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LDPC Codes Based on New Combinatorial Designs

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Abstract

Earlier binary (k, r)-regular LDPC codes have been constructed using balanced incomplete block designs, mutually orthogonal Latin rectangles, partial geometries, group divisible designs, resolvable group divisible designs and finite geometries. Here we have constructed LDPC codes from certain triangular and L_2-type designs which are free of 4–cycles.

Key words: LDPC Codes; Association schemes; Partially balanced incomplete block designs; Triangular designs; L_2-type designs

AMS Subject Classifications: 62K10, 05B05

1. Introduction

1.1. LDPC codes

A binary (k, r)-regular low–density parity–check *(LDPC) code* is the null space of a $s \times t$ sparse parity-check matrix H (*i.e.* the majority of entries must be zero) over a Galois field $GF(2)$ of order 2 such that each row has r nonzero elements and each column has k nonzero elements where $r \ll t$ and $k \ll s$. The minimum distance of a code is equal to the minimum number of nonzero columns in the parity-check matrix such that a nontrivial linear combination of these columns sums to zero over $GF(2)$ [see Wicker (1995), p. 84 and Johnson and Weller (2003), p. 1416].

Parity-check matrices (or LDPC codes) may be represented as *Tanner bipartite graphs* with vertex set $V \cup W$ where V is comprised of code bits and W is comprised of parity-check equations. There exists an edge $\{v, w\}, v \in V$ and $w \in W$, in this bipartite graph if and only if v is a term in the check equation w. A *cycle* in a graph is a sequence of connected vertices which start and end at the same vertex in the graph and no other vertices occur more than once. The length of the cycle is the number of edges it contains and the *girth* of a graph is the length of its smallest cycle. Since the Tanner graph is bipartite, the length of a cycle must be even and at least 4.

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An LDPC code performs well with iterative decoding provided the corresponding Tanner graph have a reasonably large girth, *i.e.*, the graph should be free of short cycles. The cycles that affect the performance the most are the cycles of length four. These short cycles severely limit the performance of iterative decoding. For codes with these short cycles, iterative decoding becomes correlated after two iterations. Therefore, cycles of length four must be avoided in code construction [see Bonello *et al.* (2011) and Xu *et al.* (2005)].

An LDPC code is free of 4-cycles if no two distinct columns (or two distinct rows) of H have more than one nonzero component in common or the inner product of any two distinct rows or any two distinct columns of the parity-check matrix H is less than or equal to 1. This constraint on H is known as the row-column constraint (or RC constraint), see Diao *et al.* (2013). The RC-constraint confirms that the girth of the LDPC codes generated by such H is at least six.

1.2. Balanced incomplete block design

A *balanced incomplete block design* (BIBD) or a $2-(v,k,\lambda)$ design is an arrangement of v elements into $b = (\lambda(v^2 - v))/((k^2 - k))$ blocks, each of size $k(< v)$ such that every element is replicated r times and any two distinct elements occur together in λ blocks.

A BIBD is *resolvable* if the b blocks each of size k can be partitioned into r resolution classes such that

(i) Each resolution class contains b/r blocks;
(ii) Every element is replicated exactly once in each resolution class.

A BIBD with $k = 3$ and $\lambda = 1$ is usually known as Steiner's triple system (STS) or Steiner 2–design and a resolvable Steiner's triple system is known as Kirkman triple system (KTS), see Raghavarao (1971), Johnson and Weller (2001) and Ray–Chaudhuri and Wilson (1971).

Example 1: Consider a resolvable BIBD with parameters: $v = 9, b = 12, r = 4, k = 3, \lambda = 1$ whose resolution classes are:
RI: [(1 2 3) (4 5 6) (7 8 9)]; RII: [(1 4 7) (2 5 8) (3 6 9)]; RIII: [(1 5 9) (2 6 7) (3 4 8)]; RIV: [(1 6 8) (2 4 9) (3 5 7)].

1.3. Association scheme

A relationship defined on a set of v elements is called an *association scheme* with two associate classes if it satisfies the following conditions:

(a) Any two distinct elements are either 1st or 2nd associates of each other and any element is the 0-th associate of itself,
(b) Each element has n_j; j-th associates ($j = 0, 1, 2$) and
(c) For every pair of elements which are j-th associates of each other, there are $p^j_{u,w}$ elements that are u-th associates of one and w-th associates of the other ($j, u, w = 0, 1, 2$).

