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Stress-Strength Reliability Analysis of Power Function and Nakagami Distributions using Comparative Sampling

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Abstract

In this article, we examine the probability of disaster in a stress-strength model where item strength follows the power function distribution and stress follows the Nakagami distribution. We compare simple random sampling (SRS) and ranked set sampling (RSS) approaches to assess their efficiency and accuracy in this context. Our study contributes to stress-strength modeling literature by introducing a novel application of ranked set sampling with Nakagami and Power function distributions. We also performed cost function optimization in this context.

Key words: Nakagami distribution; Power function distribution; Stress-strength model; Simple random sampling; Ranked set sampling.

AMS Subject Classifications: 62K05, 05B05

The video recording of the paper made under the SSCA's Online Lecture series is available at the Youtube channel URL `https://youtu.be/MRNbU0_F9mI`.

1. Introduction

The stress-strength paradigm stands as an essential methodology in reliability analysis, represented as $P = Pr(Y > X)$, where X and Y symbolize the stress and strength variables, respectively. This model is instrumental in evaluating the probability of system survival under varying operational loads. Another formulation articulates reliability as $P = Pr(X > \theta)$, where θ represents the upper threshold of the strength distribution. This metric is valuable in scenarios where system failure is contingent upon exceeding a critical strength limit.

In this research article, we investigate the reliability dynamics of a stress-strength model by examining the probability of failure through the measure $P = Pr(Y > X)$, where the stress follows the Nakagami and strength follows the Power function distributions respectively. Here, strength is taken as Power function distribution to justify the fact that

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the strength of the items is always finite. The related probability's of disaster, that occurs when $P(X > \theta)$ is studied. Our methodology encompasses both SRS and RSS approaches to derive reliability estimators, followed by a comprehensive comparative analysis of their statistical attributes, with particular emphasis on bias and mean squared error evaluation. Through rigorous mathematical formulation, we demonstrate the superior performance of RSS over conventional SRS in reliability estimation, particularly highlighting its enhanced precision and operational efficiency.

Furthermore, we develop an optimization framework for a linear cost function based on the established parametric relationships, yielding practical implications for manufacturing processes and quality control protocols. This investigation extends the existing stress-strength modeling literature by presenting an innovative implementation of ranked set sampling methodology in conjunction with Nakagami and Power function distributions. Our analytical framework offers new perspectives on reliability estimation in complex systems, potentially advancing the field of reliability engineering and quality assessment.

In the literature, early contributions to the field include the foundational studies by Basu (1964) and Barlow and Proschan (1967), which laid the groundwork for many subsequent investigations. The 1970's saw significant advancements, with notable works by Church and Harris (1970), Enis and Geisser (1971), Downton (1973), and Tong and *et al.* (1974) expanding the theoretical framework and practical applications of reliability analysis.

Further developments emerged in the late 1970's and early 1980's, with key contributions from Kelley *et al.* (1976), Sinha and Kale (1980), Sathe and Shah (1981), and Chao (1982). These researchers explored various aspects of stress-strength models and their statistical properties.

The field continued to evolve through the 1980's and 1990's, with important works by Awad and Gharraf (1986), Constantine *et al.* (1986), and Bain (2017). These studies further refined the methodologies and expanded the scope of reliability analysis.

More recent contributions, such as those by Chaturvedi and Sharma (2007), Kumar and Vaish (2014), and Kumar *et al.* (2020) have built upon this strong foundation, introducing new perspectives and addressing contemporary challenges in reliability engineering. This diverse body of literature reflects the ongoing importance and evolution of stress-strength modeling and reliability analysis in various engineering and statistical applications.

The seminal work of Meniconi and Barry (1996) provided compelling evidence for the efficacy of Power function distribution in electrical reliability analysis. Their comparative study, which evaluated multiple probability models including Exponential, Log-normal, and Weibull distributions, demonstrated through reliability measures and hazard function analysis that the Power function distribution offers superior modeling capabilities for electrical component reliability assessment.

This research presents a detailed exploration of stress-strength reliability analysis. The theoretical foundation is established in Section 3, where expressions for the probability of failure are formulated. In Section 4, the stress-strength reliability measure $P = Pr(Y > X)$ is derived for Power function and Nakagami distributions. Section 5 employs the maximum likelihood estimation method to estimate $P = Pr(Y > X)$ for both SRS and RSS. A simulation study is conducted in Section 6 to numerically validate the theoretical results.

Section 7 provides a comprehensive discussion of the findings. In Section 8, an illustrative example is presented to optimize the cost function for RSS in comparison to SRS. The article concludes with Section 9, summarizing the key insights and contributions of the study.

