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Constant Block-Sum Designs Through Confounded Factorials

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Abstract

Confounded factorial designs are shown to provide a rich class of constant block-sum designs. The approach also provides a direct and straightforward proof of the necessary condition for existence of constant block-sum designs given recently by Khattree (2022).

Key words: Balanced incomplete block design; Group divisible design; Treatment contrast.

1. Introduction

Constant block-sum designs for quantitative treatment levels have been recently introduced by Khattree (2019a,b). In these designs, the sum of the treatment levels in each block is constant. Several methods of their construction have been presented by Khattree (2020). A general approach to determine whether or not a given design can be transformed into a constant block-sum design and its construction if it exists has been developed in Khattree (2022). He also discussed several individual examples, including two-associate class group divisible (GD) designs. Non-existence of constant block-sum balanced incomplete designs was established by Khattree (2019a, 2022). Bansal and Garg (2022) and Khattree (2022) derived some conditions for existence of partially balanced constant block-sum designs and gave further combinatorial methods of their construction. Gupta (2021) gave general results for GD designs with respect to the property of constant block-sum. He established non-existence of semi-regular and regular GD constant block-sum designs. He also discussed construction of singular GD constant block-sum designs and gave several illustrative examples.

Motivated by the results presented by Khattree (2022), the purpose of this paper is to study construction of constant block-sum designs using factorial designs. It is shown that the method of confounding provides a rich class of constant block-sum designs. The approach also provides a direct and straightforward proof of the necessary condition for existence of constant block-sum designs given by Khattree (2022).

2. Method of Construction

Consider an equireplicate confounded block design with parameters v, b, r, k, and let $\boldsymbol{\tau} = (\tau_1, \tau_2, \cdots, \tau_v)'$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_b)'$ respectively denote the $v \times 1$ and $b \times 1$ vectors

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of treatment and block parameters. Let $\boldsymbol{h}'\boldsymbol{\tau}$ denote a treatment contrast that is partially or completely confounded in the design, $\boldsymbol{h}'\mathbf{1}_v = 0$, where $\mathbf{1}_a$ denotes a $a \times 1$ vector of 1's. Further, $\boldsymbol{s}'\boldsymbol{\tau}$ denotes a treatment contrast that is estimated with full efficiency in the design, i.e. it is not confounded in any of the replications of the design, $\boldsymbol{s}'\mathbf{1}_v = 0$. We will refer to factorial effects that are estimated with full efficiency as completely unconfounded effects.

To motivate the method of construction, we replace the ith treatment in the confounded design by the ith element of $\boldsymbol{h}$ and $\boldsymbol{s}$. In other words, the treatments in the design are replaced by the corresponding coefficients of the confounded and unconfounded contrasts. This is illustrated with the help of the following example.

Example 1: Consider the 2^3 partially confounded design of Table 1 having parameters $v = 8$, $b = 4$,, $r = 2$, $k = 4$. The designed is obtained by confounding the three-factor interaction $F_1F_2F_3$ in one replication and the two-factor interaction F_2F_3 in the other replication.

Table 1

$F_1F_2F_3$ confounded		F_2F_3 confounded	
Block 1	Block 2	Block 3	Block 4
000	001	000	001
101	010	011	010
110	100	100	101
011	111	111	110

Let $\boldsymbol{u}_1$, $\boldsymbol{u}_2$, $\boldsymbol{u}_3$, $\boldsymbol{u}_{12}$, $\boldsymbol{u}_{13}$, $\boldsymbol{u}_{23}$, and $\boldsymbol{u}_{123}$ be the contrast coefficient vectors for the F_1, F_2, F_3 main effects and F_1F_2, F_1F_3, F_2F_3, $F_1F_2F_3$ interactions respectively,

$$\begin{bmatrix} \boldsymbol{u}_1' \\ \boldsymbol{u}_2' \\ \boldsymbol{u}_3' \\ \boldsymbol{u}_{12}' \\ \boldsymbol{u}_{13}' \\ \boldsymbol{u}_{23}' \\ \boldsymbol{u}_{123}' \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & -1 & +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 & -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 & -1 & +1 & -1 & +1 \\ +1 & +1 & -1 & -1 & -1 & -1 & +1 & +1 \\ +1 & -1 & +1 & -1 & -1 & +1 & -1 & +1 \\ +1 & -1 & -1 & +1 & +1 & -1 & -1 & +1 \\ -1 & +1 & +1 & -1 & +1 & -1 & -1 & +1 \end{bmatrix} .$$

