

On Renewal Processes - Some Bounds and Characterisations

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Abstract

Bounds for the distribution/ survival function and a few other characterisations associated with renewal processes based on ageing properties of the generating inter-arrival times are obtained. A few results are also obtained by comparing renewal processes with respect to some stochastic orderings.

Key words: Ageing classes; Convex transform order; Log-concavity; Renewal process.

1. Introduction

Let $\{N(t), t \geq 0\}$ be a renewal process with arrival and inter-arrival times denoted by the sequences $\{S_1, S_2, \dots\}$ and $\{X_1, X_2, \dots\}$, respectively, where $S_n = X_1 + \dots + X_n$, and, the inter-arrival times are continuous, non-negative, independent and identically distributed (iid) random variables (rvs) with common distribution function (df) $F(\cdot)$ and survival function (sf) $\bar{F}(\cdot) = 1 - F(\cdot)$. We assume that $F(0) = 0$ and the right extremity $r(F) = \sup\{x : F(x) < 1\} = \infty$. The distribution of $N(t)$ is given by

$$P(N(t) \geq n) = P(S_n \leq t) = F_n(t), n \geq 1, t \geq 0, \quad (1)$$

where F_n is the n -fold convolution of F with itself (see Ross (1995)). Unless otherwise specified, the preceding notations are used throughout this article. The n -fold convolution of a df is of great interest in applied probability theory, especially in reliability and renewal theory. However, to obtain analytical expression for the n -fold convolution is mathematically challenging and sometimes impossible. F_n can be obtained in analytical form for some dfs such as uniform, exponential and gamma with integer shape parameter. Some bounds are proposed for F_n , which, in turn, provide some bounds for the df and sf of $N(t)$, using ageing properties of the renewal df F being concave / log-concave, convex / log-convex (Bergstrom and Bagnoli (2005)), IFR / DFR, NBU / NWU (Barlow and Proschan (1975)). A few results are also obtained here by comparing renewal processes having different renewal distributions

with respect to convex (concave) transform order and super-additive / sub-additive order (for more details on stochastic orders, see Shaked and Shanthikumar (2007)). The bounds obtained here appear to be new and more strict compared to the previously obtained bounds by Barlow and Proschan (1975). This is illustrated with some numerical examples in Section 2.1.

Definitions used in this article are from Barlow and Proschan (1975) and Shaked and Shanthikumar (2007) and these have been given in Appendix for reference.

The article is organized as follows: Section 2 comprises of a few new bounds with detailed proofs and associated comments, remarks, corollary, and some numerical illustrations. Section 3 proposes a few results by comparing renewal processes with respect to some stochastic orderings. Finally Section 4 comprehensively concludes the paper with some future directions of research.

2. On bounds for renewal processes

Let $R(t) = -\ln \bar{F}(t)$ and $K(t) = -\ln F(t)$, $t > 0$, denote the hazard function (hf) and reversed hazard function (rhf) associated with the df F , respectively. We first state and prove a known lemma which will be used to prove many of the results of this section.

Lemma 1: For $\lambda_n \geq 0$, $n \geq 1$, with $\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = 0$, $\lim_{n \rightarrow \infty} \sum_{j=0}^n e^{-\lambda_n} \frac{\lambda_n^j}{j!} = 1$.

Proof: For arbitrary $k > 0$,

$$\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = 0 \Rightarrow \lim_{n \rightarrow \infty} \frac{\lambda_n + k\sqrt{\lambda_n}}{n} = 0 \Rightarrow \lambda_n + k\sqrt{\lambda_n} < n \text{ for all } n \text{ large.} \quad (2)$$

We have, for arbitrary $k > 0$,

$$\begin{aligned} 1 &\geq \lim_{n \rightarrow \infty} \sum_{j=0}^n e^{-\lambda_n} \frac{\lambda_n^j}{j!} = \lim_{n \rightarrow \infty} \sum_{j=0}^n P(X_n = j), \text{ where } X_n \sim \text{Poisson}(\lambda_n) \\ &= \lim_{n \rightarrow \infty} P(X_n \leq n) \\ &\geq \lim_{n \rightarrow \infty} P\left(X_n \leq \lambda_n + k\sqrt{\lambda_n}\right) \text{ by (2)} \\ &\geq \lim_{n \rightarrow \infty} P\left(|X_n - \lambda_n| \leq k\sqrt{\lambda_n}\right) \\ &\geq \lim_{n \rightarrow \infty} \left(1 - \frac{1}{k^2}\right) \text{ by Chebyshev's inequality} \\ &= 1 - \frac{1}{k^2}. \end{aligned}$$

Therefore $\lim_{n \rightarrow \infty} \sum_{j=0}^n e^{-\lambda_n} \frac{\lambda_n^j}{j!} = 1$. □

The following result provide some bound for the probability distribution of a renewal process when the underlying renewal distribution F is NBU/ NWU.

Theorem 1: If F is NBU (NWU), then

$$P(N(t) < n) \geq (\leq) e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!}, n \geq 1, t > 0. \quad (3)$$

In particular, if F is exponential df with mean $\frac{1}{\mu}, \mu > 0$, then equality holds as $\{N(t), t > 0\}$ is a Poisson process with rate μ .

