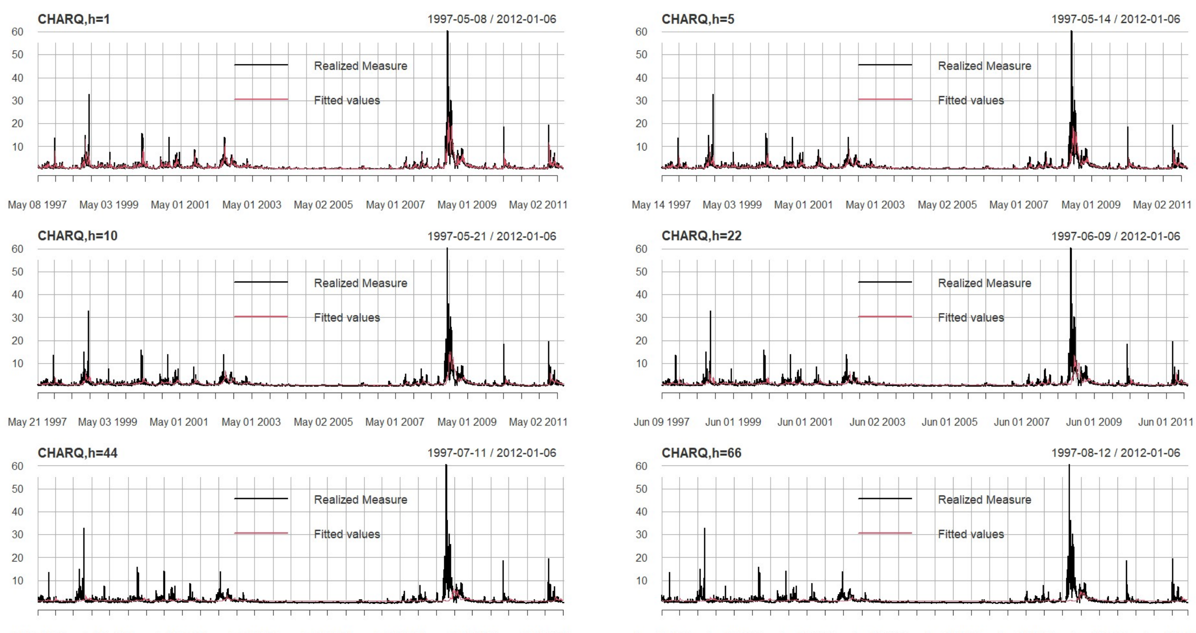

Figure 8: Plots for the fitted CHARQ model at different horizons

5. Concluding remarks

HAR models were studied along with their extensions for dynamic modeling of realised variance behaviour and its advantages over intraday were brought out which is widely used in high frequency data structure in order to capture the noise present in the intraday.

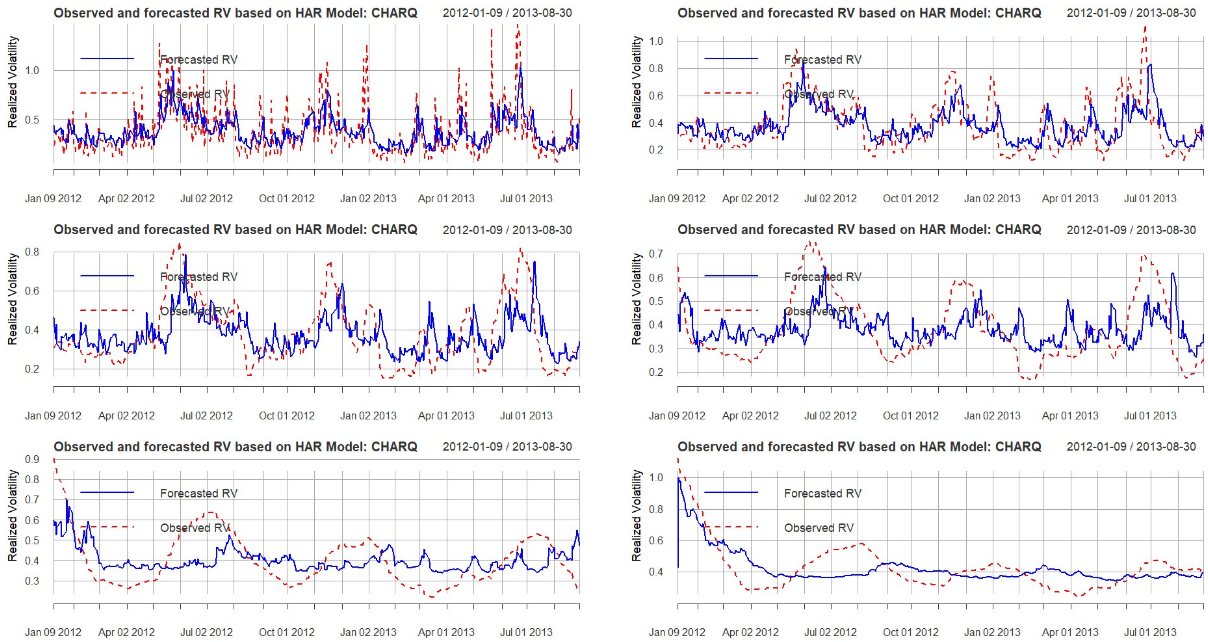

Figure 9: Plots for the forecasted CHARQ model at different horizons

An attempt has been done on a real dataset relating to Standard and Poor's 500 (S&P 500) stock market high frequency data and its volatility was estimated by using HAR models and its extensions and were compared on different horizons with their volatility were studied. The In-sample estimation results revealed that CHAR-Q models performed well in the estimation period compared to all other models. The out-sample forecasting results revealed that extensions of HAR-type models using BPV, jump and quarticity components tend to have the good prediction accuracies, more so for short run periods. In short, by way of an example, volatility on monetary policy announcement today will be more sensitive to the market mood on the pre-announcement day than on other days.

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