Also, the vector of treatment parameters can be written as,

$$\boldsymbol{\tau}' = (\tau_{000} \;\; \tau_{001} \;\; \tau_{010} \;\; \tau_{011} \;\; \tau_{100} \;\; \tau_{101} \;\; \tau_{110} \;\; \tau_{111}) \; ,$$

with τ_x denoting the effect of the treatment combination x. Now we replace the treatment combinations in each block by the corresponding $F_1F_2F_3$ contrast coefficients and obtain the design displayed in Table 2. The block sums are given in the last row of the table.

Table 2

	Replace treatment combinations by the corresponding $F_1F_2F_3$ contrast coefficients			
	Block 1	Block 2	Block 3	Block 4
	-1	$+1$	-1	$+1$
	-1	$+1$	-1	$+1$
	-1	$+1$	$+1$	-1
	-1	$+1$	$+1$	-1
Block sums	-4	$+4$	0	0

Similarly, Tables 3 and 4 give the designs obtained by replacing the treatment combinations in each block of the design by respectively the F_2F_3 and F_1F_2 contrast coefficients. Note that $F_1F_2F_3$ and F_2F_3 are partially confounded whereas F_1F_2 is not confounded and it is estimated without any loss of information.

Table 3

	Replace treatment combinations by the corresponding F_2F_3 contrast coefficients			
	Block 1	Block 2	Block 3	Block 4
	$+1$	-1	$+1$	-1
	-1	-1	$+1$	-1
	-1	$+1$	$+1$	-1
	$+1$	$+1$	$+1$	-1
Block sums	0	0	$+4$	-4

Table 4

	Replace treatment combinations by the corresponding F_1F_2 contrast coefficients			
	Block 1	Block 2	Block 3	Block 4
	$+1$	$+1$	$+1$	$+1$
	-1	-1	-1	-1
	$+1$	-1	-1	-1
	-1	$+1$	$+1$	$+1$
Block sums	0	0	0	0

Block sums are constant, being equal to zero, for the design of Table 4 corresponding to the F_1F_2 interaction estimated with full efficiency in the design. It can be verified that the block sums are also constant, being equal to zero, for the designs constructed similarly corresponding to the other four unconfounded effects F_1, F_2, F_3, and F_1F_3 respectively. However, block sums are not constant for the designs of Tables 2 and 3 corresponding to the partially confounded interactions $F_1F_2F_3$ and F_2F_3 respectively.

The pattern in block sums with respect to confounded and unconfounded contrasts observed in the above example holds true in general. A completely unconfounded contrast $\boldsymbol{s}'\boldsymbol{\tau}$ is estimated from within block comparisons, *i.e.* it is estimated orthogonal to blocks. Clearly, its contrast coefficients falling in any block must sum to zero in order for the corresponding block effect to be canceled out from within block comparisons. Thus, as observed in the above example, block sum for a completely unconfounded contrast must be zero for each and every block. Conversely, a partially or completely confounded contrast $\boldsymbol{h}'\boldsymbol{\tau}$ is mixed up with some block contrast implying non-constancy of block sums.

Lemma: Let block contents of a partially confounded design be replaced by corresponding coefficients of a treatment contrast. Then the property of constant block sum being equal to zero holds for all contrasts that are estimated with full efficiency. Furthermore, this property does not hold for the treatment contrasts that are partially or completely confounded in the design.

Although, neither the block contents of $+1$ and -1 nor the block sum of zero are helpful from a practical point of view, as will be seen later, useful constant block sum designs can be easily derived through this approach.