Proof: Since NWU proof is similar, we give proof only when F is NBU. If X is NBU then $X \leq_{su} Y$, where Y has df $G(t) = 1 - e^{-\lambda t}, t > 0, \lambda > 0$, and $X \leq_{su} Y$ means $G^{-1}(F(t))$ is super-additive (su). If $W(t) = G^{-1}(F(t))$ then

$$W(t) = -\frac{\ln(1 - F(t))}{\lambda} = -\frac{\ln \bar{F}(t)}{\lambda} = \frac{R(t)}{\lambda}, t \geq 0,$$

so that $W(\cdot)$ is super-additive/ sub-additive iff $R(t)$ is super-additive/ sub-additive. If F is NBU, $R(\cdot)$ is super-additive. Therefore, $R^{-1}(\cdot)$ is sub-additive (Østerdal (2006)), that is, $R^{-1}(x + y) \leq R^{-1}(x) + R^{-1}(y), x, y \geq 0$. Let $Y_1, Y_2, \dots$ be iid rvs with df $H(x) = 1 - e^{-x}, x \geq 0$. Then, with $X_j = R^{-1}(Y_j), j \geq 1$, $X_1, X_2, \dots$ are iid rvs with df F , as $P(X_j \leq x) = P(R^{-1}(Y_j) \leq x) = P(Y_j \leq R(x)) = H(R(x)) = F(x)$. Also, $X_1, X_2, \dots$ generate the renewal process $\{N(t), t \geq 0\}$ with renewal df F whereas the iid inter-arrival times $Y_1, Y_2, \dots$ generate a Poisson process with rate 1. Therefore,

$$\begin{aligned} P(N(t) < n) &= P\left(\sum_{j=1}^n X_j > t\right) \\ &= P\left(\sum_{j=1}^n R^{-1}(Y_j) > t\right) \\ &\geq P\left(R^{-1}\left(\sum_{j=1}^n Y_j\right) > t\right) \\ &= P\left(\sum_{j=1}^n Y_j > R(t)\right) \\ &= \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!} e^{-R(t)} = e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!}, \end{aligned}$$

since $\sum_{j=1}^n R^{-1}(Y_j) \geq R^{-1}\left(\sum_{j=1}^n Y_j\right)$ as $R^{-1}(t)$ is sub-additive completing the proof. $\square$

The following result provides bounds for $P(N(t) < n)$ when F is IFR/ DFR.

Theorem 2:

(i) If F is IFR, then

$$e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!} \leq P(N(t) < n) \leq e^{-nR\left(\frac{t}{n}\right)} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!}, n \geq 1. \quad (4)$$

(ii) If F is DFR, then

$$e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!} \leq P(N(t) < n) \leq e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!}, n \geq 1. \quad (5)$$

Proof: We first prove that when F is IFR (DFR), then

$$P(N(t) < n) \leq (\geq) e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!}, n \geq 1, t > 0. \quad (6)$$

We give the proof when F is IFR and the proof is similar in the DFR case.

Let F be IFR and $G(x) = 1 - e^{-x}, x > 0$, be the df of standard exponential rv Y . With $\lambda_n = nR(\frac{t}{n}), n \geq 1$, note that $\lim_{n \rightarrow \infty} \frac{\lambda_n}{n} = 0$ as $R(0) = 0$. Hence by Lemma 1, we have

$$\begin{aligned} P(N(t) < n) &\leq e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!} \\ \Leftrightarrow P(N(t) \geq n) &\geq e^{-nR(\frac{t}{n})} \sum_{j=n}^{\infty} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!}. \end{aligned}$$

Using (1), it is enough to show that $F_n(t) \geq G_n\left(nR\left(\frac{t}{n}\right)\right), t \geq 0$, where G_n is n -fold convolution of G with itself. For $n = 1$, equality holds since $G_1(R(t)) = G(-\ln \bar{F}(t)) = 1 - \bar{F}(t) = F_1(t)$. Assuming that the inequality holds for $n - 1$, we have

$$F_n(t) = \int_0^t F_{n-1}(t-x) dF(x) \geq \int_0^t G_{n-1}\left((n-1)R\left(\frac{t-x}{n-1}\right)\right) dG(R(x)). \quad (7)$$

If X_1 is IFR, then X_1 is IFRA and by 5.4. Results. (e) in page 107 of Barlow and Proschan (1975), this is equivalent to $X \leq_c Y$, where $X \leq_c Y$ means $G^{-1}(F(t))$ is convex, that is, X is less than Y in convex order (see Appendix). We then have

$$W(t) = G^{-1}(F(t)) = -\ln(1 - F(t)) = -\ln \bar{F}(t) = R(t), t \geq 0.$$

Hence, $W(t)$ is convex/ concave iff $R(t)$ is convex/ concave. Since $R(\cdot)$ is convex, we have

$$\begin{aligned} \frac{t}{n} &= \left(\frac{n-1}{n}\right) \left(\frac{t-x}{n-1}\right) + \frac{x}{n} \\ \Rightarrow R\left(\frac{t}{n}\right) &\leq \left(\frac{n-1}{n}\right) R\left(\frac{t-x}{n-1}\right) + \frac{R(x)}{n} \end{aligned}$$

$$\begin{aligned}
\Rightarrow (n-1)R\left(\frac{t-x}{n-1}\right) &\geq nR\left(\frac{t}{n}\right) - R(x) \\
\Rightarrow F_n(t) &\geq \int_0^t G_{n-1}\left(nR\left(\frac{t}{n}\right) - R(x)\right) dG(R(x)) \quad \text{using (7)} \\
\Rightarrow F_n(t) &\geq \int_0^{R(t)} G_{n-1}\left(nR\left(\frac{t}{n}\right) - u\right) dG(u) \\
&\geq \int_0^{nR\left(\frac{t}{n}\right)} G_{n-1}\left(nR\left(\frac{t}{n}\right) - u\right) dG(u) \\
&= G_n\left(nR\left(\frac{t}{n}\right)\right),
\end{aligned}$$

as $R\left(\frac{t}{n}\right) \leq \frac{R(t)}{n}$ since $R(0) = 0$ and $R(\cdot)$ is convex. Therefore, (6) holds for n and by induction, the proof is complete. Since $IFR \Rightarrow IFRA \Rightarrow NBU$, from (6) and (3), we get (4), and since $DFR \Rightarrow DFRA \Rightarrow NWU$, from (6) and (3), we get (5). $\square$

Remark 1: Note that if F is IFR, then it is IFRA and hence $R(\cdot)$ is star-shaped which means that $\frac{R(t)}{t}$ increases as t increases. Hence, for $n \geq 1$, $nR\left(\frac{t}{n}\right) \leq R(t)$ and $e^{-nR\left(\frac{t}{n}\right)} \geq e^{-R(t)}$, $t > 0$. But, the left hand side in (4) is less than or equal to the right hand side. A similar remark holds for (5).