1.4. Partially balanced incomplete block (PBIB) design

Given an association scheme with two associate classes on a set of v elements, a *PBIB design* based on this association scheme is a block design with v elements and b blocks satisfying the following conditions:

(i) Each element appears at most once in a block,
(ii) Each block has a fixed number of elements, say k,
(iii) Each element appears in a fixed number of blocks, say r, and
(iv) Every pair of elements which are j-th ($j = 1, 2$) associates of each other appear together in λ_j blocks ($\lambda_1 \neq \lambda_2$).

Some special classes of PBIB designs known as group divisible, triangular and L_2-type Latin square designs are described below:

1.5. Group divisible design

Let $v = mn(m, n \geq 2)$ elements be arranged in an $m \times n$ array, say M. A *group divisible (GD) association scheme* on these $v = mn$ elements is defined as follows: two elements are first associates if they occur in the same row of M and second associates, otherwise.

A PBIB design based on GD association scheme is said to be GD design. The integers: $v = mn, b, r, k, \lambda_1$ and λ_2 are known as parameters of the GD design and they satisfy the relations: $bk = vr$;$(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1)$. Furthermore, if $r - \lambda_1 = 0$ then the GD design is singular (S); if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$ then it is semi–regular (SR) and if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$ then the design is regular (R).

Example 2: Consider the following resolvable solution of an SRGD design SR9 with parameters: $v = 8, b = 16, r = 4, k = 2, \lambda_1 = 0,\ \lambda_2 = 1, m = 2, n = 4$ as given in Clatworthy (1973):
RI: [(1 5) (2 6) (3 7) (4 8)]; RII: [(2 7) (1 8) (4 5) (3 6)]; RIII: [(4 6) (3 5) (2 8) (1 7)]; RIV: [(3 8) (4 7) (1 6) (2 5)].
The arrangement of $v = 8$ elements in 2×4 array is given as: $\begin{matrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{matrix}$.

1.6. Triangular design

A *triangular association scheme* is an arrangement of $v = (s(s-1))/2$ elements in an $s \times s$ array such that the positions on the principal diagonal are left blank, the $(s(s-1))/2$ positions above and below the principal diagonal are filled with the v elements in such a way that the resultant arrangement is symmetric about the principal diagonal. Then any two elements which occur in the same row or same column are first associates; otherwise they are second associates. A PBIB design based on triangular association scheme is called a triangular design. The integers $v = (s(s-1))/2, b, r, k, \lambda_1$ and λ_2 are known as parameters of the triangular design and they satisfy the relations: $bk = vr$; $2(s-2)\lambda_1 + ((s-2)(s-3))/2\lambda_2 = r(k-1)$.

Example 3: Consider a triangular design T9 given in Clatworthy (1973) with parameters: $v = b = 10, r = k = 3, \lambda_1 = 1, \lambda_2 = 0$ whose blocks are given as: (1 2 5); (8 9 10); (2 3 8); (5 7 9); (2 4 9); (5 6 8); (3 4 10); (6 7 10); (1 4 7); (1 3 6)

The arrangement of 10 elements in 5×5 array is given as: $\begin{array}{ccccc} * & 1 & 2 & 3 & 4 \\ 1 & * & 5 & 6 & 7 \\ 2 & 5 & * & 8 & 9 \\ 3 & 6 & 8 & * & 10 \\ 4 & 7 & 9 & 10 & * \end{array}.$

1.7. L_2-type design

An L_2-*association scheme* is an arrangement of $v = s^2$ elements into an $s \times s$ array such that any two elements in the same row or in the same column of the array are 1st associates; otherwise they are 2nd associates. A PBIB design based on L_2-association scheme is known as an L_2- type design. The integers $v = s^2, b, r, k, \lambda_1$ and λ_2 are known as parameters of the L_2-type design and they satisfy the relations: $bk = vr$; $2(s-1)\lambda_1 + (s-1)^2\lambda_2 = r(k-1)$.

Example 4: Consider an L_2-type design as given in Clatworthy (1973) with parameters: LS26: $v = b = 9, r = k = 4, n_1 = n_2 = 4, \lambda_1 = 1, \lambda_2 = 2$ whose blocks are given as:
(1 2 6 9); (2 4 6 8); (1 4 8 9); (2 5 7 9); (2 3 4 7); (3 4 5 9); (1 5 6 7); (3 6 7 8); (1 3 5 8)

The arrangement of $v = 9$ elements in 3×3 array is given as: $\begin{array}{ccc} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{array}.$

SRX, TX and LSX numbers are from Clatworthy (1973). For details on BIB, GD, triangular and L_2-type designs, we refer to Dey (2010), Raghavarao (1971), Raghavarao and Padgett (2005).