2. Preliminary

The mathematical formulation of the Nakagami distribution (NAD) is defined as follows:

$$f(x,\,\xi,\,\psi)=\frac{2}{\Gamma(\xi)}\left(\frac{\xi}{\psi}\right)^{\xi}x^{2\xi-1}\exp\left(-\frac{\xi}{\psi}\,x^{2}\right);\quad x>0,\psi>0,\xi>0.5 \tag{1}$$

$$F(x,\,\xi,\,\psi)=\frac{\gamma\left(\xi,\,\frac{\xi}{\psi}\,x^{2}\right)}{\Gamma(\xi)};\quad x>0,\psi>0,\xi>0.5 \tag{2}$$

where, $f(.)$ and $F(.)$ denotes the probability density function(pdf) and cumulative distribution function(cdf) respectively and $\gamma(x,a)$ is lower incomplete gamma function. The pdf $g(.)$ and cdf $G(.)$ of Power function distribution is taken as:

$$g(y,\,\mu,\,\theta)=\frac{\mu}{\theta}\left(\frac{y}{\theta}\right)^{\mu-1};\quad 0<y<\theta,\mu>0 \tag{3}$$

$$G(y,\,\mu,\,\theta)=\left(\frac{y}{\theta}\right)^{\mu};\quad 0<y<\theta,\mu>0 \tag{4}$$

3. Disaster probability ($\boldsymbol{\alpha = P(X>\theta)}$)

Theorem 1: If X follows the Nakagami distribution (1) and Y follows the power function distribution (3), then the parameter α is given by the expression:

$$\alpha=\frac{1}{\sqrt{\xi}}\Gamma\left[\xi,\frac{\xi}{\beta}\right] \tag{5}$$

where $\beta=\dfrac{\psi}{\theta^2}$

Proof: As we know

$$\begin{aligned}\alpha &= P(X>\theta)\\ &=\int_{\theta}^{\infty}\frac{2}{\Gamma(\xi)}\left(\frac{\xi}{\psi}\right)^{\xi}x^{2\xi-1}\exp\left(-\frac{\xi}{\psi}\,x^{2}\right)dx\end{aligned} \tag{6}$$

on taking $\dfrac{\xi}{\psi}x^2=u$ in Eq. (6), we get

$$\begin{aligned}&=\frac{1}{\sqrt{\xi}}\int_{\frac{\xi\theta^2}{\psi}}^{\infty}u^{\xi-1}e^{-u}du\\ \alpha &= \frac{1}{\sqrt{\xi}}\Gamma\left[\xi,\frac{\xi}{\beta}\right]\end{aligned}$$

where, $\beta = \dfrac{\psi}{\theta^2}$ and $\Gamma[x, a]$ is upper incomplete gamma function.
Hence, the theorem follows.

3.1. Numerical investigation of disaster risk probability ($\boldsymbol{\alpha}$)

Numerical evaluation of the probability measure $\alpha = P(X > \theta)$, derived from expression (5), exhibits systematic variations across different parametric combinations of β and ξ, as presented in Table 1. The analytical results reveal a direct correlation between the disaster probability and parameter β. In the context of stress-strength modeling, where system failure is characterized by $X > \theta$ [Alam and Roohi (2003)], this relationship provides crucial insights for system reliability optimization.

Table 1: Numerical values for probability of disaster $\boldsymbol{\alpha = P(X > \theta)}$

β	ξ=0.6	ξ=1.5	ξ=2	ξ=2.6	ξ=3.5
2	0.985141	0.493692	0.520260	0.694699	1.483699
1.5	0.846234	0.414194	0.434913	0.580485	1.244489
0.2	0.037432	0.001315	0.000353	0.000096	0.000020
0.1	0.001476	0.000001	0.000000	0.000000	0.000000
0.05	0.000003	0.000000	0.000000	0.000000	0.000000

Alternatively, we may also obtain the numerical values of β for fixed ξ at different tolerance level α from Eq. (5). Further, these values are used to obtain the optimum cost for manufacturing of item at desired tolerance level.

Table 2: Values of $\boldsymbol{\beta}$ at different tolerance level $\boldsymbol{\alpha}$

			ξ=2.5		
α	0.1	0.05	0.01	0.005	0.001
β	0.570651	0.470841	0.340964	0.306055	0.248595

4. Stress-strength reliability $\boldsymbol{P = Pr(Y > X)}$ for nakagami and power function distributions

Theorem 2: Let stress (X) and strength (Y) be distributed according to $f(x, \xi, \psi)$ and $g(y, \mu, \theta)$, respectively. The probability $P = Pr(Y > X)$ is formulated as

$$P = \frac{1}{\Gamma\xi}\left[\gamma\left(\xi, \frac{\xi}{\beta}\right) - \left(\frac{\beta}{\xi}\right)^{\mu/2}\gamma\left(\xi + \frac{\mu}{2}, \frac{\xi}{\beta}\right)\right] \tag{7}$$

Proof: According to stress-strength reliability

$$P = \int_{x=0}^{\theta}\int_{y=x}^{\theta} f(x,\, \xi,\, \psi)\, g(y,\, \mu,\, \theta)\, dy\, dx \tag{8}$$