An interesting property for the bound of $P(N(t) < n)$ when F is IFR/ DFR is obtained as follows.

Theorem 3: If F is IFR (DFR), then

$$P(N(t) < n) \leq (\geq) P(Y_n \leq n-1), n \geq 1, t > 0, \quad (8)$$

where $Y_n \sim \text{Poisson}\left(-\ln \bar{F}^n\left(\frac{t}{n}\right)\right)$, with equality if $\bar{F}(t) = e^{-\mu t}$, $t > 0, \mu > 0$. Also,

$$Y_n \xrightarrow{d} Y \quad (9)$$

where $\xrightarrow{d}$ denotes convergence in distribution and $Y \sim \text{Poisson}(tf(0))$, provided F is absolutely continuous with probability density function (pdf) f .

Proof: Let $Y_n \sim \text{Poisson}\left(-\ln \bar{F}^n\left(\frac{t}{n}\right)\right)$, $n \geq 1$, so that

$$P(Y_n = j) = \frac{\left(-\ln \bar{F}^n\left(\frac{t}{n}\right)\right)^j}{j!} \bar{F}^n\left(\frac{t}{n}\right), j \geq 0. \quad (10)$$

If F is IFR (DFR), then from Theorem 2, $P(N(t) < n) \leq (\geq) P(Y_n \leq n-1)$, $n \geq 1$. Since $nR\left(\frac{t}{n}\right) = -n \ln \bar{F}\left(\frac{t}{n}\right) = -\ln \bar{F}^n\left(\frac{t}{n}\right)$ and $e^{-nR\left(\frac{t}{n}\right)} = \bar{F}^n\left(\frac{t}{n}\right)$, we obtain (8). If F has pdf f , for fixed $t > 0$, $\bar{F}\left(\frac{t}{n}\right) \uparrow 1$ as $n \rightarrow \infty$ so that $\lim_{n \rightarrow \infty} n \ln \bar{F}\left(\frac{t}{n}\right) = \lim_{n \rightarrow \infty} \frac{\ln \bar{F}\left(\frac{t}{n}\right)}{\frac{1}{n}} = \lim_{x \rightarrow 0} \frac{\ln \bar{F}(tx)}{x} = \lim_{x \rightarrow 0} \frac{-tf(tx)}{F(tx)} = -tf(0)$, using L'Hospital's rule. Thus $\lim_{n \rightarrow \infty} \bar{F}^n\left(\frac{t}{n}\right) = e^{-tf(0)}$. Since $\ln \bar{F}\left(\frac{t}{n}\right) \uparrow 0$ as $n \rightarrow \infty$, $n \log \bar{F}\left(\frac{t}{n}\right) \downarrow -tf(0)$ as $n \rightarrow \infty$ and hence $-\ln \bar{F}^n\left(\frac{t}{n}\right) \uparrow tf(0)$ as $n \rightarrow \infty$, proving (9). $\square$

Example 1: If $\bar{F}(t) = e^{-\lambda t}, t > 0, \lambda > 0$, then $P(N(t) < n) = P(Y_n \leq n-1), n \geq 1, t > 0$, where $Y_n \stackrel{d}{=} Y \sim \text{Poisson}(\lambda t)$, $\stackrel{d}{=}$ denoting equality in distribution. This follows trivially as $\{N(t), t \geq 0\}$ is a Poisson process.

Now assume that F is concave (convex). We give an upper (lower) bound for $P(N(t) \geq n)$ as the following theorem.

Theorem 4: If F is concave (convex) and $nF^{-1}\left(\frac{k}{n}\right) \leq t < nF^{-1}\left(\frac{k+1}{n}\right), 0 \leq k \leq n-1, n \geq 1$, then

$$P(N(t) \geq n) \leq (\geq) \frac{1}{n!} \sum_{j=0}^k (-1)^j \binom{n}{j} \left(nF\left(\frac{t}{n}\right) - j \right)^n, \quad (11)$$

with $F^{-1}(t) = \inf \{x : F(x) \geq t\}, 0 \leq t \leq 1$.

Proof: If F is concave, then $F\left(\frac{S_n}{n}\right) \geq \frac{F(X_1)+F(X_2)+\dots+F(X_n)}{n}$ and

$$\begin{aligned} P(N(t) \geq n) &= P(S_n \leq t) \\ &= P\left(F\left(\frac{S_n}{n}\right) \leq F\left(\frac{t}{n}\right)\right) \\ &\leq P\left(\frac{F(X_1) + \dots + F(X_n)}{n} \leq F\left(\frac{t}{n}\right)\right) \\ &= P\left(F(X_1) + \dots + F(X_n) \leq nF\left(\frac{t}{n}\right)\right) \\ &= F_U^{n*}\left(nF\left(\frac{t}{n}\right)\right), \end{aligned}$$

where F_U^{n*} is the n -fold convolution of F_U with itself, U being uniform rv on $(0,1)$ with df F_U . The last equality follows from the fact that, $F(X_i), i = 1, \dots, n$ are iid $U(0,1)$ rvs as F is assumed to be continuous. Therefore, from the well known analytical expression of the n -fold convolution of F_U with itself,

$$F_U^{n*}(t) = \begin{cases} 0, & t < 0, \\ \frac{1}{n!} \sum_{j=0}^k (-1)^j \binom{n}{j} (t-j)^n, & k \leq t < k+1, \quad 0 \leq k \leq (n-1), \end{cases}$$

we get (11). The reverse inequality in (11) can be proved using similar arguments. $\square$