The above lemma is closely related to the main result of Khattree (2022). He proved that a necessary condition for existence of a constant block-sum design is that $\boldsymbol{w} \neq \mathbf{1}_v$ is an eigenvector of $\boldsymbol{A}$ corresponding to a zero eigenvalue, where

$$\boldsymbol{A} \,=\, \boldsymbol{N}\boldsymbol{N}' \,-\, \frac{rk}{v}\mathbf{1}_v\mathbf{1}_v' \,,$$

and $\boldsymbol{N}$ is the incidence matrix of the design. Gupta (2021) showed that the term $(rk/v)\mathbf{1}_v\mathbf{1}_v'$ in the expression of $\boldsymbol{A}$ is in fact redundant. Thus equivalently, a necessary condition for existence of constant block-sum design is that $\boldsymbol{w} \neq \mathbf{1}_v$ is an eigenvector of $\boldsymbol{N}\boldsymbol{N}'$ corresponding to a zero eigenvalue. Note that a treatment contrast is estimated with full efficiency if and only if its contrast coefficient vector is an eigenvector of $\boldsymbol{N}\boldsymbol{N}'$ with zero eigenvalue. Thus, estimation of a treatment contrast orthogonal to blocks provides a direct and straightforward proof of the necessary condition for existence of a constant block-sum design.

We now discuss constructions of constant block-sum designs. Let q denote the number of treatment contrasts that are estimated with full efficiency in a factorial design, and let these contrasts be denoted by

$$\boldsymbol{U}'\boldsymbol{\tau} \,=\, \begin{bmatrix} \boldsymbol{u}_1' \\ \boldsymbol{u}_2' \\ \vdots \\ \boldsymbol{u}_q' \end{bmatrix} \boldsymbol{\tau} \;\;,$$

where $\boldsymbol{u}_i' = (u_{i1} \;\; u_{i2} \;\; \cdots \; u_{iv})$, with $\boldsymbol{u}_i'\mathbf{1}_v = 0$, $i = 1,\, 2, \cdots ,\; q$. Consider θ_u, a linear function of the q contrasts given by

$$\theta_u \,=\, \boldsymbol{C}'\boldsymbol{U}'\boldsymbol{\tau} \,=\, \sum_{i=1}^{q} (c_i\boldsymbol{u}_i')\,\boldsymbol{\tau} \,=\, \boldsymbol{t}_u'\boldsymbol{\tau} \;\;,$$

where

$$\boldsymbol{C}' = (\,c_1 \quad c_2 \quad \cdots \quad c_q\,)\ ,$$
$$\boldsymbol{t}'_u = \left(\sum_{i=1}^{q} c_i u_{i1} \quad \sum_{i=1}^{q} c_i u_{i2} \quad \cdots \quad \sum_{i=1}^{q} c_i u_{iv}\right)$$
$$= (t_{u1} \quad t_{u2} \quad \cdots \quad t_{uv})\ ,$$

and c_i's are some constants chosen such that all the elements of $\boldsymbol{t}_u$ are different from each other. Being a linear function of treatment contrasts that are estimated with full efficiency, the treatment contrast θ_u is also estimated with full efficiency in the design. Thus, using the Lemma, the property of constant block-sum holds when block contents of the design are replaced by corresponding coefficients of the treatment effects in the linear function θ_u, i.e. by the corresponding elements of $\boldsymbol{t}_u$. The $\boldsymbol{t}_u$ being a contrast coefficient vector, $\sum_{i=1}^{v} t_{ui} = 0$, which means that not all the t_{ui}'s are greater than zero. However, it is easily seen, *cf.* Khattree (2022), that the property of constant block-sum still holds if we add a constant value, say c_0, to all the elements of $\boldsymbol{t}_u$. Let $\boldsymbol{t}^*_u = (t_{u1} + c_0 \;\; t_{u2} + c_0 \;\; \cdots \;\; t_{uv} + c_0)$, where c_0 is chosen such that all the elements of $\boldsymbol{t}^*_u$ are greater than zero. Finally, the treatment combinations in the design are then replaced by the corresponding elements of $\boldsymbol{t}^*_u$ to arrive at a constant block-sum design. For illustration, we again consider the 2^3 partially confounded design of Example 1.