2. Earlier constructions

Low-density parity-check (LDPC) codes were introduced by Gallager (1962). LDPC codes can be divided into two types: random codes and structured codes. Random LDPC codes are constructed by computer search while structured LDPC codes are constructed by algebraic and combinatorial methods. Earlier constructions of regular LDPC codes from combinatorial designs may be summarized below in Table 1:

A recent survey on algebraic constructions of LDPC codes may also be found in Saurabh and Sinha (2023). The purpose of this paper is to construct binary regular LDPC codes based on triangular and L_2- type designs. The incidence matrix of such block design is used as parity-check matrix of the code which satisfies row-column constraint which ensures that the girth of the proposed code is at least six and the corresponding LDPC code (or Tanner graph) is free of 4-cycles. We are describing below the method to obtain LDPC codes from BIB and GD designs:

2.1. LDPC codes from BIB and GD designs

The following Lemmas [see Saurabh and Sinha (2023)] describe the constructions of LDPC codes from BIB and two associate class PBIB designs:

Lemma 1: The existence of two associate classes PBIB design with parameters: v, b, r, k, λ_1, $\lambda_2 \in 0, 1$ implies the existence of a (k, r)-regular LDPC codes free of four cycles with code length b and code rate about $1 - k/r(k < r)$.

Table 1: Combinatorial structures and corresponding LDPC codes

No.	Combinatorial Structure	Reference
1	BIB designs	Ammar *et al.* (2004), Lan *et al.* (2008)
2	Resolvable BIB designs	Johnson and Weller (2001, 2003)
3	Group divisible designs	Shan and Li (2013)
4	Resolvable group divisible designs	Xu *et al.* (2015)
5	$\alpha(>1)$-resolvable group divisible designs	Saurabh and Sinha (2023)
6	Semipartial geometries	Li *et al.* (2008)
7	Partial geometries	Johnson and Weller (2004), Diao *et al.* (2016), Xu *et al.* (2019)
8	Finite geometries	Kou *et al.* (2001)
9	Mutually orthogonal Latin rectangles	Vasic *et al.* (2002)
10	Euclidean geometries and partial BIBDs	Mahadevan and Morris (2002)
11	Oval designs	Weller and Johnson (2003)
12	Cyclic 2-(v,3,1) designs	Vasic and Milenkovic (2004)
13	Mutually orthogonal Latin squares	Zhang *et al.* (2010)
14	Difference covering arrays	Donovan *et al.* (2022)
15	Cubic semi–symmetric graphs	Crnkovic *et al.* (2022)

As a special case of Lemma 1, we can obtain the following result:

Lemma 2: The existence of a BIB design with parameters: $v, b, r, k, \lambda = 1$ implies the existence of a (k, r)-regular LDPC codes free of four cycles with code length b and code rate about $1 - k/r(k < r)$.

2.2. LDPC codes from resolvable BIB and GD designs

2.2.1. LDPC codes from resolvable BIB designs

Johnson and Weller (2003) used following series of Kirkman triple systems (KTSs) in the construction of LDPC codes:
Series 1: $v = 3(4t + 1), b = (4t + 1)(6t + 1), r = 6t + 1, k = 3, \lambda = 1$.
Series 2: $v = 3(6t + 1), b = (6t + 1)(9t + 1), r = 9t + 1, k = 3, \lambda = 1$;
where $s = 6t + 1$ is a prime or prime power.

Since the series I and II of KTSs are resolvable BIB designs, their incidence matrices N may be partitioned in to 'r' submatrices as $N = (N_1|N_2|N_3|N_4|\ldots|N_r)$ where each N_i is $v \times (v/k)$ matrix such that each row sum of N_i $(1 \leq i \leq r)$ is one. Further juxtaposing set of any $p(4 \leq p \leq r)$ submatrices of N we obtain series of LDPC codes with length $vp/3$ and code rate about $1 - 3/p$ [see Saurabh and Sinha (2023)]. This method may also be used to obtain LDPC codes from a resolvable BIB design with $\lambda = 1$ other than above Series (1 and 2) of resolvable BIB designs.