Example 2: Let F be truncated Weibull df with shape parameter $= \frac{1}{2}$ and scale parameter $= 1$. Therefore,

$$F(t) = \begin{cases} 0, & t < 0, \\ \frac{1-e^{-\sqrt{t}}}{1-e^{-3}}, & 0 \leq t < 9, \\ 1, & t \geq 9. \end{cases}$$

Since $\frac{d^2F}{dt^2} = -\frac{e^{-\sqrt{t}}}{4t(1-e^{-3})} \left(1 + \frac{1}{\sqrt{t}}\right) < 0$ for $t > 0$, F is concave. However F is neither IFR nor DFR as the failure rate function $r(t) = \frac{1}{2\sqrt{t}(1-e^{-\sqrt{t}-3})}, 0 < t < 9$ is not monotone. Also F can not be NWU as there exist a $t \geq 9$ such that $F(t) = 1$. Furthermore, it is observed that F is not NBU as

1. $0 = \bar{F}(9) = P(T > 7 + 2) < P(T > 7)P(T > 2) = \bar{F}(7)\bar{F}(2) = \frac{(1-e^{-\sqrt{7}})(1-e^{-\sqrt{2}})}{(1-e^{-\sqrt{3}})^2}$
2. $\bar{F}(2) = P(T > 1 + 1) > P(T > 1)P(T > 1) = (\bar{F}(1))^2$

Therefore, F is none of IFR, DFR, NBU and NWU distributions, although F is concave and hence bounds given in Theorem 1 and Theorem 2 can not be used for this distribution. However we get an upper bound of $P(N(t) \geq n)$ for $t \geq 0$ and $n = 1, 2, \dots$ by Theorem 4. For example, taking $n = 2$ in (11), we obtain

$$P(N(t) \geq 2) = \begin{cases} \frac{2+2e^{-2\sqrt{\frac{t}{2}}}-4e^{-\sqrt{\frac{t}{2}}}}{(1-e^{-3})^2}, & 0 \leq t < 0.8309, \\ \frac{4e^{-\sqrt{\frac{t}{2}}-3}-2e^{-2\sqrt{\frac{t}{2}}+1}-2e^{-3}-e^{-6}}{(1-e^{-3})^2}, & 0.8309 \leq t < 18, \\ 1, & t \geq 18. \end{cases}$$

We now assume that F is log-concave/ log-convex. Bounds for $P(N(t) < n)$ in this case are given below.

Theorem 5:

- (i) If F is log-concave, for $t > 0$ and $n \geq 1$,

$$e^{-nK(\frac{t}{n})} \sum_{j=n}^{\infty} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!} \leq P(N(t) < n) \leq e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!}. \quad (12)$$

- (ii) If F is log-convex, for $t > 0$ and $n \geq 1$,

$$e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!} \leq P(N(t) < n) \leq e^{-nK(\frac{t}{n})} \sum_{j=n}^{\infty} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!}. \quad (13)$$

Proof: We first prove that if F is log-concave (log-convex), for $t \geq 0$ and $n \geq 1$,

$$P(N(t) \geq n) \leq (\geq) e^{-nK(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!}. \quad (14)$$

Let F is log-concave, or equivalently, $\ln F$ is concave so that

$$\ln F\left(\frac{S_n}{n}\right) \geq \frac{\ln F(X_1) + \dots + \ln F(X_n)}{n}$$

and

$$\begin{aligned}
P(N(t) \geq n) &= P(S_n \leq t) \\
&= P\left(\ln F\left(\frac{S_n}{n}\right) \leq \ln F\left(\frac{t}{n}\right)\right) \\
&\leq P\left(\frac{\ln F(X_1) + \cdots + \ln F(X_n)}{n} \leq \ln F\left(\frac{t}{n}\right)\right) \\
&= P\left(\ln F(X_1) + \cdots + \ln F(X_n) \leq n \ln F\left(\frac{t}{n}\right)\right) \\
&= P\left(-\ln F(X_1) - \cdots - \ln F(X_n) \geq -n \ln F\left(\frac{t}{n}\right)\right) \\
&= 1 - F_Y\left(nK\left(\frac{t}{n}\right)\right),
\end{aligned}$$

where $K(t) = -\ln F(t)$, and $Y = -\ln F(X_1) - \cdots - \ln F(X_n)$ is the sum of iid standard exponential rvs as F is assumed to be continuous and hence has Gamma distribution with shape parameter n and scale parameter 1. Therefore, we get (14). When F is log-convex, the reverse inequality can be proved using similar arguments.

Since log-concave $\Rightarrow$ IFR, from (4) and (14), we get (12), and since log-convex $\Rightarrow$ DFR, from (5) and (14), we get (13), completing the proof. $\square$

The following corollaries provide stricter bounds for $P(N(t) \geq n)$ when F is log-concave or log-convex.