Example 1 contd.: Here we have five completely unconfouned contrasts, *i.e.* $q = 5$, given by

$$\boldsymbol{U} = (\,\boldsymbol{u}_1 \quad \boldsymbol{u}_2 \quad \boldsymbol{u}_3 \quad \boldsymbol{u}_{12} \quad \boldsymbol{u}_{13}\,)\ ,$$

and let $\boldsymbol{T}$ denote the vector of treatment combinations arranged in the lexicographic order, i.e. in increasing numerical order,

$$\boldsymbol{T}' = (000 \;\; 001 \;\; 010 \;\; 011 \;\; 100 \;\; 101 \;\; 110 \;\; 111).$$

Taking $\boldsymbol{C}' = (\,0.44 \quad -0.10 \quad -0.08 \quad 0.18 \quad -0.20\,)$ and $c_0 = 1.2$ gives,

$$\boldsymbol{t}^{*\prime}_u = (\,0.92 \;\; 1.16 \;\; 0.36 \;\; 0.60 \;\; 1.84 \;\; 1.28 \;\; 2.00 \;\; 1.44\,)\ .$$

Replacing the ith element of $\boldsymbol{T}$ in Table 1 by the ith element of $\boldsymbol{t}^{*\prime}_u$, $i = 1, 2, \cdots, v$, yields a design with a constant block-sum of $4c_0 = 4.8$. A very large number of distinct constant block-sum designs can be constructed in this fashion by choosing different values of $\boldsymbol{C}$ and the constant c_0. Tables 5 and 6 list five more solutions for the vector of treatment levels $\boldsymbol{t}^{*\prime}_u$ obtained by trial and error. Many more solutions can be easily constructed in this way.

Table 5: Further solutions for Example 1

$\boldsymbol{t}^{*\prime}_u$ No.	$\boldsymbol{t}^{*\prime}_u$
1	0.56 1.12 0.40 0.96 1.48 1.24 2.04 1.80
2	1.09 0.89 0.99 0.79 0.55 1.07 1.85 2.37
3	0.21 0.71 1.17 1.67 0.39 2.29 0.63 2.53
4	1.07 1.57 0.83 1.33 1.13 3.03 0.17 2.07
5	0.72 1.92 0.48 1.68 1.08 2.28 0.12 1.32

Table 6: The $\boldsymbol{C}'$ and $\boldsymbol{c}_0$ corresponding to $\boldsymbol{t}_u^{*\prime}$ listed in Table 5

$\boldsymbol{t}_u^{*\prime}$ No.	c_1	c_2	c_3	c_4	c_5	c_0
1	0.44	0.10	0.08	0.18	-0.20	1.2
2	0.26	0.30	0.08	0.35	0.18	1.2
3	0.26	0.30	0.60	-0.18	0.35	1.2
4	0.20	-0.30	0.60	-0.18	0.35	1.4
5	0.00	-0.30	0.60	-0.18	0.00	1.2

The next two examples further illustrate the richness of confounded factorials as constant block-sum designs.

Example 2: We now consider a 2^4 partially confounded design presented in Table 7, having parameters $v = 16$, $b = 8$, $r = 2$, $k = 4$, obtained by confounding $F_1F_2F_3$ and $F_2F_3F_4$ in one replication and $F_1F_2F_4$ and $F_1F_3F_4$ in the other replication. Note that the generalized interactions F_1F_4 and F_2F_3 are also partially confounded in the design.

Table 7

$F_1F_2F_3$, $F_2F_3F_4$, F_1F_4 confounded				$F_1F_2F_4$, $F_1F_3F_4$ F_2F_3 confounded			
Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
0000	0001	1001	1000	0000	0010	0011	0001
0110	0111	1111	1110	0111	0101	0100	0110
1011	1010	0010	0011	1001	1011	1010	1000
1101	1100	0100	0101	1110	1100	1101	1111

As before, let $\boldsymbol{T}$ be the vector of treatment combinations arranged in the lexicographic order. Further, let

$$\boldsymbol{J}_0 = \begin{pmatrix} -1 \\ +1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{J}_2 = \begin{pmatrix} +1 \\ +1 \end{pmatrix}.$$