Substituting $y = vx$ in Eq. (8),we get

$$P = \int_{x=0}^{\theta} \int_{y=x}^{\theta/x} f(x,\, \xi,\, \psi)\, g(vx,\, \mu,\, \theta)\, dv\, dx$$
$$= \frac{1}{\theta^{\mu}} \frac{2}{\Gamma(\xi)} \left(\frac{\xi}{\psi}\right)^{\xi} \int_0^{\sqrt{\psi/\beta}} x^{2\xi+\mu-1} \exp\left(-\frac{\xi}{\psi}\, x^2\right) \left[\left(\frac{\sqrt{\psi/\beta}}{x}\right)^{\mu} - 1\right] dx \tag{9}$$

on taking $\frac{\xi}{\psi} x^2 = u$, in Eq. (9), we get

$$P = \frac{1}{\theta^{\mu}} \frac{2}{\Gamma(\xi)} \left(\frac{\xi}{\psi}\right)^{\xi} \left[\frac{1}{2}\left(\frac{\psi}{\xi}\right)^{\xi} \int_0^{\xi/\beta} u^{\xi-1} e^{-u}\, du - \frac{1}{2}\left(\frac{\psi}{\xi}\right)^{\xi+\mu/2} \int_0^{\xi/\beta} u^{\xi+\mu/2-1} e^{-u}\, du\right]$$
$$= \frac{1}{\theta^{\mu}\Gamma\xi} \left[\left(\frac{\psi}{\beta}\right)^{\mu/2} \int_0^{\xi/\beta} u^{\xi-1} e^{-u}\, du - \left(\frac{\psi}{\xi}\right)^{\mu/2} \int_0^{\xi/\beta} u^{\xi+\mu/2-1} e^{-u} du\right]$$

Putting $\theta = \left(\frac{\psi}{\beta}\right)^{1/2}$

$$P = \frac{1}{\Gamma\xi} \left[\gamma\left(\xi, \frac{\xi}{\beta}\right) - \left(\frac{\beta}{\xi}\right)^{\mu/2} \gamma\left(\xi + \frac{\mu}{2}, \frac{\xi}{\beta}\right)\right]$$

where, $\gamma(a, x) = \int_0^x t^{a-1} e^{-t} dt$ is lower incomplete gamma function.
Hence, the theorem follows.

Table 3: The Stress-strength reliability of an item for fixed value of $\xi = 2.5$

$\downarrow\mu$ $\beta\rightarrow$	0.001	0.01	0.1	1	5	10
0.1	0.299133	0.213614	0.117660	0.018746	0.000812	0.000162
1.0	0.969910	0.904847	0.699099	0.150043	0.006822	0.001373
1.5	0.994576	0.969498	0.828477	0.202057	0.009395	0.001896
2.5	0.999812	0.996653	0.940481	0.278586	0.013449	0.002726
5.0	1.000000	0.999982	0.994223	0.385308	0.019856	0.004057

5. Likelihood maximization for parameter estimation

5.1. In the case of simple random sampling (SRS)

Theorem 3: For a simple random sample x_1, x_2,. . .,x_n from Nakagami distribution having pdf Eq.(1), the maximum likelihood estimate (MLE) of ξ and ψ is

$$\hat{\psi}_{srs} = \frac{\sum_{i=1}^{n} x_i^2}{n} \qquad \text{and} \qquad \hat{\xi}_{srs} = \text{From Eq.(13)}$$

Proof: If we take a random sample x_1, x_2,...,x_n from the Nakagami (ξ, ψ) of size n, then the likelihood function of the Nakagami distribution NAD (ξ, ψ) is given by

$$L(x,\, \xi,\, \psi) = \frac{(2\xi^{\xi})^n}{(\Gamma\xi)^n (\psi^{\xi})^n} \prod_{i=1}^{n} (x_i)^{(2\xi-1)} \exp\left(-\frac{\xi}{\psi}\sum_{i=1}^{n} x_i^2\right) \tag{10}$$

We can formulate the likelihood function as follows

$$\log\ L = n\log\ 2 - n\log\Gamma\xi + n\xi\log\ \xi - n\xi\log\psi + (2\xi-1)\sum_{i=1}^{n}\log\ x_i - \frac{\xi}{\psi}\sum_{i=1}^{n} x_i^2 \tag{11}$$

Differentiating the log likelihood function of NAD (ξ, ψ) given in Eq. (11) with respect to ψ in the case when ξ is known and with respect to ξ in the case when ψ is known. Then equating the resulting equations equal to zero, we get

$$\hat{\psi}_{srs} = \frac{\sum_{i=1}^{n} x_i^2}{n} \tag{12}$$

and

$$\frac{\partial logL}{\partial\xi} = \log\xi - \left(\frac{\partial}{\partial\xi}\log\Gamma\xi\right) - \log\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) + \frac{1}{n}\sum_{i=1}^{n}\log x_i^2 = 0 \tag{13}$$

ML estimator of ξ can be obtained from Eq. (13) by using Newton-Raphson method because Eq. (13) does not yield a closed-form solution.

Theorem 4: For a simple random sample y_1, y_2,...,y_n from Power function distribution having pdf Eq. (3), the MLE's of θ and μ are given respectively

$$\hat{\theta}_{srs} = y_{(n)}$$

and

$$\hat{\mu}_{srs} = \frac{n}{n\log\ y_{(n)} - \sum_{i=1}^{n}\log\ y_i}$$

where, $y_{(n)}$ is n^{th} order statistics.

