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Transmuted Sine - G Family of Distributions: Theory and Applications

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Abstract

In this paper, we introduced transmuted sine - G family and it's mathematical properties. Recently, the statistical relevance and applicability of trigonometric distributions got much attention among researchers for modeling various real-time phenomena. This paper contributes to a core area of statistics by investigating a new trigonometric family of probability distributions defined from the alliance of the families known as transmuted and sine - G family with the inspiring name of transmuted sine generating (TS - G) family. The characteristics of this new family are studied through analytical, graphical and numerical approaches. In addition, we observe the fact that the TS - G family can generate original, simple and pliant trigonometric models for statistical purposes. This fact is revealed with the special TS - G model based on the Weibull model and discussed maximum likelihood estimation with real time application.

Key words: Sine - G family; Transmuted family; Weibull distribution; Maximum likelihood estimation.

AMS Subject Classifications: 60E05, 62E10, 62F10

1. Introduction

Statistical distribution is very useful in describing and predicting real-world phenomena. Life time distributions are playing a vital role in many area of research such as economics, engineering, finance, medicine, biological science, amongst others. Further analyzing life time data are imperative. Recent developments focus on designing and generating new families of probability distributions that extend well-known probability distributions and at the same time provide great flexibility in modeling real time data. In recent years, the generalization of probability distributions has attracted many statisticians. For example, exponentiated family of distributions proposed by Gupta *et al.* (1999). The exponentiated family of probability distributions provides flexibility by adding one more parameter to the base distribution. Several classes of probability distributions have been introduced by adding one or more parameters to generate new family of probability distributions in the statistical literature. Examples of such families are the exp - G family by Gupta *et al.* (2001),

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Weibull - G family by Bourguignon *et al.* (2014), Topp - Leone generated (TL - G) family by Al-Shomrani *et al.* (2016), a new extended alpha power transformed - G by Ahmad *et al.* (2020), a new alpha power transformed - G by Elbatal *et al.* (2019), truncated inverted Kumaraswamy - G by Bantan *et al.* (2019) , type II general inverse exponential - G by Jamal *et al.* (2020), exponentiated truncated inverse Weibull - G by Almarashi *et al.* (2020) and type II power TL - G by Bantan *et al.* (2020). The quadratic rank transmutation map was introduced by Shaw *et al.* (2007) to generate new family of distributions. Recent studies have highlighted the statistical relevance and applicability of trigonometric distributions for modeling many phenomena. Kharazmi and Saadatinik (2016) introduced Hyperbolic Cosine - F (HCF) family and Sakthivel *et al.* (2020) proposed Hyperbolic Cosine Rayleigh distribution and studied some of it's mathematical properties with application to breaking stress of carbon fibers. Kumar *et al.* (2015) and Souza (2015) introduced the sine - G family with use of the sine function. This paper introduced a new family of distribution namely transmuted sine - G (TS - G) family. The characteristics of this family are studied through graphical and numerical approaches.

In this paper, we introduce a new probability distribution namely transmuted sine - G family and studied some of it's properties. In section 2, we present the transmuted probability models. Section 3 discuss sine - G family. In section 4, we propose some transmuted sine - G family of distributions. In section 5, present statistical properties of transmuted sine Weibull distribution. The reliability analysis of proposed model is discussed in section 6. The maximum likelihood estimation for the parameters of transmuted sine Weibull distribution is presented given in section 7. The application of transmuted sine Weibull (TS Weibull) distribution is studied using real data set in section 8 and conclusions of this work is presented in section 9.

2. Transmuted Distribution

The quadratic rank transmutation map introduced by Shaw *et al.* (2007). The cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$ are defined as follows:
The cdf of transmuted family of distribution is defined as

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x)\,; \qquad |\lambda| \leq 1 \tag{1}$$

The pdf of transmuted family of distribution is defined as

$$f(x) = g(x)[(1+\lambda) - 2\lambda G(x)]\,; \qquad |\lambda| \leq 1 \tag{2}$$

where λ is a parameter of transmutation; $G\,(x)$ is the cdf and $g\,(x)$ is the pdf of the baseline distribution respectively.

3. Sine - G Family

The method of generating new family of probability distributions using sine transformation was introduced by Kumar *et al.* (2015) and Souza (2015). The cdf and pdf of sine -

G family distribution can be obtained as follows:
The cdf of sine - G distribution is defined as

$$G(x) = \sin\left(\frac{\pi}{2}H(x)\right) \tag{3}$$

The pdf of sine - G distribution is defined as

$$g(x) = \frac{\pi}{2}h(x)\cos\left(\frac{\pi}{2}H(x)\right) \tag{4}$$

where $h(x)$ and $H(x)$ are pdf and cdf of baseline distribution respectively.

4. Proposed Model

4.1. Transmuted sine family

This paper contributes to the subject by investigating a new trigonometric family of probability distributions defined from the alliance of the families known as transmuted distribution and sine - G family and it is named as transmuted sine - G family (TS - G).
The cdf of transmuted sine - G family is defined as

$$F(x) = (1+\lambda)\sin\left(\frac{\pi}{2}H(x)\right) - \lambda\left[\sin\left(\frac{\pi}{2}H(x)\right)\right]^2 ; \qquad |\lambda| \leq 1 \tag{5}$$

The pdf of transmuted sine - G family is

$$f(x) = \frac{\pi}{2}h(x)\cos\left(\frac{\pi}{2}H(x)\right)\left[(1+\lambda) - 2\lambda\sin\left(\frac{\pi}{2}H(x)\right)\right] ; \qquad |\lambda| \leq 1 \tag{6}$$

where λ is a parameter of transmutation. The $h(x)$ and $H(x)$ are pdf and cdf of baseline distribution respectively.