The contrast coefficient vectors $\boldsymbol{u}_i$, $\boldsymbol{u}_{i_1i_2}$, $\boldsymbol{u}_{i_1i_2i_3}$, and $\boldsymbol{u}_{1234}$ for the main effects and interactions, $i,\ i_1 < i_2 < i_3 = 1, 2, 3, 4$, are given by $\boldsymbol{f}_1 \otimes \boldsymbol{f}_2 \otimes \boldsymbol{f}_3 \otimes \boldsymbol{f}_4$ as below:

$$\boldsymbol{f}_1\otimes\boldsymbol{f}_2\otimes\boldsymbol{f}_3\otimes\boldsymbol{f}_4 = \left[\begin{array}{l} \boldsymbol{u}_i \\ \boldsymbol{u}_{i_1i_2} \\ \boldsymbol{u}_{i_1i_2i_3} \\ \boldsymbol{u}_{1234} \end{array} \quad \text{where } \boldsymbol{f}_j = \boldsymbol{J}_0 \left\{ \begin{array}{l} \text{for } j = i \\ \text{for } j = i_1, i_2 \\ \text{for } j = i_1, i_2, i_3 \\ \text{for } j = 1,2,3,4 \end{array} \right\}, \text{ and } f_j = \begin{array}{l} \boldsymbol{J}_2 \text{ otherwise} \\ j = 1,2,3,4 \end{array} \right]$$

The completely unconfounded $q = 9$ contrast coefficient vectors are given by,

$$\boldsymbol{U} = (\boldsymbol{u}_1 \;\; \boldsymbol{u}_2 \;\; \boldsymbol{u}_3 \;\; \boldsymbol{u}_4 \;\; \boldsymbol{u}_{12} \;\; \boldsymbol{u}_{13} \;\; \boldsymbol{u}_{24} \;\; \boldsymbol{u}_{34} \;\; \boldsymbol{u}_{1234}).$$

For instance, taking $\boldsymbol{C}' = (-0.22,\ 0.30\ \ -0.25\ \ 0\ \ 0\ \ 0\ \ 0\ \ -0.30\ \ -0.25)$ and $c_0 = 1.2$ gives,

$$\boldsymbol{t}_u^{*\prime} = (0.79\ \ 1.95\ \ 1.45\ \ 0.29\ \ 1.89\ \ 2.05\ \ 1.55\ \ 1.39\ \ 0.85\ \ 1.01\ \ 0.51\ \ 0.35\ \ 0.95\ \ 2.11\ \ 1.61\ \ 0.45),$$

which yields a design given in Table 8 with a constant block-sum of $4c_0 = 4.8$.

Table 8

Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7	Block 8
0.79	1.95	1.01	0.85	0.79	1.45	0.29	1.95
1.55	1.39	0.45	1.61	1.39	2.05	1.89	1.55
0.35	0.51	1.45	0.29	1.01	0.35	0.51	0.85
2.11	0.95	1.89	2.05	1.61	0.95	2.11	0.45

Five more solutions are given in Tables 9 and 10.

Table 9: Further solutions for Example 2

$\boldsymbol{t}_u^{*\prime}$ No.	$\boldsymbol{t}_u^{*\prime}$
1	0.64 2.30 1.80 0.14 2.24 1.90 1.40 1.74 1.20 0.86 0.36 0.70 0.80 2.46 1.96 0.30
2	1.99 1.91 3.41 1.49 3.51 1.59 3.09 3.01 2.41 0.49 0.99 0.91 2.09 2.01 2.51 0.59
3	1.02 1.88 2.28 1.42 2.92 3.38 3.78 3.32 1.92 2.38 0.78 0.32 4.02 4.88 3.28 2.42
4	1.02 1.88 2.28 1.42 0.92 1.38 1.78 1.32 1.92 2.38 0.78 0.32 2.02 2.88 1.28 0.42
5	1.27 0.19 0.43 1.03 0.47 2.07 1.83 0.71 1.17 2.29 2.53 0.93 2.57 1.97 1.73 2.81

Table 10: The $\boldsymbol{C}'$ and c_0 corresponding to $\boldsymbol{t}_u^{*\prime}$ listed in Table 9