Corollary 1: If F is log-concave, for $t > 0$ and $n \geq 1$,

$$P(N(t) \geq n) \leq \min \left\{ e^{-nK\left(\frac{t}{n}\right)} \sum_{j=0}^{n-1} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!}, 1 - e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!} \right\}. \quad (15)$$

Proof: Since F log-concave implies that F is IFR (Lemma 5.9, page 77, Barlow and Proschan (1975)), by Theorem 5, we get

$$P(N(t) \geq n) \leq e^{-nK\left(\frac{t}{n}\right)} \sum_{j=0}^{n-1} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!}, \quad (16)$$

and by Theorem 2(i), we get

$$e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!} \leq P(N(t) < n) \Leftrightarrow P(N(t) \geq n) \leq 1 - e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!}. \quad (17)$$

Combining (16) and (17), we get (15). $\square$

Corollary 2: If F is log-convex, for $t > 0$ and $n \geq 1$,

$$P(N(t) \geq n) \geq \max \left\{ e^{-nK\left(\frac{t}{n}\right)} \sum_{j=0}^{n-1} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!}, 1 - e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!} \right\}. \quad (18)$$

Proof: Since F log-convex implies that F is DFR (Lemma 5.9, page 77, Barlow and Proschan (1975)), by Theorem 5, we get

$$P(N(t) \geq n) \geq e^{-nK(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!}, \quad (19)$$

and by Theorem 2(ii), we get

$$e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!} \geq P(N(t) < n) \Leftrightarrow P(N(t) \geq n) \geq 1 - e^{-R(t)} \sum_{j=0}^{n-1} \frac{(R(t))^j}{j!}. \quad (20)$$

Combining (19) and (20), we get (18). $\square$

Example 3: When F is Fréchet df with $F(t) = e^{-t^{-\alpha}}, t > 0, \alpha > 0$, $\frac{d \ln F}{dt} = \alpha t^{-\alpha-1}$ and $\frac{d^2 \ln F}{dt^2} = -\alpha(\alpha+1)t^{-\alpha-2} < 0$, so that F is log-concave and by Theorem 5(i), we get the bounds for the renewal process generated by F as

$$e^{-nK(\frac{t}{n})} \sum_{j=n}^{\infty} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!} \leq P(N(t) < n) \leq e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!},$$

where $R\left(\frac{t}{n}\right) = -\ln\left(1 - e^{-\left(\frac{t}{n}\right)^{-\alpha}}\right)$ and $K\left(\frac{t}{n}\right) = \left(\frac{t}{n}\right)^{-\alpha}$.

Example 4: When F is truncated Gumbel df with $F(t) = \frac{e^{-e^{-t}} - e^{-1}}{1 - e^{-1}}, t > 0$, and pdf $f(t) = \frac{dF}{dt} = \frac{e^{-(t+e^{-t})}}{1 - e^{-1}}, t > 0$, $\frac{d \ln F}{dt} = e^{-t} - 1$ and $\frac{d^2 \ln F}{dt^2} = -e^{-t} < 0$. So f is log-concave and since f is continuously differentiable, by Bergstrom and Bagnoli (2005), F is also log-concave. By Theorem 5(i), we get the bounds for the renewal process generated by F as

$$e^{-nK(\frac{t}{n})} \sum_{j=n}^{\infty} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!} \leq P(N(t) < n) \leq e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!},$$

where $R\left(\frac{t}{n}\right) = \ln\left(\frac{1 - e^{-1}}{1 - e^{-\frac{t}{n}}}\right)$ and $K\left(\frac{t}{n}\right) = \ln\left(\frac{1 - e^{-1}}{e^{-e^{-\frac{t}{n}}} - e^{-1}}\right)$.

Example 5: When F is $F(n_1, n_2)$ -distribution with pdf

$$f(t) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \left(\frac{n_1}{n_2}\right)^{\frac{n_1}{2}} t^{\frac{n_1}{2}-1} \left(1 + \frac{n_1 t}{n_2}\right)^{-\frac{n_1+n_2}{2}}, t > 0,$$

if $n_1 \leq 2$, f is log-convex (ibid). Since f is continuously differentiable and log-convex on $(0, \infty)$, and $f(0) = 0$, F is also log-convex on $(0, \infty)$ (ibid). By Theorem 5(ii), we get the bounds for the renewal process generated by F as

$$e^{-nR(\frac{t}{n})} \sum_{j=0}^{n-1} \frac{\left(nR\left(\frac{t}{n}\right)\right)^j}{j!} \leq P(N(t) < n) \leq e^{-nK(\frac{t}{n})} \sum_{j=n}^{\infty} \frac{\left(nK\left(\frac{t}{n}\right)\right)^j}{j!},$$

where $R(t) = -\ln \bar{F}(t)$, $K(t) = -\ln F(t)$ and

$$F(t) = \frac{\Gamma\left(\frac{n_1+n_2}{2}\right)}{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)} \int_0^{\frac{n_1 t}{n_1 t + n_2}} x^{\frac{n_1}{2}-1} (1-x)^{\frac{n_2}{2}-1} dx.$$

2.1. Some numerical illustrations

Example 6: Consider the df

$$F(t) = \begin{cases} 0, & t < 0, \\ 2t - t^2, & 0 \leq t < 1, \\ 1, & t \geq 1. \end{cases}$$

Since $\frac{d^2 F}{dt^2} = -2 < 0$, F is concave and since the failure rate function $r(t) = \frac{2}{1-t}$, $0 \leq t < 1$ is increasing, F is also IFR. For $n = 2$, we obtain

$$F_2(t) = \begin{cases} 0, & t < 0, \\ 2t^2 - \frac{4t^3}{3} + \frac{t^4}{6}, & 0 \leq t < 1, \\ -\frac{5}{3} + \frac{16t}{3} - 4t^2 + \frac{4t^3}{3} - \frac{t^4}{6}, & 1 \leq t < 2, \\ 1, & t \geq 2, \end{cases}$$

Therefore $P(N(t) < 2) = P(S_2 > t) = 1 - F_2(t)$. Since F is IFR, the upper and lower bound for $P(N(t) < 2)$ is obtained as

$$u(t) = \left(1 - \frac{t}{2}\right)^4 \left(1 - \ln\left(1 - \frac{t}{2}\right)\right), 0 \leq t < 2 \quad (21)$$

and

$$l_1(t) = (1-t)^2(1-2\ln(1-t)), 0 \leq t < 1 \quad (22)$$

respectively, using (4). Since F is concave, the lower bound for $P(N(t) < 2)$ is obtained as

$$l_2(t) = \begin{cases} 1 - (2t^2 - t^3 + t^4/8), & 0 \leq t \leq 0.5858, \\ 2(1-t+t^2/4)^2, & 0.5858 \leq t < 2, \\ 0, & t \geq 2, \end{cases} \quad (23)$$

using (11).