Proof: The log likelihood function is given as:

$$\log L = n\left(\log\mu - \mu\log\theta\right) + (\mu-1)\sum_{i=1}^{n}\log y_i \tag{14}$$

The first-order condition reveals that n^{th} order statistic is MLE of θ i.e.

$$\hat{\theta}_{srs} = y_{(n)} \tag{15}$$

Differentiating Eq.(14) with respect to μ and equating the resulting equation to zero, we get the MLE of μ i.e.

$$\hat{\mu}_{srs} = \frac{n}{n \log\, y_{(n)} - \sum_{i=1}^{n} \log\, y_i} \tag{16}$$

Theorem 5: If $x_1,\ x_2, \ldots, x_n$ is a simple random sample from Nakagami distribution and $y_1,\ y_2, \ldots, y_m$ is a simple random sample from Power function distribution then the MLE of stress-strength reliability $P = Pr(Y > X)$ is given by

$$\hat{P}_{(srs)} = \frac{1}{\Gamma\hat{\xi}_{srs}}\left[\gamma\left(\hat{\xi}_{srs}, \frac{\hat{\xi}_{srs}}{\hat{\beta}_{srs}}\right) - \left(\frac{\hat{\beta}_{srs}}{\hat{\xi}_{srs}}\right)^{\hat{\mu}_{srs}/2} \gamma\left(\hat{\xi}_{srs} + \frac{\hat{\mu}_{srs}}{2}, \frac{\hat{\xi}_{srs}}{\hat{\beta}_{srs}}\right)\right] \tag{17}$$

Proof: Let us take a simple random sample $x_1,\ x_2, \ldots, x_n$ from Nakagami distribution with parameters $(\xi,\ \psi)$ and $y_1,\ y_2, \ldots, y_m$ is a simple random sample from Power function distribution with parameters $(\mu,\ \theta)$. The MLE's of ψ, ξ, θ and μ can be obtained by using Theorem (3) and (4), respectively. Since ML estimation procedure holds the in-variance property. Hence, on using Theorem (3) and (4) and in-variance property of ML estimator, the ML estimate $\hat{\beta}_{srs}$ is given by

$$\hat{\beta}_{srs} = \frac{\hat{\psi}_{srs}}{\hat{\theta}^2_{srs}}$$

Using the same fact the MLE of stress-strength reliability defined in Eq. (7) is given by

$$\hat{P}_{(srs)} = \frac{1}{\Gamma\hat{\xi}_{srs}}\left[\gamma\left(\hat{\xi}_{srs}, \frac{\hat{\xi}_{srs}}{\hat{\beta}_{srs}}\right) - \left(\frac{\hat{\beta}_{srs}}{\hat{\xi}_{srs}}\right)^{\hat{\mu}_{srs}/2} \gamma\left(\hat{\xi}_{srs} + \frac{\hat{\mu}_{srs}}{2}, \frac{\hat{\xi}_{srs}}{\hat{\beta}_{srs}}\right)\right]$$

5.2. In the case of ranked set sampling (RSS)

Let us consider two ranked set samples from Nakagami distribution and Power function distribution respectively. The first sample, denoted as $x_{(ij)}$, has a size of $n_1 = r_1 m_1$, where i ranges from 1 to m_1, and j from 1 to r_1. Here, m_1 represents the set size, and r_1 the number of cycles. Similarly, the second sample, $y_{(pq)}$, has a size of $n_2 = r_2 m_2$, where k ranges from 1 to m_2, and l from 1 to r_2. In this case, m_2 is the set size, and r_2 the number of cycles. We can now express the PDF's for $X_{(ij)}$ and $Y_{(pq)}$ as follows:

$$f_i(x_{ij}) = \frac{m_1!}{(i-1)!(m_1-i)!}\left[F_X(x)\right]^{i-1}\left[1 - F_X(x)\right]^{m_1-i} f(x_{ij}) \tag{18}$$

$$g_k(y_{pq}) = \frac{m_2!}{(k-1)!(m_2-k)!}\left[F_Y(y)\right]^{k-1}\left[1 - F_Y(y)\right]^{m_2-k} g(y_{pq}) \tag{19}$$

The likelihood function is given as

$$\begin{aligned} L &= \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} f_i(x_{ij}) \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} g_k(y_{pq}) \\ &= \prod_{i=1}^{r_1}\prod_{j=1}^{m_1} \frac{m_1!}{(i-1)!(m_1-i)!}\left[F_X(x)\right]^{i-1}\left[1-F_X(x)\right]^{m_1-i} f(x_{ij}) \\ &\quad \prod_{k=1}^{r_2}\prod_{l=1}^{m_2} \frac{m_2!}{(k-1)!(m_2-k)!}\left[F_Y(y)\right]^{k-1}\left[1-F_Y(y)\right]^{m_2-k} g(y_{pq}) \end{aligned}$$

Let $u = \dfrac{m_1!}{(i-1)!(m_1-i)!}$ and $v = \dfrac{m_2!}{(k-1)!(m_2-k)!}$, we get

$$\begin{aligned} &= u\frac{(2)^{n_1}}{(\Gamma\xi)^{n_1 m_1}}\left(\frac{\xi}{\psi}\right)^{n_1\xi} \prod_{i=1}^{r_1}\prod_{j=1}^{m_1}\left[\gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right]^{i-1}\left[\Gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right]^{m_1-i} \\ &\quad x^{2\xi-1}\exp\left(-\frac{\xi}{\psi}\,x_{ij}^2\right) v\,(\theta)^{-\mu n_2 m_2}\prod_{k=1}^{r_2}\prod_{l=1}^{m_2} y^{\mu k-1}\left(\theta^{\mu}-y^{\mu}\right)^{m_2-k} \end{aligned}$$