4.2. Transmuted sine exponential distribution

The cdf of exponential distribution with parameter θ is

$$H(x) = 1 - e^{-\theta x} ; \quad x > 0 ; \theta > 0 \tag{7}$$

The pdf of exponential distribution with parameter θ is

$$h(x) = \theta e^{-\theta x} ; \quad x > 0 ; \theta > 0 \tag{8}$$

where θ is a rate parameter.
The cdf of transmuted sine exponential family is given by

$$F(x) = (1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\theta x}\right)\right) - \lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\theta x}\right)\right)\right]^2 ; \quad x > 0 ; |\lambda| \leq 1, \theta > 0 \tag{9}$$

The pdf of transmuted sine exponential family is given by

$$f(x) = \frac{\pi}{2}\left(\theta e^{-\theta x}\right)\cos\left(\frac{\pi}{2}\left(1-e^{-\theta x}\right)\right)\left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\theta x}\right)\right)\right]; \quad x>0\,;|\lambda|\leq 1, \theta>0 \tag{10}$$

where θ is a rate parameter and λ is parameter of transmutation.
The r^{th} moment is defined as

$$E\left(X^r\right) = \theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,r}-2\lambda\delta_{j,l,r}\right] \tag{11}$$

where (12)

$$\gamma_{k,r} = \sum_{k=0}^{2i}\binom{2i}{k}\frac{(-1)^k r!}{(\theta(k+1))^{r+1}}$$

and (13)

$$\delta_{j,l,r} = \sum_{j=0}^{\infty}\sum_{l=0}^{2i+2j+1}\left(\frac{\pi}{2}\right)^{2j+1}\binom{2i+2j+1}{l}\frac{(-1)^i(-1)^l}{(2j+1)!}\frac{r!}{(\theta(l+1))^{r+1}}$$

The moment generating function is defined as

$$M_X(t) = \theta\sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\frac{t^r}{r!}\left[(1+\lambda)\gamma_{k,r}-2\lambda\delta_{j,l,r}\right] \tag{14}$$

The first four moments are given below

$$E\left(X\right) = \theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,1}-2\lambda\delta_{j,l,1}\right] \tag{15}$$

$$E\left(X^2\right) = \theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,2}-2\lambda\delta_{j,l,2}\right] \tag{16}$$

$$E\left(X^3\right) = \theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,3}-2\lambda\delta_{j,l,3}\right] \tag{17}$$

$$E\left(X^4\right) = \theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,4}-2\lambda\delta_{j,l,4}\right] \tag{18}$$

The r^{th}moment of TS exponential distribution is expressed as series. One can easily verify the convergence of this series by using Cauchy ratio test.

4.3. Transmuted sine modified Weibull distribution

The modified Weibull distribution was introduced by Zaindin *et al.* (2009) and the cdf of modified Weibull distribution is given by

$$H(x) = \left(1-e^{-\alpha x-\beta x^{\theta}}\right)\;; \quad x>0\,;\alpha,\beta\geq 0,\theta>0 \tag{19}$$

such that $\alpha+\beta>0$. Here β and θ are shape parameters and α is a scale parameter. The pdf of modified Weibull distribution is given by

$$h(x)=\left(\alpha+\theta\beta x^{\theta-1}\right)e^{-\alpha x-\beta x^{\theta}}\;;\quad x>0\,;\alpha,\beta\geq 0,\theta>0 \tag{20}$$

The cdf of transmuted sine modified Weibull distribution is

$$\begin{aligned} F(x) &= (1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\alpha x-\beta x^{\theta}}\right)\right)-\lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\alpha x-\beta x^{\theta}}\right)\right)\right]^2; \\ &\qquad x>0\,;|\lambda|\leq 1,\alpha,\beta\geq 0,\theta>0 \end{aligned} \tag{21}$$

The pdf of transmuted sine modified Weibull distribution is defined as

$$\begin{aligned} f(x)=\frac{\pi}{2}&\left(\alpha+\theta\beta x^{\theta-1}\right)e^{-\alpha x-\beta x^{\theta}}\cos\left(\frac{\pi}{2}\left(1-e^{-\alpha x-\beta x^{\theta}}\right)\right) \\ &\left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\alpha x-\beta x^{\theta}}\right)\right)\right]; \\ &x>0\,;|\lambda|\leq 1,\alpha,\beta\geq 0,\theta>0 \end{aligned} \tag{22}$$

The r^{th} moment of TS modified Weibull distribution is defined as

$$E\left(X^r\right) = \sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,l,r}-2\lambda\delta_{j,m,n,r}\right] \tag{23}$$

where

$$\gamma_{k,l,r} = \sum_{k=0}^{2l}\sum_{l=0}^{\infty}\binom{2i}{k}\frac{(-1)^{k+l}(\beta k)^l}{l!}\left[\frac{(r+l\theta)!}{k\left(\alpha k\right)^{r+l\theta}}+\frac{\beta\theta\left(r+l\left(\theta+1\right)-1\right)}{\left(\alpha k\right)^{r+l(\theta+1)}}\right]$$

and

$$\begin{aligned}\delta_{j,m,n,r} = \sum_{j=0}^{\infty}\sum_{m=0}^{2i+2j+1}\sum_{n=0}^{\infty}\left(\frac{\pi}{2}\right)^{2j+1}\binom{2i+2j+1}{m}\frac{(-1)^{j+m+n}\left(\beta m\right)^n}{(2j+1)!} \\ \left[\frac{(r+n\theta)!}{m\left(\alpha m\right)^{r+n\theta}}+\frac{\beta\theta\left(r+n\theta\right)!}{\left(\alpha m\right)^{r+n(\theta+1)}}\right]\end{aligned}$$

The moment generation function of TS modified Weibull distribution is defined as

$$M_X(t) = \sum_{i=0}^{\infty}\sum_{r=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\frac{t^r}{r!}\left[(1+\lambda)\gamma_{k,l,r}-2\lambda\delta_{j,m,n,r}\right] \tag{24}$$

The first four moments are given below

$$E\left(X\right) = \sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,l,1}-2\lambda\delta_{j,m,n,1}\right] \tag{25}$$

$$E\left(X^2\right) = \sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,l,2}-2\lambda\delta_{j,m,n,2}\right] \tag{26}$$

$$E\left(X^3\right) = \sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,l,3}-2\lambda\delta_{j,m,n,3}\right] \tag{27}$$

$$E\left(X^4\right) = \sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,l,4}-2\lambda\delta_{j,m,n,4}\right] \tag{28}$$

4.4. Weibull distribution

The Weibull distribution was introduced by Swedish physicist Waloddi Weibull (1951). He applied it on modelling yield strength of materials. The Weibull distribution is popular and widely used in many fields such as engineering, reliability, failure analysis, lifetime analysis, material science, quality control, physics, medicine, meteorology, hydrology and others. However, there are cases when standard Weibull distribution fails to model data adequately enough for certain types of data. Hence, it is necessary to apply generalized Weibull distribution because of it's flexibility and suitability for such type of data. Later, the importance of this type of generalization has been proved in recent years on various problems.