The graphs of exact value $P(N(t) < 2)$, upper bound $u(t)$ in (21), lower bound $l_1(t)$ in (22) and another lower bound $l_2(t)$ in (23) are given in Figure 3. It is observed from Figure 3 that lower bound in (23) is more strict than the lower bound in (22) given by Barlow and Proschan (1975). The upper and lower bounds for $P(N(t) < 3)$ is shown in Figures 1 and 2 for the df in Example 3 and 4, respectively. It is to be noted that in both cases, the exact value for $P(N(t) < n)$, $n \geq 2$ cannot be obtained explicitly and hence the bounds provide estimates, specially so for Example 4 where the bounds are close enough. The graphs are plotted using R.

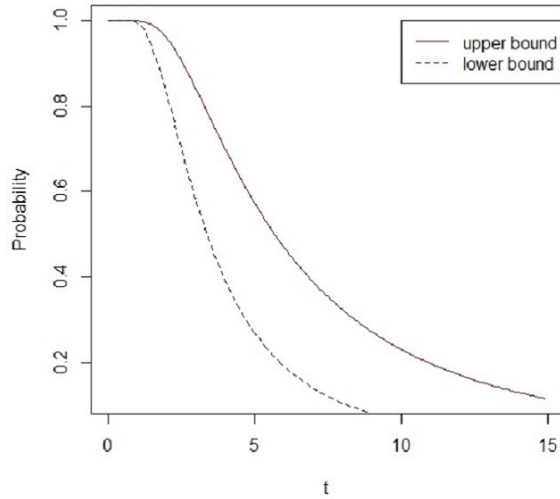


Figure 1: $P(N(t) < 3)$ in Example 3 for $\alpha = 1$

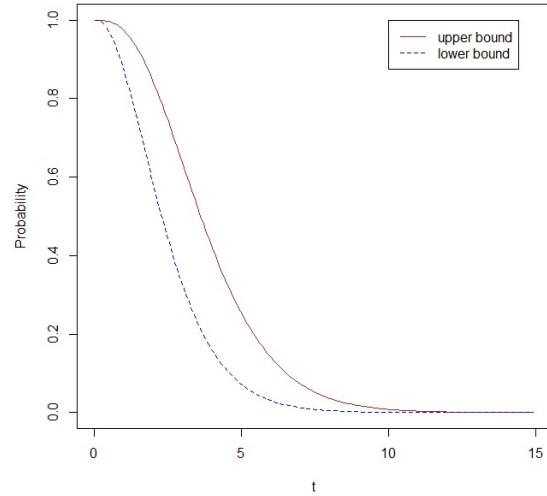


Figure 2: $P(N(t) < 3)$ in Example 4

Example 7: Let F be Weibull df with $F(t) = 1 - e^{-t^2}$, $t \geq 0$. Since $\frac{d^2 F}{dt^2} = 2(1 - 2t^2)e^{-t^2} < 0$ for $t > \frac{1}{\sqrt{2}}$ and $\frac{d^2 F}{dt^2} > 0$ for $t < \frac{1}{\sqrt{2}}$, F is not concave. However, $\frac{d^2 \ln f}{dt^2} = -2 - \frac{1}{t^2} < 0$, so f is log-concave and since f is continuously differentiable, by Bergstrom and Bagnoli (2005), F is log-concave. Therefore applying Corollary 1, we obtain the upper bound for $P(N(t) \geq 3)$

$$P(N(t) \geq 3) \leq \min \left[1 - \left(1 + t^2 + \frac{t^4}{2} \right) e^{-t^2}, \right. \\ \left. \left(1 - e^{-\frac{t^2}{9}} \right)^3 \left(1 - 3 \ln \left(1 - e^{-\frac{t^2}{9}} \right) \right) + \frac{9}{2} \left(\ln \left(1 - e^{-\frac{t^2}{9}} \right) \right)^2 \right]$$

Figure 4 displays the upper bounds from which it is observed that the upper bound in (16) is more strict than the upper bound in (17) given by Barlow and Proschan (1975).

3. Comparing renewal processes with respect to stochastic ordering

Comparison of two renewal processes generated by different renewal distributions is developed in this section on the basis of super-additive/ sub-additive order and convex/ concave transform order. Theorem 6 and Theorem 7 provide bound for the probability distribution of a renewal process $\{N_F(t), t \geq 0\}$ in the form of probability distribution of another renewal process $\{N_G(t), t \geq 0\}$ when the underlying renewal dfs F and G have super-additive/ sub-additive order and convex/ concave transform order respectively.