The log likelihood function is given by

$$\begin{aligned} \log L &= \log u - n_1 m_1 \log(\Gamma\xi) + n_1\log 2 + n_1\xi\left(\log\frac{\xi}{\psi}\right) + (2\xi-1)\log x_{ij} \\ &\quad + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\log\left[\gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right] + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i)\log\left[\Gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right] \\ &\quad -\frac{\xi}{\psi}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}^2 + \log v - \mu n_2 m_2\log\theta + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(\mu k-1)\log y_{pq} \\ &\quad + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2}(m_2-k)\log\left(\theta^{\mu}-y_{pq}^{\mu}\right) \end{aligned} \tag{20}$$

Partially differentiating Eq.(20) with respect to ξ and ψ respectively, we get

$$\begin{aligned} \frac{\partial \log L}{\partial \xi} &= n_1(\log\xi - 1) - n_1 m_1\frac{\partial}{\partial\xi}\log\Gamma\xi - n_1\log\psi + 2\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}\log x_{ij} \\ &\quad + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\frac{\partial}{\partial\xi}\log\left[\gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right] - \frac{1}{\psi}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}^2 \\ &\quad + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i)\frac{\partial}{\partial\xi}\log\left[\Gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right] \end{aligned} \tag{21}$$

and

$$\begin{aligned} \frac{\partial \log L}{\partial \psi} &= -\frac{n_1\xi}{\psi} + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(i-1)\frac{\partial}{\partial\xi}\log\left[\gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right] + \frac{\xi}{\psi^2}\sum_{i=1}^{r_1}\sum_{j=1}^{m_1}x_{ij}^2 \\ &\quad + \sum_{i=1}^{r_1}\sum_{j=1}^{m_1}(m_1-i)\frac{\partial}{\partial\xi}\log\left[\Gamma\left(\xi,\frac{\xi}{\psi}x_{ij}^2\right)\right] \end{aligned} \tag{22}$$

Partially differentiating Eq.(20) with respect to μ and θ respectively, we get

$$\frac{\partial \log L}{\partial \mu} = \sum_{k=1}^{r_2}\sum_{l=1}^{m_2} k \log y_{pq} + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2} (m_2 - k)\left[\frac{\theta^{\mu}\log\mu - y_{pq}^{\mu}\log y_{pq}}{\theta^{\mu} - y_{pq}^{\mu}}\right] - n_2 m_2 \log\theta \tag{23}$$

and

$$\frac{\partial \log L}{\partial \theta} = -\frac{\mu n_2 m_2}{\theta} + \sum_{k=1}^{r_2}\sum_{l=1}^{m_2} (m_2 - k)\frac{\mu\theta^{\mu-1}}{(\theta^{\mu} - y_{pq}^{\mu})} \tag{24}$$

The MLE's of ξ, ψ, μ and θ can be obtained from Eq. (21), (22), (23) and (24) by using the Newton-Raphson method, respectively, as the equations are not in closed form. We get the ML estimates as $\hat{\xi}_{rss}$, $\hat{\psi}_{rss}$, $\hat{\mu}_{rss}$ and $\hat{\theta}_{rss}$, respectively. By applying the in-variance property of maximum likelihood estimation, we can find the maximum likelihood estimate of the parameter 'β' and reliability parameter P based on RSS, denoted by $\hat{\beta}_{rsss}$ and $\hat{P}_{(rss)}$, respectively.

$$\hat{\beta}_{rss} = \frac{\hat{\psi}_{rss}}{\hat{\theta}^2_{rss}} \tag{25}$$

and

$$\hat{P}_{(rss)} = \frac{1}{\Gamma\hat{\xi}_{rss}}\left[\gamma\left(\hat{\xi}_{rss}, \frac{\hat{\xi}_{rss}}{\hat{\beta}_{rss}}\right) - \left(\frac{\hat{\beta}_{rss}}{\hat{\xi}_{rss}}\right)^{\hat{\mu}_{rss}/2}\gamma\left(\hat{\xi}_{rss} + \frac{\hat{\mu}_{rss}}{2}, \frac{\hat{\xi}_{rss}}{\hat{\beta}_{rss}}\right)\right] \tag{26}$$

6. Simulation study

In this section, the simulation studies are conducted to compare the performances using different sample datasets and different stress-strength dependent parameters. We mainly compare the performances of the ML estimates in terms of their biases and mean square errors (MSE) from the formula Bias$(\hat{P}) = E(\hat{P} - P)$ and MSE$(\hat{P}) = E(\hat{P} - P)^2$ respectively. In this particular case the stress variable is set to obey Nakagami distribution, the strength variable is set to obey Power function distribution. Based on the simple random sample of stress, the maximum likelihood estimations of ξ and ψ are obtained by using the R software through the Eq. (12) and (13) respectively. Similarly, based on the simple random sample of strength, the ML estimation of θ and μ are also obtained from Eq. (15) and (16), respectively. ML estimator of the combined parameter ($\beta = \frac{\psi}{\theta^2}$) is obtained from Eq. (25). The methodology to draw the ranked set sample from the population is given below:

1. A random subset of the population consisting of m^2 units is selected.
2. The m^2 units are then divided arbitrarily into m sets, each containing m units.
3. The units within each set are ranked based on either professional judgment or correlation with the variable of interest.