The cdf of Weibull distribution is given by

$$H(x) = 1 - e^{-\eta x^{\theta}}\,; \quad x > 0\,; \eta, \theta > 0 \tag{29}$$

The pdf of Weibull distribution is given by

$$h(x) = \eta\theta x^{\theta-1} e^{-\eta x^{\theta}}\,; \quad x > 0\,; \eta, \theta > 0 \tag{30}$$

where θ is a shape parameter and η is a scale parameter.

4.5. Transmuted sine Weibull family

The cdf of transmuted sine Weibull (TS Weibull) distribution is given as

$$F(x) = (1+\lambda)\sin\left(\frac{\pi}{2}\left(1 - e^{-\eta x^{\theta}}\right)\right) - \lambda\left[\sin\left(\frac{\pi}{2}\left(1 - e^{-\eta x^{\theta}}\right)\right)\right]^2; \quad x > 0\,; \eta, \theta > 0, |\lambda| \le 1 \tag{31}$$

The pdf of transmuted sine Weibull (TS Weibull) distribution is given as

$$f(x) = \frac{\pi}{2}\eta\theta x^{\theta-1} e^{-\eta x^{\theta}} \cos\left(\frac{\pi}{2}\left(1 - e^{-\eta x^{\theta}}\right)\right) \left[(1+\lambda) - 2\lambda\sin\left(\frac{\pi}{2}\left(1 - e^{-\eta x^{\theta}}\right)\right)\right]; x > 0\,; \eta, \theta > 0, |\lambda| \le 1 \tag{32}$$

where θ is a shape parameter, η is a scale parameter and λ is transmuting parameter.

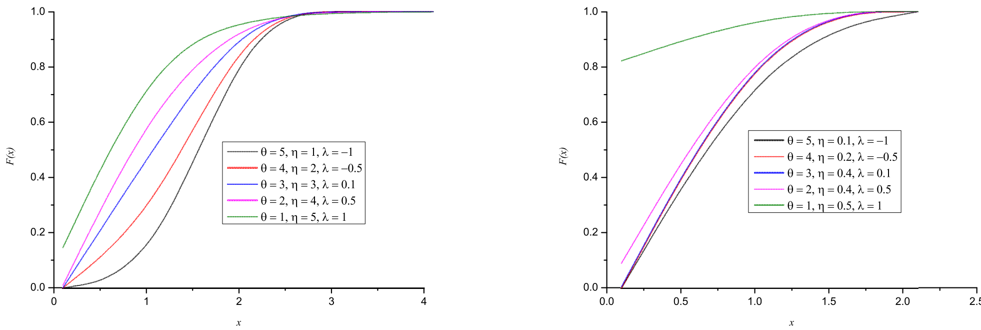

Figure 1: Plots for cdf of TS Weibull distribution for different values of the parameters.

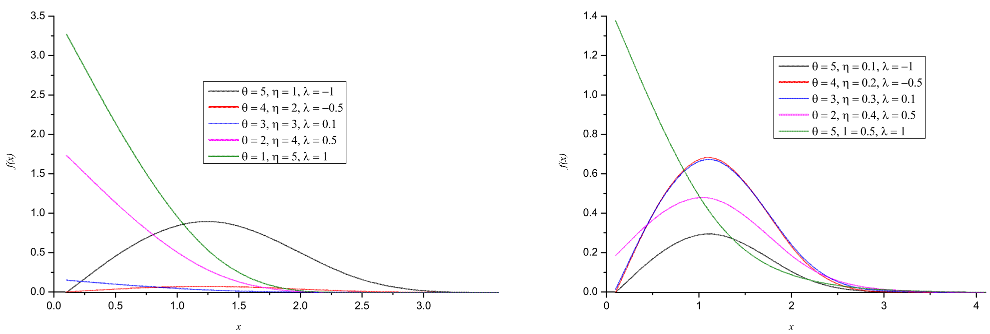

Figure 2: Plots for pdf of TS Weibull distribution for different values of the parameters.

5. Statistical Properties

5.1. Moments

We obtained an expression for the r^{th} moment of TS Weibull distribution as

$$E\left(X^r\right) = \eta\theta \sum_{i=0}^{\infty} \left(\frac{\pi}{2}\right)^{2i+1} \frac{(-1)^i}{(2i)!} \left[(1+\lambda)\gamma_{k,r} - 2\lambda\delta_{j,l,r,}\right] \tag{33}$$

where

$$\gamma_{k,r} = \sum_{k=0}^{2i} \left(\begin{array}{c} 2i \\ k \end{array} \right) \frac{(-1)^k}{\eta\theta(k+1)} \frac{\left(\frac{r}{\theta}\right)!}{(\eta(k+1))^{\frac{r}{\theta}}}$$

and

$$\delta_{j,l,r} = \sum_{j=0}^{\infty} \sum_{l=0}^{2i+2j+1} \frac{(-1)^j(-1)^l}{(2j+1)!} \left(\begin{array}{c} 2i+2j+1 \\ l \end{array} \right) \left(\frac{\pi}{2}\right)^{2j+1}$$

The first four moments are given below

$$E(X) = \eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,1}-2\lambda\delta_{j,l,1}\right] \tag{34}$$

$$E\left(X^2\right) = \eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,2}-2\lambda\delta_{j,l,2}\right] \tag{35}$$

$$E\left(X^3\right) = \eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,3}-2\lambda\delta_{j,l,3}\right] \tag{36}$$

$$E\left(X^4\right) = \eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,4}-2\lambda\delta_{j,l,4}\right] \tag{37}$$

The r^{th}moment of TS Weibull distribution is expressed as series. One can easily verify the convergence of this series by using Cauchy ratio test.