2.2.2. LDPC codes from resolvable GD designs

Xu *et al.* (2015) considered submatrices of the incidence matrix of a resolvable GD design with parameters: $v = mn, b, r, k, \lambda_1 = 0, \lambda_2 = 1$ as the parity-check matrix to con-

struct series of regular LDPC codes as follows:
Consider a resolvable GD design with parameters: $v = mn, b, r, k, \lambda_1 = 0, \lambda_2 = 1$. Then its incidence matrix N may be partitioned in to 'r' submatrices as $N = (N_1|N_2|N_3|N_4|\ldots|N_r)$ where each N_i is $v \times (v/k)$ matrix such that each row sum of $N_i (1 \leq i \leq t)$ is one. Further juxtaposing set of any $p (1 \leq p \leq r)$ submatrices of N we obtain series of LDPC codes with length vp/k and code rate about $1 - k/p$.

Xu *et al.* (2015) used the results of Assaf and Hartman (1989), Greig (1999) and Sun and Ge (2009) on the existence of resolvable GD designs for the construction of LDPC codes. Some of the series obtained as special cases of their results are given below:

Series 3 [Assaf and Hartman (1989)]: There exists a resolvable GD design with parameters: $v = 12s, b = 16s^2, r = 4s, k = 3, \lambda_1 = 0, \lambda_2 = 1, m = 3, n = 4s$.

Series 4 [Greig (1999)]: There exists a resolvable GD design with parameters: $v = 32s, b = 64s^2, r = 8s, k = 4, \lambda_1 = 0, \lambda_2 = 1, m = 4, n = 8s$.

Series 5 [Sun and Ge (2009)]: There exist resolvable GD designs with parameters: $v = s(s^2 - 1), b = (s^2 - 1)^2, r = s^2 - 1, k = s, \lambda_1 = 0, \lambda_2 = 1, m = s, n = s^2 - 1$ where s is a prime or prime power.

The following series of resolvable designs obtained as a special case of Theorem 10 of Saurabh and Sinha (2023) may also be used in LDPC codes:

Series 6: There exists a resolvable SRGD design with parameters: $v = q(q - t), b = q^2, r = q, k = q - t, \lambda_1 = 0, \lambda_2 = 1, m = q - t, n = q (1 \leq t \leq q - 1)$ where q is a prime or prime power.

3. LDPC codes from triangular and L_2-type designs

3.1. Some series of BIB designs

The following series of BIB designs may be found in Raghavarao (1971, pp. 77–78) for s being a prime or prime power:

Series 7: $v' = b' = s^2 + s + 1, r' = k' = s + 1, \lambda' = 1$.
Series 8: $v' = (s + 1)(s^2 + 1), b' = (s^2 + 1)(s^2 + s + 1), r' = s^2 + s + 1, k' = s + 1, \lambda' = 1$.
Series 9: $v' = s^2, b' = s(s + 1), r' = s + 1, k' = s, \lambda' = 1$.
Series 10: $v' = s^3, b' = s^2(s^2 + s + 1), r' = s^2 + s + 1, k' = s, \lambda' = 1$.

3.2. Some series of triangular designs

The series (10–12) of triangular designs given below may be found in Raghavarao (1971) and Dey (2010):

Series 11: For $v = (s(s - 1))/2; s \geq 5$ and block size $k = 2$, there exist triangular designs with parameters:
(i) $b = (s(s - 1)(s - 2))/8, r = 2(s - 2), \lambda_1 = 1, \lambda_2 = 0$.
(ii) $b = (s(s - 1)(s - 2)(s - 3))/8, r = ((s - 2)(s - 3))/8, \lambda_1 = 0, \lambda_2 = 1$.

Series 12: The existence of a triangular design with parameters $v = (2s - 1)s, b = (2s -$

$1)(2s-3), r = 2s-3, k = s, \lambda_1 = 0, \lambda_2 = 1$ implies the existence of another triangular design with parameters $v = (2s-1)(s-1), b = (2s-1)(2s-3), r = 2s-3, k = s-1, \lambda_1 = 0, \lambda_2 = 1$.

Series 13: The existence of a BIB design: $v' = s-1, b', r', k', \lambda = 1$ implies the existence of a triangular design with parameters: $v = (s(s-1))/2, b = sb', r = 2r', k = k', \lambda_1 = 1, \lambda_2 = 0$.

Then utilizing the series (7–10) of BIB designs respectively in Series 13, we obtain the following series (14–17) of triangular designs respectively:

Series 14: $v = ((s^2+s+1)(s^2+s+2))/2, b = 2v, r = 2(s+1), k = s+1, \lambda_1 = 1, \lambda_2 = 0$.