4. An individual quantile sample is constructed by taking the lowest ranked unit from the first set, the second lowest ranked unit from the second set, and continuing in this fashion.

5. To obtain a larger sample of size $n = r * m$, steps 1 through 4 can be repeated for r cycles.

The ranked set sampling method takes only one observation from each set in each cycle. In the first cycle, it chooses the lowest observation $x_{(11)r}$. In later cycles, it independently selects the second lowest $x_{(22)r}$ from a different set of m observations and the highest $x_{(mm)r}$ from the final set of m. Let $x_{(ii)k}, i = 1, 2, \ldots, m; k = 1, 2, \ldots, r$, be a ranked sample set with set size m and r cycles. For convenience, this paper will use the notation $x_{(i)r}$ in place of the full description.
Simulation steps are given below-

Step :1 We generate 1000 simple random samples of x_1, x_2,. . .,x_n, and y_1, y_2,. . .,y_m from Nakagami distribution and Power function distribution with the sample sizes of $(n_1, n_2) =$ (15, 15), (15, 20), (15, 25), (20, 20), (20, 25), (25, 25) in Case 1 and (20, 20), (20, 30), (20, 40), (30, 30), (30, 40), (40, 40) in Case 2, respectively.

Step :2 We generate 1000 ranked set samples of x_{11}, x_{22},. . .,$x_{m_1r_1}$ and y_{11}, y_{22},. . .,$y_{m_2r_2}$ from Nakagami distribution and from Power function distribution for the first case when the number of cycles is taken as $r_1 = r_2 = 5$ with set sizes $m_1 = m_2 = 3, 4, 5$ and for the second case when the number of cycles is taken as $r_1 = r_2 = 10$ with set sizes $m_1 = m_2 = 2, 3, 4$, respectively.

Step :3 To generate the SRS sample and RSS sample for the stress variable we consider the parametric value of Nakagami distribution as $\psi = 0.5$ and $\xi = (0.6, 1.5, 3.0)$. Similarly for the strength variable we take the parametric value of Power function distribution as $\theta = 2.23$ and $\mu = (0.5, 1.5, 3.0)$. We take the single values of the parameters ψ and θ to fix the parameter $\beta = 0.1$ which is equal to $\dfrac{\psi}{\theta^2} = \dfrac{0.5}{2.23^2} = 0.1$.

Step :4 The Biases, MSES and relative efficiency (RE) are presented in the Table 4.

Table 4: Biases, MSES and RE of P under SRS and RSS when the combined parameter $\beta = 0.1$ and $r_1 = r_2 = 5$

Case 1					SRS			RSS		
(ξ, μ)	(n_1, n_2)	(m_1, m_2)	P_{true}	$\hat{P}_{srs}$	Bias	MSE	$\hat{P}_{rss}$	Bias	MSE	RE
(0.6, 0.5)	(15,15)	(3,3)	0.5227125	0.591339	0.068626	0.010507	0.556335	0.033623	0.010297	1.0204
	(15,20)	(3,4)		0.588641	0.065928	0.009249	0.550926	0.028214	0.008555	1.0812
	(15,25)	(3,5)		0.592958	0.070245	0.008456	0.545796	0.023083	0.007778	1.0871
	(20,20)	(4,4)		0.592077	0.069364	0.008230	0.548540	0.025828	0.008066	1.0203
	(20,25)	(4,5)		0.593785	0.071073	0.007818	0.545833	0.023120	0.006213	1.2584
	(25,25)	(5,5)		0.593685	0.070973	0.007670	0.542317	0.019604	0.006451	1.1890
(1.5, 1.5)	(15,15)	(3,3)	0.8322674	0.856630	0.024362	0.005255	0.838484	0.006217	0.003951	1.3299
	(15,20)	(3,4)		0.849698	0.017430	0.004190	0.836275	0.004008	0.002382	1.7594
	(15,25)	(3,5)		0.846347	0.014080	0.003431	0.837606	0.005338	0.001767	1.9420
	(20,20)	(4,4)		0.852620	0.020353	0.003971	0.838037	0.005769	0.002269	1.7500
	(20,25)	(4,5)		0.848200	0.015933	0.003263	0.836344	0.004077	0.001616	2.0192
	(25,25)	(5,5)		0.849415	0.017147	0.003098	0.835449	0.003181	0.001590	1.9479
(3.0,3.0)	(15,15)	(3,3)	0.9646058	0.982819	0.018213	0.000861	0.970658	0.006052	0.000617	1.3948
	(15,20)	(3,4)		0.983515	0.018909	0.000686	0.968606	0.004000	0.000554	1.2388
	(15,25)	(3,5)		0.984642	0.020036	0.000622	0.968473	0.003867	0.000445	1.3987
	(20,20)	(4,4)		0.983449	0.018843	0.000700	0.968972	0.004366	0.000515	1.3595
	(20,25)	(4,5)		0.984943	0.020338	0.000637	0.968501	0.003896	0.000395	1.6099
	(25,25)	(5,5)		0.985938	0.021332	0.000648	0.967211	0.002606	0.000458	1.4141