Variance

$$\begin{aligned} V(X) = {} & \eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,2}-2\lambda\delta_{j,l,2}\right] \\ & -\left[\eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,1}-2\lambda\delta_{j,l,1}\right]\right]^2 \end{aligned} \tag{38}$$

The moment generating function of TS Weibull distribution is given below

$$M_X(t) = \sum_{r=0}^{\infty}\frac{t^r}{r!}\eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,1}-2\lambda\delta_{j,l,r}\right] \tag{39}$$

The cumulant generating function of TS Weibull distribution is given below

$$K_X(t) = \log\left[\sum_{t=0}^{\infty}\frac{t^t}{r!}\eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,1}-2\lambda\delta_{\mathrm{j},l,r}\right]\right] \tag{40}$$

The characteristic function of TS Weibull distribution is given below

$$\phi_X(t) = \sum_{r=0}^{\infty}\frac{(it)^r}{r!}\eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,1}-2\lambda\delta_{j,l,r}\right] \tag{41}$$

5.2. Quantile function

The quantile function of TS Weibull distribution is given by

$$Q(p) = \left[\frac{\begin{array}{c}\lambda\sum_{i=0}^{\infty}\sum_{k=0}^{4i+2}\frac{(-1)^{i+k}}{(2i+1)!}\left(\frac{\pi}{2}\right)^{4i+2}\binom{4i+2}{k}(\eta k)\\ -(1+\lambda)\sum_{i=0}^{\infty}\sum_{k=0}^{2i+1}\frac{(-1)^{i+k}}{(2i+1)!}\left(\frac{\pi}{2}\right)^{2i+1}\binom{2i+1}{k}(\eta k)\end{array}}{(\log p)}\right]^{1/\theta} \tag{42}$$

5.3. Generalized entropy

The generalized entropy of TS Weibull distribution is given below

$$GE(w,\psi) = \frac{\eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,\psi}-2\lambda\delta_{j,l,\psi}\right]}{\psi(\psi-1)\left[\eta\theta\sum_{i=0}^{\infty}\left(\frac{\pi}{2}\right)^{2i+1}\frac{(-1)^i}{(2i)!}\left[(1+\lambda)\gamma_{k,1}-2\lambda\delta_{j,l,1]}\right]\right]^{\psi}} - 1 \tag{43}$$

5.4. Asymptotic behaviours

The asymptotic behaviours of transmuted sine Weibull distribution is given below

$$\begin{aligned}
\lim_{x\to 0} f(x:\eta,\theta,\lambda) &= 0 \text{ and } \lim_{x\to\infty} f(x:\eta,\theta,\lambda) = 0 \\
\lim_{x\to 0} f(x:\eta,\theta,\lambda) &= \frac{\pi}{2}\eta\theta\lim_{x\to 0} x^{\theta-1}e^{-\eta x^{\theta}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right) \\
&\quad \times\left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right] \\
&= \frac{\pi}{2}\eta\theta\times(0)=0 \\
\Rightarrow \lim_{x\to 0} f(x:\eta,\theta,\lambda) &= 0
\end{aligned} \tag{44}$$

$$\begin{aligned}
\lim_{x\to\infty} f(x:\eta,\theta,\lambda) &= \frac{\pi}{2}\eta\theta\lim_{x\to\infty} x^{\theta-1}\lim_{x\to\infty} e^{-\eta x^{\theta}} \\
&\quad \times\lim_{x\to\infty}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right) \\
&\quad \times\lim_{x\to\infty}\left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right] \\
&= \frac{\pi}{2}\eta\theta\times(0)=0 \\
\Rightarrow \lim_{x\to\infty} f(x:\eta,\theta,\lambda) &= 0
\end{aligned} \tag{45}$$

5.5. Order statistics

The pdf of j^{th} order statistic of TS Weibull distribution is given by

$$\begin{aligned}
f_{X_{(j)}}(x) = {} & \frac{n!}{(j-1)!(n-j)!}\frac{\pi}{2}\eta\theta x^{\theta-1}e^{-\eta x^{\theta}} \\
& \cos\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right) \\
& \left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right] \\
& \left[(1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)-\lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right]^2\right]^{j-1} \\
& \left[1-\left[(1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)-\lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right]^2\right]^{n-j}\right]
\end{aligned}$$

The pdf of largest order statistic $X_{(n)}$ is given by

$$\begin{aligned} f_{X_{(n)}}(x) &= n\frac{\pi}{2}\eta\theta x^{\theta-1}e^{-\eta x^{\theta}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right) \\ &\left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right] \\ &\left[(1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)-\lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right]^{2}\right]^{n-1} \end{aligned}$$

The pdf of smallest order statistic $X_{(1)}$ is given by

$$\begin{aligned} f_{X_{(1)}}(x) &= n\frac{\pi}{2}\eta\theta x^{\theta-1}e^{-\eta x^{\theta}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right] \\ &\left[1-\left[(1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)-\lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x^{\theta}}\right)\right)\right]^{2}\right]\right]^{n-1} \end{aligned}$$

5.6. Stochastic ordering

Stochastic ordering of positive continuous random variables are an important tool for judging their comparative behaviour. A random variable X is said to be smaller than a random variable Y then
(1) Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x
(2) Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x
(3) Mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \leq m_Y(x)$ for all x
(4) Likelihood ratio order ($X \leq_{lr} Y$) if $f_X(x)/f_Y(x)$ decreases in x.

The following implications based on these properties are illustrated by Yadav *et al.* (2019) and Shaked *et al.* (1995).

$$(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Rightarrow (X \leq_{mrl} Y)$$

and hence

$$(X \leq_{hr} Y) \Rightarrow (X \leq_{st} Y)$$

The following theorem shows that the TS Weibull random variable is ordered with respect to the strongest likelihood ratio ordering.

Theorem 1:

Let $X \sim TSW(\eta_1, \theta_1, \lambda_1)$ and $Y \sim TSW(\eta_2, \theta_2, \lambda_2)$ the following under conditions
(i) $\eta_1 = \eta_2, \lambda_1 = \lambda_2$ and $\theta_1 > \theta_2$
(ii) $\eta_1 = \eta_2, \lambda_1 > \lambda_2$ and $\theta_1 = \theta_2$
(iii) $\eta_1 > \eta_2, \lambda_1 = \lambda_2$ and $\theta_1 = \theta_2$
then ($X \leq_{lr} Y$) and hence it implies other ordering.