Theorem 6: Let $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ be two renewal processes generated respectively by dfs F and G which are continuous on $[0, \infty)$ with $F(0) = G(0) = 0$ and $F \leq_{su} (\leq_{sa}) G$, that is, $G^{-1}(F(\cdot))$ is superadditive (subadditive). If $W(t) = G^{-1}(F(t))$, $t > 0$, then, for $n \geq 1$,

$$P(N_F(t) \geq n) \leq (\geq) P(N_G(W(t)) \geq n).$$

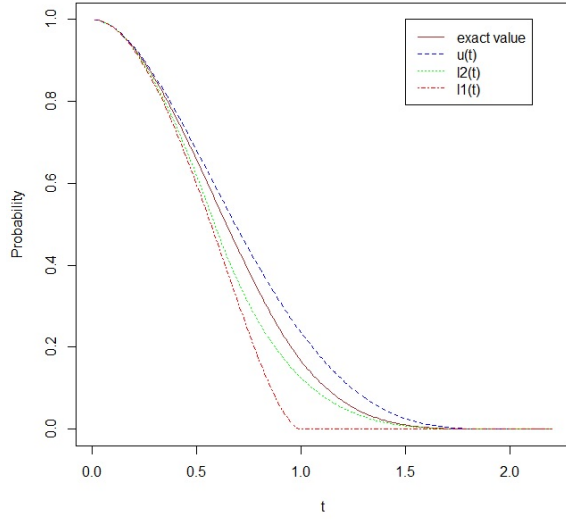


Figure 3: $P(N(t) < 2)$ in Example 6

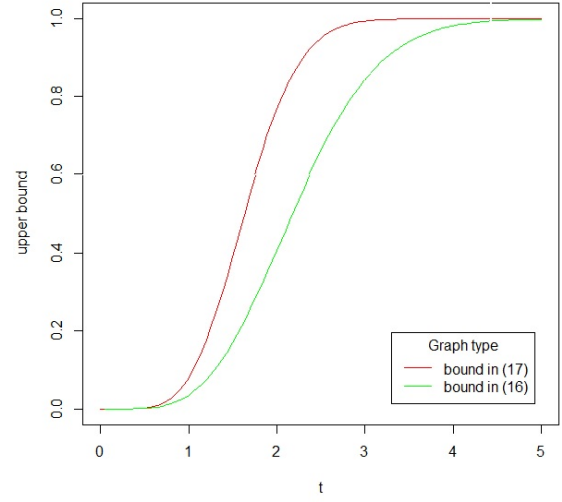


Figure 4: $P(N(t) \geq 3)$ in Example 7

Proof: Since $F \leq_{su} G$, $W^{-1}(\cdot)$ is sub-additive (Østerdal (2006)), that is, $W^{-1}(x + y) \leq W^{-1}(x) + W^{-1}(y)$, $x, y \geq 0$. Let $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ be sequences of iid rvs denoting the inter-arrival times associated with the renewal processes $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ respectively. Then $P(N_F(t) \geq n) = P(S_n \leq t) = F_n(t)$, the n -fold convolution of F with itself, where $S_n = X_1 + \dots + X_n$ is distributed as F_n . Similarly, $P(N_G(t) \geq n) = G_n(t)$. It is enough to show that $F_n(t) \leq G_n(W(t))$. We have $P(X_j \leq t) = F(t) = G(W(t)) = P(Y_j \leq W(t)) = P(W^{-1}(Y_j) \leq t)$ so that

$$\begin{aligned}
 F_n(t) &= P(N_F(t) \geq n) \\
 &= P\left(\sum_{j=1}^n X_j \leq t\right) \\
 &= P\left(\sum_{j=1}^n W^{-1}(Y_j) \leq t\right) \\
 &\leq P\left(W^{-1}\left(\sum_{j=1}^n Y_j\right) \leq t\right) \\
 &= P\left(\sum_{j=1}^n Y_j \leq W(t)\right) \\
 &= G_n(W(t))
 \end{aligned}$$

since $W^{-1}(t)$ is sub-additive. Using similar arguments, it can be proved that, if $F \leq_{sa} G$, that is, $G^{-1}(F(\cdot))$ is sub-additive, then for $t > 0$ and $n \geq 1$,

$$P(N_F(t) \geq n) \geq P(N_G(W(t)) \geq n),$$

completing the proof. □

Theorem 7: Let $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ be two renewal processes generated respectively by two dfs F and G which are continuous on $[0, \infty)$ with $F(0) = G(0) = 0$ and $F \leq_c G$. If $W(t) = G^{-1}(F(t)), t > 0$, then, for $n \geq 1$,

$$P(N_G(W(t)) < n) \leq P(N_F(t) < n) \leq P\left(N_G\left(nW\left(\frac{t}{n}\right)\right) < n\right). \quad (24)$$

The inequalities will reverse when $G \leq_c F$.

Proof: Let $F \leq_c G$ and $W(t) = G^{-1}(F(t)), t > 0$. We first prove that

$$P(N_F(t) < n) \leq P\left(N_G\left(nW\left(\frac{t}{n}\right)\right) < n\right). \quad (25)$$

Note that $W(\cdot)$ is a convex function by definition and $P(N_F(t) \geq n) = P(S_n \leq t) = F_n(t)$, the n -fold convolution of F with itself, $P(N_G(t) \geq n) = G_n(t)$ and it is sufficient to show that $F_n(t) \geq G_n\left(nW\left(\frac{t}{n}\right)\right)$ as in the proof of Theorem 2. Equality holds for $n = 1$ by the definition of $W(\cdot)$. Assuming that the inequality holds for $n - 1$, we have

$$F_n(t) = \int_0^t F_{n-1}(t-x) dF(x) \geq \int_0^t G_{n-1}\left((n-1)W\left(\frac{t-x}{n-1}\right)\right) dG(W(x)). \quad (26)$$