Table 5: Biases, MSES and RE of P under SRS and RSS when the combined parameter $\beta = 0.1$ and $r_1 = r_2 = 10$

Case 2					SRS			RSS		
(ξ, μ)	(n_1, n_2)	(m_1, m_2)	P_{true}	$\hat{P}_{srs}$	Bias	MSE	$\hat{P}_{rss}$	Bias	MSE	RE
(0.6, 0.5)	(20,20)	(2,2)	0.522713	0.552325	0.029613	0.008021	0.530530	0.007817	0.005433	1.4763
	(20,30)	(2,3)		0.545121	0.022408	0.005856	0.534239	0.011527	0.003404	1.7204
	(20,40)	(2,4)		0.543142	0.020429	0.004903	0.533177	0.010465	0.002741	1.7885
	(30,30)	(3,3)		0.545193	0.022481	0.005481	0.538708	0.015995	0.003038	1.8042
	(30,40)	(3,4)		0.545384	0.022671	0.004278	0.535602	0.012890	0.002183	1.9590
	(40,40)	(4,4)		0.546516	0.023803	0.004058	0.535543	0.012830	0.001909	2.1260
(1.5, 1.5)	(20,20)	(2,2)	0.832267	0.850239	0.017972	0.003988	0.836635	0.004368	0.003431	1.1623
	(20,30)	(2,3)		0.845973	0.013705	0.002882	0.835628	0.003360	0.001993	1.4467
	(20,40)	(2,4)		0.842181	0.009914	0.002301	0.833979	0.001712	0.001265	1.8186
	(30,30)	(3,3)		0.843625	0.011358	0.002692	0.835259	0.002992	0.001631	1.6509
	(30,40)	(3,4)		0.840826	0.008558	0.002008	0.832193	-0.000074	0.001201	1.6725
	(40,40)	(4,4)		0.839773	0.007505	0.002061	0.836465	0.004197	0.000997	2.0671
(3.0,3.0)	(20,20)	(2,2)	0.964606	0.981699	0.017093	0.000835	0.969461	0.004855	0.000506	1.6491
	(20,30)	(2,3)		0.983959	0.019353	0.000638	0.967085	0.002479	0.000352	1.8115
	(20,40)	(2,4)		0.984990	0.020384	0.000569	0.967147	0.002541	0.000290	1.9611
	(30,30)	(3,3)		0.985509	0.020903	0.000650	0.966986	0.002381	0.000372	1.7469
	(30,40)	(3,4)		0.984823	0.020217	0.000569	0.966024	0.001418	0.000266	2.1379
	(40,40)	(4,4)		0.984142	0.019536	0.000535	0.967573	0.002968	0.000276	1.9361

Table (4) and (5) clearly indicates that the relative efficiency exceeds one in all instances; thus, it can be concluded that ranked set sampling demonstrates higher efficiency compared to simple random sampling for estimating stress-strength reliability.

7. Discussion

In manufacturing, when the strength of a device follows a Power function distribution, its maximum feasible operational value often has an upper threshold, denoted as θ_0. For instance, a turbine's rotational speed must not exceed its engineered safety limit to prevent mechanical failure. Suppose θ_α is the target operational value at a predefined tolerance α. If $\theta_\alpha < \theta_0$ manufacturers can derive the corresponding parameter μ_α (From Table 3) to design the item with a strength distribution defined by $(\mu_\alpha, \theta_\alpha)$ ensuring reliability. Conversely, if $\theta_\alpha > \theta_0$, adjustments to α or alternative designs become necessary. A real-world analogy is elevator systems: exceeding their maximum load capacity (θ_0) necessitates either reducing passenger limits (α) or upgrading components to meet safety standards.

8. An illustrative example

Without loss of generality, Let the ranking cost per unit be C_R and the measurement cost per unit be C_M, where typically $C_R < C_M$ since visual ordering or quick assessments are generally less expensive than precise measurements. For stress-strength models, we need to consider measurements for both stress (X) and strength (Y) components.
For a fixed budget B, consider these constraints:

- In RSS, we need to rank mx^2 units for stress and my^2 units for strength to obtain samples of sizes mx and my respectively.
- In SRS, we can directly measure $n * x$ units for stress and $n * y$ units for strength.
- The total cost must not exceed the budget B

The efficiency of RSS relative to SRS can be measured through the ratio of their respective mean squared errors (MSE) in estimating R. Let MSE($\hat{P}_{rss}$) be the mean squared error of the RSS reliability estimator MSE($\hat{P}_{srs}$) be the mean squared error of the SRS reliability estimator. The relative efficiency (RE) under perfect ranking is given by RE = MSE($\hat{P}_{srs}$)/MSE($\hat{P}_{rss}$).

For stress-strength models with underlying distributions RE $\approx$ $[(mx + 1)(my + 1)]/4$.(Dell and Clutter (1972)) This shows that RSS can potentially provide greater efficiency gains for reliability estimation compared to mean estimation, as we benefit from improved estimation in both X and Y samples.