Proof:

$$\frac{f_X(x)}{f_Y(y)}=\frac{\frac{\pi}{2}\eta_1\theta_1 x^{\theta_1-1}e^{-\eta x^{\theta_1}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_1}}\right)\right)\left[(1+\lambda_1)-2\lambda_1\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_1}}\right)\right)\right]}{\frac{\pi}{2}\eta_2\theta_2 x^{\theta_2-1}e^{-\eta_2 x^{\theta_2}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{\theta_2}}\right)\right)\left[(1+\lambda_2)-2\lambda_2\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{\theta_2}}\right)\right)\right]}$$

Taking logarithm of both sides, we can write

$$\log\frac{f_X(x)}{f_Y(y)}=\log\left[\frac{\frac{\pi}{2}\eta_1\theta_1 x^{\theta_1-1}e^{-\eta x^{\theta_1}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_1}}\right)\right)\left[(1+\lambda_1)-2\lambda_1\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_1}}\right)\right)\right]}{\frac{\pi}{2}\eta_2\theta_2 x^{\theta_2-1}e^{-\eta_2 x^{\theta_2}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{\theta_2}}\right)\right)\left[(1+\lambda_2)-2\lambda_2\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{\theta_2}}\right)\right)\right]}\right].$$

Taking partial derivative on both sides, we write

$$\begin{aligned}\frac{d}{dx}\log\frac{f_X(x)}{f_Y(y)} &= \frac{(\theta_1-1)\,x^{\theta_1-2}}{x^{\theta_1-1}}-\frac{(\theta_2-1)\,x^{\theta_2-2}}{x^{\theta_2-1}}-\eta_1\theta_1 x^{\theta_1-1}+\eta_2\theta_2 x^{\theta_2-1}\\ &-\frac{\frac{\pi}{2}\eta_1\theta_1 x^{\theta_1-1}e^{-\eta x^{\theta_1}}\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_1}}\right)\right)}{\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_2}}\right)\right)}\\ &+\frac{2\lambda_1\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_1}}\right)\right)\frac{\pi}{2}\eta_1\theta_1 x^{\theta_1-1}e^{-\eta x^{\theta_1}}}{1+\lambda_1-2\lambda_1\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_1 x^{\theta_1}}\right)\right)}\\ &+\frac{\frac{\pi}{2}\eta_2\theta_2 x^{\theta_2-1}e^{-\eta_2 x^{\theta_2}}\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{2}}\right)\right)}{\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{\theta_2}}\right)\right)}\\ &+\frac{2\lambda_2\cos\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{\theta_2}}\right)\right)\frac{\pi}{2}\eta_2\theta_2 x^{\theta_2-1}e^{-\eta_2 x^{\theta_2}}}{1+\lambda_2-2\lambda_2\sin\left(\frac{\pi}{2}\left(1-e^{-\eta_2 x^{\theta_2}}\right)\right)}\end{aligned}$$

It can be easily verified that under conditions (i), (ii) and (iii) then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(y)}<0$. This means that $(X\leq_{lr}Y)$ and hence $(X\leq_{hr}Y),(X\leq_{mrl}Y)$ and $(X\leq_{st}Y)$.

6. Reliability Analysis

The survival function of TS Weibull distribution is defined as

$$R(t) = 1-\left[(1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\eta t^{\theta}}\right)\right)-\lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\eta t^{\theta}}\right)\right)\right]^2\right] \tag{46}$$

The hazard rate function is

$$h(t) = \frac{\frac{\pi}{2}\eta\theta t^{\theta-1}e^{-\eta t^{\theta}}\cos\left(\frac{\pi}{2}\left(1-e^{-\eta t^{\theta}}\right)\right)\left[(1+\lambda)-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta t^{\theta}}\right)\right)\right]}{1-\left[(1+\lambda)\sin\left(\frac{\pi}{2}\left(1-e^{-\eta t^{\theta}}\right)\right)-\lambda\left[\sin\left(\frac{\pi}{2}\left(1-e^{-\eta t^{\theta}}\right)\right)\right]^2\right]} \tag{47}$$

(a) $\theta=2,\eta=1$ and $\lambda=1$ hazard shape is linear.
(b) $\theta=1,\eta=1$ and $\lambda=1$ hazard shape is increasing decreasing increasing function.
(c) $\theta=0.5,\eta=1$ and $\lambda=-1$ hazard shape is decreasing function.
(d) $\theta=0.5,\eta=1$ and $\lambda=1$ hazard shape is increasing function.
(e) $\theta=1.2,\eta=1$ and $\lambda=-1$ hazard shape is inverse bathtab function.
(f) $\theta=1.2,\eta=1.5$ and $\lambda=-1$ hazard shape is unimodal function.

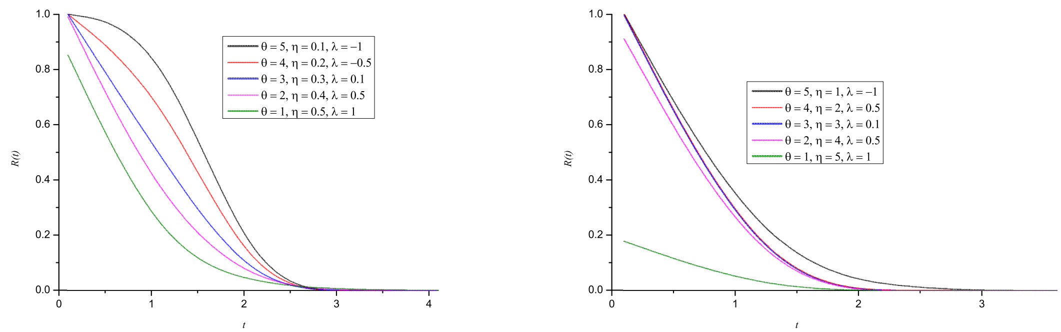

Figure 3: Plots for reliability function of TS Weibull distribution for different values of the parameters.