$\boldsymbol{t}_u^{*\prime}$ No.	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_0
1	-0.22	0.30	-0.25	0	0	0	0	-0.33	-0.50	1.3
2	-0.50	0.30	0	-0.50	0	-0.25	0	0	-0.46	2.0
3	0	1.00	-0.30	0	0.15	-0.50	0	-0.33	-0.10	2.5
4	0	0	-0.30	0	0.15	-0.50	0	-0.33	-0.10	1.5
5	0.50	0.27	0	0	0	0	0.12	-0.13	0.55	1.5

Example 3: 3^2 partially confounded factorial design with parameters $v = 9$, $b = 6$, $r = 2$, $k = 3$. Here the two main effects F_1 and F_2 have 2 $d.f.$ each, and the two-factor interaction F_1F_2 has 4 $d.f.$ The treatment combinations vector is given by,

$$\boldsymbol{T}' = (00 \;\; 01 \;\; 02 \;\; 10 \;\; 11 \;\; 12 \;\; 20 \;\; 21 \;\; 22)\ .$$

The 4 $d.f.$ F_1F_2 interaction has two components: the 2 $d.f.$ F_1F_2 component and the 2 $d.f.$ $F_1F_2^2$ component. The design of Table 11 below is obtained by confounding the 2 $d.f.$ F_1F_2 component in one replication and the 2 $d.f.$ $F_1F_2^2$ component in the other replication.

Table 11

2 $d.f.$ F_1F_2 confounded			2 $d.f.$ $F_1F_2^2$ confounded		
Block 1	Block 2	Block 3	Block 4	Block 5	Block 6
00	10	02	00	21	01
12	01	20	11	10	12
21	22	11	22	02	20

The four contrasts corresponding to the two main effects that are completely unconfounded in the design are given by,

$$\boldsymbol{U} = \begin{bmatrix} \boldsymbol{u}'_{1\ell} \\ \boldsymbol{u}'_{1q} \\ \boldsymbol{u}'_{2\ell} \\ \boldsymbol{u}'_{2q} \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 & 0 & 0 & 0 & +1 & +1 & +1 \\ +1 & +1 & +1 & -2 & -2 & -2 & +1 & +1 & +1 \\ -1 & 0 & +1 & -1 & 0 & +1 & -1 & 0 & +1 \\ +1 & -2 & +1 & +1 & -2 & +1 & +1 & -2 & +1 \end{bmatrix} ,$$

where ℓ and q respectively denote the linear and quadratic components. Taking $\boldsymbol{C}' = (0.50 \ \ 0 \ \ -0.20 \ \ -0.19)$ and $c_0 = 1.6$ yields a design with constant block-sum of $3c_0 = 4.8$. The treatment levels vector $\boldsymbol{t}_u^{*\prime}$, arranged in the order of treatment combinations in T is given by,

$$\boldsymbol{t}_u^{*\prime} = (1.09 \ \ 0.90 \ \ 0.71 \ \ 2.19 \ \ 2.00 \ \ 1.81 \ \ 2.09 \ \ 1.90 \ \ 1.71) \ .$$

Five more solutions for this example are listed in Table 12.

Table 12: Further solutions for Example 3

	$t_u^{*\prime}$	c_1	c_2	c_3	c_4	c_0
1	1.09 1.50 0.71 1.59 2.00 1.21 2.09 2.50 1.71	0.50	0.00	-0.19	-0.20	1.6
2	1.12 1.62 1.52 1.30 1.80 1.70 1.48 1.98 1.88	0.18	0.00	0.20	-0.10	1.6
3	0.30 0.80 0.70 1.30 1.80 1.70 2.30 2.80 2.70	1.00	0.00	0.20	-0.10	1.6
4	0.47 2.92 0.87 0.65 3.10 1.05 0.83 3.28 1.23	0.18	0.00	0.20	-0.75	1.6
5	2.17 0.12 2.57 2.35 0.30 2.75 2.53 0.48 2.93	0.18	0.00	0.20	0.75	1.8

The constant block-sum designs of this paper are derived by searching for a treatment levels vector $\boldsymbol{t}_u^*$ through trial and error. Also, in practice treatment levels would be determined by subject matter specialists based on their study objectives. Therefore, a systmatic method of finding $\boldsymbol{t}_u^*$ with treatment levels in line with the study objectives is highly important from a practical point of view and deserves further research.

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