Cost functions

Let the total cost functions be:

- For RSS: $C_{rss} = (mx^2 * C_R + mx * C_M) + (my^2 * C_R + my * C_M)$
- For SRS: $C_{srs} = nx * C_M + ny * C_M$

Optimization problem

The optimization problem can be formulated as follows:
Minimize MSE($\hat{P}_{rss}$) subject to:

1. $(mx^2 * C_R + mx * C_M) + (my^2 * C_R + my * C_M) \leq B$
2. $mx, my \geq 2$ (to ensure meaningful ranking)
3. Cost ratio $r = C_M/C_R \geq 1$
4. Desired precision: MSE($\hat{P}_{rss}$) $\leq \epsilon$

For a reliability study, we consider a balanced design with ranking cost C_R = ₹3, measurement cost C_M = ₹15, total budget B = ₹2000, cost ratio $r = C_M/C_R = 5$, and desired reliability threshold $P(Y > X) \geq 0.99$. Using equal set sizes $mx = my = m$, the total units ranked are $2m^2$ (m^2 each for X and Y), with $2m$ total units measured (m each for X and Y). The number of cycles (k) is determined by the budget constraint through the total cost function $C_{RSS} = k(2m^2C_R + 2mC_M) \leq B$. Analysis is performed for set sizes m = 2, 3, 4, 5, calculating: maximum possible cycles (k) under budget, total sample sizes ($nX = nY = mk$), relative efficiency RE = $[(m+1)(m+1)]/4$, and total cost.

Set Size (m)	Cycle k	Sample size per component	RE	Total Cost
2	30	60	2.25	1980
3	15	45	4.00	1980
4	8	32	6.25	1824
5	5	25	9.00	1650

Detailed calculations for $m = 3$:

1. Cost per cycle = (2 × 3^2 × ₹3) + (2 × 3 × ₹15) = 54 + 90 = ₹144
2. Maximum cycles = floor(₹2000/₹144) = 15
3. Sample size per component = 3 × 15 = 45
4. RE = [(3+1)(3+1)]/4 = 16/4 = 4.0
5. Total cost = 15 × ₹144 = ₹1980

Comparison with SRS Under the same budget (₹2000):

- Cost per unit = ₹15 (measurement only)
- Maximum total sample size = ₹2000/₹15 ≈ 133
- Sample size per component = 66

Reliability estimation

For m = 3 RSS design:

1. Variance reduction in each component:
   - $\text{Var}(\bar{X}_{rss}) = \text{Var}(\bar{X}_{srs})/4 = (4/45)/(4) = 0.0222$
   - $\text{Var}((\bar{Y}_{rss}) = \text{Var}(\bar{Y}_{srs})/4 = (4/45)/(4) = 0.0222$
2. Approximate variance of reliability estimator:
   - $\text{Var}(\hat{P}_{rss}) \approx 0.0156$ (using delta method)
   - $\text{Var}(\hat{P}_{srs}) \approx 0.0624$ (for equivalent SRS)
3. 95% Confidence interval for P using RSS: $\hat{P}_{rss} \pm 1.96\sqrt{0.0156} = \hat{P}_{rss} \pm 0.245$

Based on the calculations, $m = 3$ emerges as the optimal choice, offering 4 times the efficiency of SRS, maintaining adequate sample size (45 per component), and using the budget effectively (₹1980 of ₹2000). This choice is practically sound as ranking 9 units (3^2) at a time is manageable, with reasonable ranking error risks and sufficient sample size for normal approximations. While larger set sizes ($m = 4$, 5) offer higher theoretical efficiency, they become impractical due to increased ranking difficulty, higher error risks, smaller final sample sizes, and potential departure from asymptotic properties. Therefore, we recommend using $m = 3$ with 15 cycles to achieve reliable estimation of $P(Y > X)$, cost-effective budget utilization, manageable ranking requirements, and sufficient sample sizes for inference.

9. Conclusion

This study presents a comprehensive analysis of stress-strength reliability estimation using ranked set sampling (RSS) compared to simple random sampling (SRS). The investigation yields several significant findings. First, the maximum likelihood estimation under RSS demonstrates superior efficiency compared to SRS. This efficiency gain is particularly noteworthy in stress-strength applications where measurement costs are high. In examining finite strength cases within the stress-strength model, we derived explicit expressions for the disaster probability and analyzed its behavior under various parameter configurations. This analysis provides crucial insights for reliability engineers and quality control practitioners in assessing system failure risks. The numerical evaluations of α across different parameter values offer practical guidelines for system design and maintenance protocols. The optimization framework developed for cost-efficient implementation of RSS in stress-strength reliability estimation addresses the practical challenges of sampling design. By balancing statistical efficiency against economic constraints, we demonstrated that moderate set sizes (particularly $m = 3$) often provide the most practical solution, achieving four-fold efficiency gains while maintaining manageable ranking requirements and adequate sample sizes for inference. These findings have significant implications for reliability testing and quality control applications, particularly in scenarios where testing costs are substantial or destructive testing is required. The demonstrated efficiency gains of RSS over SRS suggest that its implementation could lead to considerable cost savings while maintaining or improving estimation precision in stress-strength reliability assessment.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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