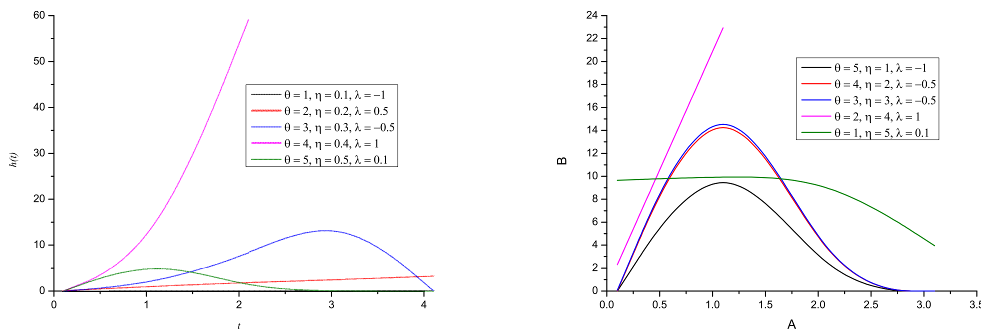

Figure 4: Plots for hazard function of TS Weibull distribution for different values of the parameters.

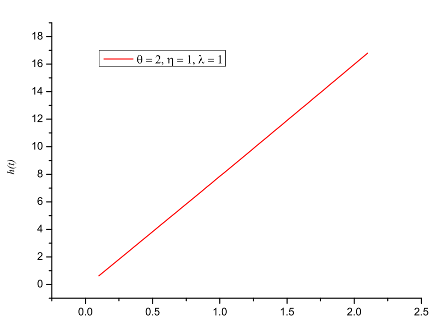

Figure 5: (a)

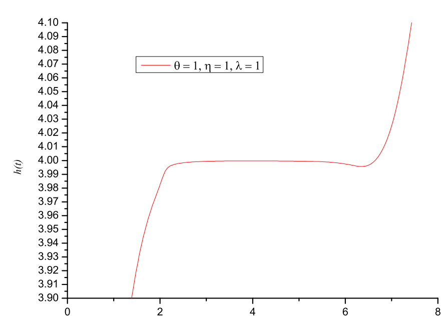

Figure 6: (b)

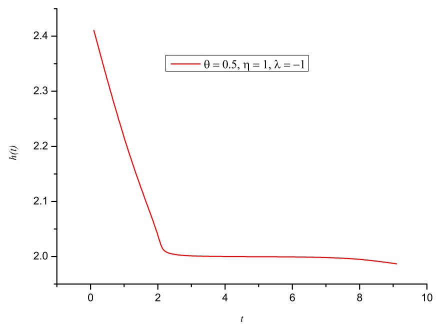

Figure 7: (c)

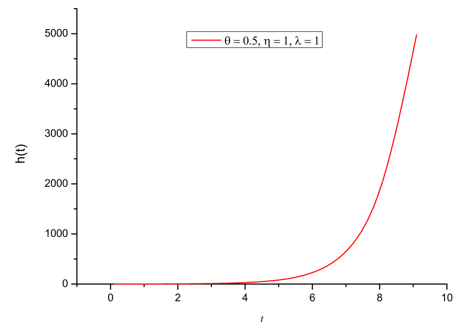

Figure 8: (d)

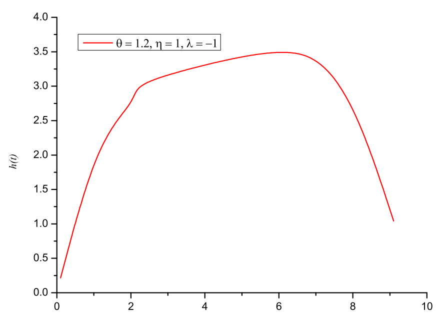

Figure 9: (e)

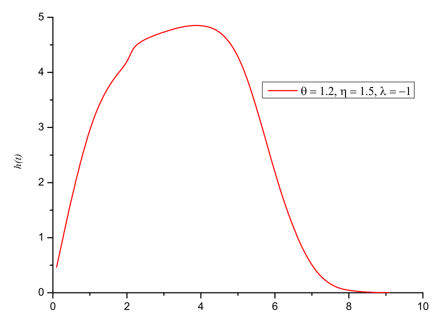

Figure 10: (f)

The above figures 5 - 10 of hazard function of TS Weibull distribution for different values of the parameters takes shapes as (a) linear, (b) increasing decreasing increasing (IDI), (c) decreasing, (d)increasing, (e) inverse bathtub and (f) unimodal shapes respectively.

7. Maximum Likelihood Estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from transmuted sine Weibull distribution. The likelihood function is given by

$$\begin{aligned} L(x:\eta,\theta,\lambda) &= \prod_{i=1}^{n} \frac{\pi}{2}\eta\theta x_i^{\theta-1} e^{-\eta x_i^\theta} \cos\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right) \\ &\quad \left[(1+\lambda) - 2\lambda \sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right] \end{aligned} \tag{48}$$

$$\begin{aligned} l(x:\eta,\theta,\lambda) &= \log\left(\frac{\pi}{2}\right) + \log(\eta) + \log(\theta) + (\theta-1)\sum_{i=1}^{n} x_i - \eta \sum_{i=1}^{n} x_i^\theta \\ &\quad + \sum_{i=1}^{n} \log\left[\cos\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right] \\ &\quad + \sum_{i=1}^{n} \log\left[(1+\lambda) - 2\lambda \sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right] \end{aligned} \tag{49}$$

$$\begin{aligned} \frac{\partial l(x:\eta,\theta,\lambda)}{\partial \eta} =& \frac{1}{\eta} - \sum_{i=1}^{n} x_i^\theta - \sum_{i=1}^{n} \frac{\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\left(\frac{\pi}{2}\right)\right) x_i^\theta e^{-\eta x_i^\theta}}{cos\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)} \\ & - \sum_{i=1}^{n} \frac{2\lambda \cos\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\left(\frac{\pi}{2}\right) x_i^\theta e^{-\eta x_i^\theta}}{1+\lambda-2\lambda\ sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)} \end{aligned} \tag{50}$$

$$\begin{aligned} \frac{\partial l(x:\eta,\theta,\lambda)}{\partial \theta} =& \frac{1}{\theta} + \sum_{i=1}^{n} x_i - \eta \sum_{i=1}^{n} x_i^\theta log x_i - \sum_{i=1}^{n} \frac{\frac{\pi}{2}\eta x_i^\theta log x_i\ sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)}{cos\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)} \\ & \times - \sum_{i=1}^{n} \frac{2\lambda(\frac{\pi}{2}) e^{-\eta x_i^\theta} \eta x_i^\theta log x_i\ cos\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)}{1+\lambda-2\lambda\ sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)} \end{aligned} \tag{51}$$

$$\frac{\partial l(x:\eta,\theta,\lambda)}{\partial \lambda} = \sum_{i=1}^{n} \frac{1-2\ sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)}{1+\lambda-2\lambda\ sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)} \tag{52}$$