Series 15: $v = \{(s+1)(s^2+1)+1\}\{(s+1)(s^2+1)\}/2, b = \{(s+1)(s^2+1)+1\}\{(s^2+1)(s^2+s+1)\}, r = 2(s^2+s+1), k = s+1, \lambda_1 = 1, \lambda_2 = 0$.

Series 16: $v = (s^2(s^2+1))/2, b = s(s^2+1)(s+1), r = 2(s+1), k = s, \lambda_1 = 1, \lambda_2 = 0$.

Series 17: $v = (s^3(s^3+1))/2, b = s^2(s^3+1)(s^2+s+1), r = 2(s^2+s+1), k = s, \lambda_1 = 1, \lambda_2 = 0$.

3.3. Some series of L_2-type designs

Consider the following series of L_2-type design as given in Raghavarao (1971) and Dey (2010):

Series 18: The existence of a BIB design with parameters: $v' = s$, b', r', k', $\lambda = 1$ implies the existence of an L_2-type design with parameters: $v = s^2, b = 2sb'$, $r = 2r'$, $k = k'$, $\lambda_1 = 1$, $\lambda_2 = 0$.

Further applying the series (7–10) of BIB designs respectively in series 18, we obtain the following series (19–22) of L_2-type designs respectively:

Series 19: $v = (s^2+s+1)^2, b = 2v, r = 2(s+1), k = s+1, \lambda_1 = 1, \lambda_2 = 0$.

Series 20: $v = (s+1)^2(s^2+1)^2, b = 2(s+1)(s^2+1)^2(s^2+s+1), r = 2(s^2+s+1), k = s+1, \lambda_1 = 1, \lambda_2 = 0$.

Series 21: $v = s^4, b = 2s^3(s+1), r = 2(s+1), k = s, \lambda_1 = 1, \lambda_2 = 0$.

Series 22: $v = s^6, b = 2s^5(s^2+s+1), r = 2(s^2+s+1), k = s, \lambda_1 = 1, \lambda_2 = 0$.

Example 5: Using BIB design: $v = 4, b = 6, r = 3, k = 2, \lambda = 1$ in Series 13, we obtain a triangular design $T1 : v = 10, b = 30, r = 6, k = 2, \lambda_1 = 1, \lambda_2 = 0$ whose incidence matrix N is:

$$\begin{bmatrix}
1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0\\
1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0\\
0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&1&0&0&0&0&0&0&0&0&1\\
0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&1&0&0&0&0\\
0&0&0&0&1&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0\\
0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0\\
0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0
\end{bmatrix}$$

Since any two distinct columns (or rows) of N intersect in at most one element, N satisfies RC-constraint. Hence we obtain a (2,6)-regular LDPC code free of 4-cycles with code length 30 and code rate about 0.66.

Further using BIB design: $v = 4, b = 6, r = 3, k = 2, \lambda = 1$ in Series 18, we obtain an L_2-type design $LS3 : v = 16, b = 48, r = 6, k = 2, \lambda_1 = 1, \lambda_2 = 0$. This design may be used to obtain (2,6)-regular LDPC code free of 4-cycles with code length 48 and code rate about 0.66.

A correspondence between resolvable BIB/GD designs, triangular/ L_2-type designs and LDPC codes is given in Table 2 where p denotes the number of resolution classes:

Table 2: Resolvable BIB/GD designs, Triangular/ L_2-type designs and LDPC codes

No.	**Resolvable BIB designs**	**Code length**	**Code rate**
1	Series 1	$p(4m + 1)$	$1 - (3/p); 4 \leq p \leq 6m + 1$
2	Series 2	$p(6m + 1)$	$1 - (3/p); 4 \leq p \leq 9m + 1$
3	Series 3	$4sp$	$1 - (3/p)(4 \leq p \leq 4s)$
4	Series 4	$8sp$	$1 - (4/p)(5 \leq p \leq 8s)$
5	Series 5	$p(s^2 - 1)$	$1 - (s/p); (s + 1 \leq p \leq s^2 - 1)$
6	Series 6	pq^2	$1 - ((q - t))/p;$ q is a prime or prime power
7	Series 11 (i)	$\frac{s(s-1)(s-2)}{8}$	$\frac{(s-3)}{(s-2)}$
8	Series 11 (ii)	$\frac{s(s-1)(s-2)(s-3)}{8}$	$\frac{(s^2-5s-10)}{(s-2)(s-3)}$
9	Series 14	$(s^2 + s + 1)(s^2 + s + 2)$	0.5
10	Series 15	$\{(s + 1)(s^2 + 1) + 1\}\times$ $(s^2 + 1)(s^2 + s + 1)$	$\frac{(2s^2+s+1)}{2(s^2+s+1)}$
11	Series 16	$s(s^2 + 1)(s + 1)$	$\frac{(s+2)}{2(s+1)}$
12	Series 17	$s^2(s^3 + 1)(s^2 + s + 1)$	$\frac{(2s^2+s+2)}{2(s^2+s+1)}$
13	Series 19	$2(s^2 + s + 1)^2$	0.5
14	Series 20	$2(s + 1)(s^2 + 1)^2 \times (s^2 + s + 1)$	$\frac{(2s^2+s+1)}{2(s^2+s+1)}$
15	Series 21	$2s^3(s + 1)$	$\frac{(s+2)}{2(s+1)}$
16	Series 22	$2s^5(s^2 + s + 1)$	$\frac{(2s^2+s+2)}{2(s^2+s+1)}$

4. Discussion and conclusion

It is observed that LDPC codes obtained from series 1 of resolvable BIB designs have shorter code length than those obtained from series 2 for the same code rate. Hence the codes obtained from series1 are better than those obtained from series 2. Further by putting $p = 4s$ in series 3 we obtain LDPC codes with code length $16s^2$ and code rate $R_1 = 1-(3/4s)$ whereas by putting $p = 2s$ in series 4 we obtain LDPC codes with same code length $16s^2$ but different code rate $R_2 = 1 - (2/s) = 1 - (8/4s) < R_1$. Thus the codes obtained from series 3 are better than those obtained from series 4.

LDPC codes obtained from series 11(i) have shorter code length in comparison to the codes obtained from series 11 (ii) with better code rates if $s < 19$ as their differences is

$\frac{(s-3)}{(s-2)} - \frac{(s^2-5s-10)}{(s-2)(s-3)} = \frac{-(s-19)}{(s-2)(s-3)}$.

Further LPDC codes obtained from series (14-17) of triangular designs have shorter code length in comparison to the codes obtained from series (19-22) of L_2-type designs w.r.t. fixed code rate. Hence the codes obtained from series (14-17) of triangular designs are better than those obtained from series (19-22) of L_2-type designs respectively. For example consider LDPC codes obtained from series 14 and 19. The two codes have same code rate 0.5 but the differences of their code lengths is $2(s^2+s+1)^2-(s^2+s+1)(s^2+s+2) = s(s+1)(s^2+s+1) > 0$.

Similarly differences of code lengths between series 15 and 20 is $s(s^2+s+1) > 0$, series 16 and 21 is $s(s+1)(s^2-1) > 0$ and series 17 and 22 is $(s^2+s+1)(s^3-1) > 0$. Also the differences of code lengths obtained from series 15 and 17 is $\{(s+1)(s^2+1)+1\}(s^2+1)(s^2+s+1) - s^2(s^3+1)(s^2+s+1) > 0$ with almost same code. Hence series 17 yields better LDPC codes in comparison to series 15.

Apart from the above discussed designs, PBIB designs based on partial geometry may also be used in LDPC codes. For example the PBIB designs: PG2, PG5, PG6a from Clatworthy (1973) yield LDPC codes free of 4–cycles and positive code rates. Some LDPC codes with shorter code lengths and higher code rates from Table 2 are given below in Table 3:

Table 3: LPDC codes with shorter code length and higher code rate from Table 1

No.	Designs	Code length	Code rate
1	Series 1	$p(4m+1)$	$1-(3/p); 4 \leq p \leq 6m+1$
2	Series 3	$16s^2$	$1-(3/4s)$
3	Series 5	$p(s^2-1)$	$1-(s/p); (s+1 \leq p \leq s^2-1)$
4	Series 6	pq^2	$1-((q-t))/p$; q is a prime or prime power
5	Series 11 (i)	$\frac{s(s-1)(s-2)}{8}$	$\frac{(s-3)}{(s-2)}; s < 19$
6	Series 14	$(s^2+s+1)(s^2+s+2)$	0.5
7	Series 16	$s(s^2+1)(s+1)$	$\frac{(s+2)}{2(s+1)}$
8	Series 17	$s^2(s^3+1)(s^2+s+1)$	$\frac{(2s^2+s+2)}{2(s^2+s+1)}$

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