The maximum likelihood estimate $\hat{\Theta} = (\hat{\eta}, \hat{\theta}, \hat{\lambda})$ of $\Theta = (\eta, \theta, \lambda)$ Also as $n \to \infty$ the asymptotic distribution of the MLEs (η, θ, λ) are given by, see Rahman *et al.* (2018) and Zaindin *et al.* (2009).

$$\begin{pmatrix} \hat{\eta} \\ \hat{\theta} \\ \hat{\lambda} \end{pmatrix} \sim N\left[\begin{pmatrix} \eta \\ \theta \\ \lambda \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}\right]$$

Where $\hat{V_{ij}} = V_{ij}$ the asymptotic variance-covariance matrix V of the estimates is obtained by inverting Hessian matrix. See Appendix. An approximate $100\,(1-\alpha)\,\%$ two sided confidence intervals for η, θ and λ are respectively is given by

$$\eta \in \left[\hat{\eta} - Z_{\frac{\alpha}{2}}\sqrt{V_{11}^{-1}}, \hat{\eta} + Z_{\frac{\alpha}{2}}\sqrt{V_{11}^{-1}}\right] \tag{53}$$

$$\theta \in \left[\hat{\theta} - Z_{\frac{\alpha}{2}}\sqrt{V_{22}^{-1}}, \hat{\theta} + Z_{\frac{\alpha}{2}}\sqrt{V_{22}^{-1}}\right] \tag{54}$$

$$\lambda \in \left[\hat{\lambda} - Z_{\frac{\alpha}{2}}\sqrt{V_{33}^{-1}}, \hat{\lambda} + Z_{\frac{\alpha}{2}}\sqrt{V_{33}^{-1}}\right] \tag{55}$$

Where Z_α is the α^{th} percentile of the standard normal distribution.

7.1. Simulation study

In this section, a simulation study is performed to test the performance of MLEs. We generate a random sample of size $n = 50, 100, 200$ and 500 from TS Weibull distribution for the values of $\theta = 2, \eta = 4$ and $\lambda = 0.5$. With replicated 1000 times. It is observed that the mean squared error (MSE) and average bias decreases when the sample size increases. Therefore, the maximum likelihood estimate converges to true value of the parameters of TS Weibull distribution.

Table 1: The MSE and average bias for the above given values of parameters

n	θ	η	λ
50	0.0138	0.0805	0.0223
	0.0185	0.0364	0.0194
100	0.0064	0.0390	0.0111
	0.0092	0.0193	0.0103
200	0.0032	0.0194	0.0055
	0.0054	0.0083	0.0044
500	0.0016	0.0097	0.0028
	0.0023	0.0047	0.0025

8. Application

The following data represents lifetimes of Kevlar 49/epoxy strands subjected to constant sustained pressure at 90 percent stress level until the strand failure studied by Barlow *et al.* (1984) and Pobocikova *et al.* (2018).
0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Table 2: Descriptive statistics

Min	Max	Mean	Vari	Lower Quantile	Median	Upper Quantile	Skewness	Kurtosis
0.0251	**9.0960**	**1.9592**	**2.4774**	**0.8982**	**1.7362**	**2.3041**	**2.0196**	**5.6004**

Table 3: ML estimates of the parameters with measures for model selection

Distribution	Parameter estimates	Log-lik	AIC	AICC	BIC
$\mathrm{W}(a,b)$	$\hat{a}$=1.3256, $\hat{b}$=2.1328	-122.5247	249.0494	249.2094	253.7108
$\mathrm{W}(a,b,c)$	$\hat{a}$=1.3169, $\hat{b}$=2.1228, $\hat{c}$=0.0058	-122.5141	251.0282	251.3525	258.0204
$\mathrm{W}(a,b,\lambda)$	$\hat{a}$=1.0509, $\hat{b}$=1.4419, $\hat{\lambda}$=-0.7955	-121.4300	248.8600	249.1843	255.8522
$\mathrm{SW}(\eta,\lambda)$	$\hat{\eta}$=1.2564257, $\hat{\lambda}$=0.2210405	-122.4717	248.9434	249.1078	253.6049
$\mathrm{TSW}(\eta,\theta,\lambda)$	$\hat{\eta}$=0.40025, $\hat{\theta}$=0.98819, $\hat{\lambda}$=-0.80737	-121.2775	242.5552	248.5552	255.5472

In Table 2, we observed that the empirical distribution is right skewed. Table 3 presents the ML estimates of the parameters along with the log-likelihood, AIC, AICC and BIC values. Further, we observed that the TS Weibull distribution provides better fits compared to 2-parameter Weibull, 3-parameter Weibull, transmuted Weibull distributions and sine-Weibull distribution. Hence, we can conclude that the TS Weibull distribution provides a better fit to the data than the other four suitable probability distributions.

9. Conclusion

In the present study, we have introduced transmuted sine Weibull distribution and we have derived some statistical properties such as moments, mean and variance for the proposed distribution. The behaviour of the pdf, cdf, reliability function and hazard function are explained through the graphical methods. We have discussed certain statistical properties like generalized entropy, asymptotic behaviour, order statistics and stochastic ordering for TS Weibull distribution. The parameter estimation is performed using the maximum likelihood method for the proposed distribution. Finally, we have shown TS Weibull distribution provides a better fit compared to 2-parameter Weibull, 3-parameter Weibull, transmuted Weibull and sine - Weibull distribution for real time data set.

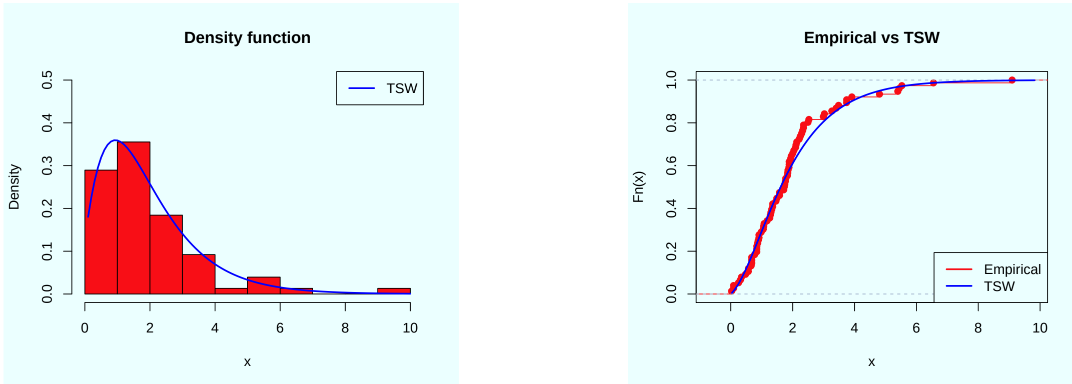

Figure 11: Plot of the empirical pdf and cdf of TS Weibull distribution.

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APPENDIX

The Hessian matrix is given as

$$H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}$$

Where the variance - covariance matrix V is obtained by

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}^{-1}$$

$$H_{11} = E\left[-\frac{\partial logL}{\partial \eta^2}\right], H_{22} = E\left[-\frac{\partial logL}{\partial \theta^2}\right], H_{33} = E\left[-\frac{\partial logL}{\partial \lambda^2}\right],$$

$$H_{12} = H_{21} = E\left[-\frac{\partial^2 logL}{\partial \eta \partial \theta}\right], H_{13} = H_{31} = E\left[-\frac{\partial^2 logL}{\partial \eta \partial \lambda}\right], H_{23} = H_{32} = E\left[-\frac{\partial^2 logL}{\partial \theta \partial \lambda}\right].$$

$$H_{11} = \frac{1}{\eta^2} + \sum_{i=1}^{n} x_i^2 \left(\frac{\pi}{2}\right) e^{\eta x_i^\theta} \left(x_i^\theta\right) \left[\frac{A+B}{\left(e^{-\eta x_i^\theta}\right)}\right]$$

$$\begin{aligned} A &= \cos\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right) \\ &\quad \left(\cos\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right) + \sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right) \end{aligned}$$

$$B = \left(\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right)^2 \left(\frac{\pi}{2}\right)\left(1-e^{-\eta x_i^\theta}\right)$$

$$\begin{aligned} H_{22} = &= \frac{1}{\theta^2}\eta \sum_{i=1}^{n} x_i^\theta (\log x_i)^2 + \sum_{i=1}^{n}\left(\frac{\pi}{2}\right)\eta\left(\log x_i\right)^2 x_i^\theta \left[\frac{C}{\left(\cos\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\right)^2}\right] \\ &\quad - \sum_{i=1}^{n} 2\lambda \log x_i e^{-\eta x_i^\theta} x_i^\theta \log x_i \left[\frac{D}{\left(1+\lambda-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right)^2}\right] \end{aligned}$$

$$\begin{aligned} C = \cos\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)&\left[\eta x_i^\theta e^{-\eta x_i^\theta}\cos\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right) + \sin\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\right] \\ + x_i^\theta \sin\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)^2 \frac{\pi}{2} e^{\eta x_i^\theta} \end{aligned}$$

$$D = \left(1+\lambda-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\right)$$
$$\left[\left(\cos\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)-x_i\eta\right)-\eta x_i^\theta\frac{\pi}{2}e^{-\eta x_i^\theta}\sin\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\right]$$
$$+2\lambda\left(\eta x_i^\theta\frac{\pi}{2}e^{-\eta x_i^\theta}\sin\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\right)^2 e^{\eta x_i^\theta}x_i^\theta$$

$$H_{33} = \sum_{i=1}^{n}\frac{\left(1-2sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\right)^2}{\left(1+\lambda-2\lambda sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\right)^2}$$

$$H_{21} = H_{12} = \sum_{i=1}^{n}x_i^\theta\log x_i+\sum_{i=1}^{n}\left[\frac{E}{\left(\cos\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\right)^2}\right]$$
$$-\sum_{i=1}^{n}\left[\frac{F}{\left(1+\lambda-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right)^2}\right]$$

$$E = \left(\frac{\pi}{2}x_i^\theta\log x_i\right)\left[\left(\cos\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right)^2\right]\left[\frac{\pi}{2}\eta e^{\eta x_i^\theta}+1\right]$$
$$+n\left(\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right)^2 e^{-\eta x_i^\theta x_i^\theta}$$

$$F = \left(\left(\frac{\pi}{2}e^{-\eta x_i^\theta\log x_i}\right)\right)\left[-\left(1+\lambda-2\lambda\sin\left(\frac{\pi}{2}\left(1-e^{-\eta x_i^\theta}\right)\right)\right)\right]$$
$$\left[2\lambda\eta\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\left(\frac{\pi}{2}\right)e^{-\eta x_i^\theta}\right]$$
$$+\left[2\lambda\cos\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)\left(1-\eta x_i^\theta\right)\right]$$
$$\left[2\lambda\cos\left(\frac{\pi}{2}\left(1-e^{-nx_i^\theta}\right)\right)^2\eta\left(\frac{\pi}{2}e^{-\eta x_i^\theta}x_i^\theta\right)\right]$$

$$H_{13} = \sum_{i=1}^{n}\left[\frac{G-H}{\left(\left(1+\lambda-2\lambda sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\right)\right)^2}\right]$$

$$G = \left(1+\lambda-2\lambda sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\right)\left(2x_i^\theta(\frac{\pi}{2}e^{-\eta x_i^\theta}cos\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\right)$$

$$H = 2\lambda x_i^\theta(\frac{\pi}{2})e^{-\eta x_i^\theta}cos\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\left(1-2sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\right)$$

$$H_{23} = H_{32} = \left[\sum_{i=1}^{n}\frac{2cos\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)(\frac{\pi}{2})e^{-\eta x_i^\theta logx_i}}{\left(1+\lambda-2\lambda sin\left(\frac{\pi}{2}(1-e^{-\eta x_i^\theta})\right)\right)^2}\right]$$