Since $W(\cdot)$ is a convex function,

$$\begin{aligned} \frac{t}{n} &= \left(\frac{n-1}{n}\right) \left(\frac{t-x}{n-1}\right) + \frac{x}{n} \\ \Rightarrow W\left(\frac{t}{n}\right) &\leq \left(\frac{n-1}{n}\right) W\left(\frac{t-x}{n-1}\right) + \frac{W(x)}{n} \\ \Rightarrow (n-1)W\left(\frac{t-x}{n-1}\right) &\geq nW\left(\frac{t}{n}\right) - W(x) \\ \Rightarrow F_n(t) &\geq \int_0^t G_{n-1}\left(nW\left(\frac{t}{n}\right) - W(x)\right) dG(W(x)) \\ &= \int_0^{W(t)} G_{n-1}\left(nW\left(\frac{t}{n}\right) - u\right) dG(u) \\ &\geq \int_0^{nW\left(\frac{t}{n}\right)} G_{n-1}\left(nW\left(\frac{t}{n}\right) - u\right) dG(u) \\ &= G_n\left(nW\left(\frac{t}{n}\right)\right) \end{aligned}$$

using (26) and since $W\left(\frac{t}{n}\right) \leq \frac{W(t)}{n}$ as $W(0) = 0$ and $W(\cdot)$ is convex. Therefore, the inequality also holds for n , and by induction, the proof is complete. Using similar arguments, it can be proved that if $G \leq_c F$, equivalently, $G^{-1}F$ is concave, then for $t > 0$ and $n \geq 1$,

$$P(N_F(t) < n) \geq P\left(N_G\left(nW\left(\frac{t}{n}\right)\right) < n\right).$$

Since $F \leq_c G \Rightarrow F \leq_{su} G$, applying Theorem 6, we get

$$P(N_F(t) < n) \geq P(N_G(W(t)) < n). \quad (27)$$

Combining (25) and (27), we get the (24). Using similar arguments, it can be proved that if $G \leq_c F$ or in other words, $G^{-1}(F(\cdot))$ is concave, then for $t > 0$ and $n \geq 1$,

$$P\left(N_G\left(nW\left(\frac{t}{n}\right)\right) < n\right) \leq P(N_F(t) < n) \leq P(N_G(W(t)) < n),$$

completing the proof. $\square$

Corollary 3: If G in Theorem 7 is exponential so that $\{N_G(t), t \geq 0\}$ is a Poisson process and F is log-concave (log-convex), then the bounds in Theorem 7 are the same as those in Theorem 5(i) (Theorem 5(ii)).

Remark 2: Under the assumption that $W(t)$ is either convex for all $t > 0$ or concave for all $t > 0$, the reverse implications in Theorems 6, 7, are trivially true giving rise to characterization results.

Example 8: Let us consider two renewal processes: $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ where F is Weibull (μ_1, γ) and G is exponential (μ_2) with $\gamma > 1, \mu_1 > 0, \mu_2 > 0$. Then we obtain the following bounds: for $t > 0$ and $n \geq 1$,

$$e^{-\mu_1 t^\gamma} \sum_{j=0}^{n-1} \frac{(\mu_1 t^\gamma)^j}{j!} \leq P(N_F(t) < n) \leq e^{-\left(\frac{\mu_1 t^\gamma}{n^{\gamma-1}}\right)} \sum_{j=0}^{n-1} \frac{\left(\frac{\mu_1 t^\gamma}{n^{\gamma-1}}\right)^j}{j!}.$$

The inequalities will reverse when $0 < \gamma < 1$.

We have $F(t) = 1 - e^{-\mu_1 t^\gamma}$, $G(t) = 1 - e^{-\mu_2 t}$, $W(t) = G^{-1}(F(t)) = -\frac{\ln(1-F(t))}{\mu_2} = \left(\frac{\mu_1}{\mu_2}\right) t^\gamma, t > 0$.

Therefore, $\frac{dW}{dt} = \left(\frac{\gamma \mu_1}{\mu_2}\right) t^{\gamma-1} \Rightarrow \frac{d^2 W}{dt^2} = \left(\frac{\gamma(\gamma-1)\mu_1}{\mu_2}\right) t^{\gamma-2} \Rightarrow \frac{d^2 W}{dt^2} > (<) 0$ for $t \geq 0$,

$\gamma > (<) 1, \mu_1, \mu_2 > 0$. Therefore, $W(\cdot)$ is a convex (concave) function as $\gamma > (<) 1$ and hence we can apply Theorem 7. Since $\{N_G(t), t \geq 0\}$ is a Poisson process with rate μ_2 , by Theorem 7, lower bound $= P(N_G(W(t)) < n) = \sum_{j=0}^{n-1} \frac{(\mu_2 W(t))^j}{j!} e^{-\mu_2 W(t)} = \sum_{j=0}^{n-1} \frac{(\mu_1 t^\gamma)^j}{j!} e^{-\mu_1 t^\gamma} = e^{-\mu_1 t^\gamma} \sum_{j=0}^{n-1} \frac{(\mu_1 t^\gamma)^j}{j!}$. By Theorem 7, upper bound $= P\left(N_G\left(nW\left(\frac{t}{n}\right)\right) < n\right)$

$$= \sum_{j=0}^{n-1} \frac{(\mu_2 nW(\frac{t}{n}))^j}{j!} e^{-\mu_2 nW(\frac{t}{n})} = \sum_{j=0}^{n-1} \frac{\left(\frac{\mu_1 t^\gamma}{n^{\gamma-1}}\right)^j}{j!} e^{-\left(\frac{\mu_1 t^\gamma}{n^{\gamma-1}}\right)} = e^{-\left(\frac{\mu_1 t^\gamma}{n^{\gamma-1}}\right)} \sum_{j=0}^{n-1} \frac{\left(\frac{\mu_1 t^\gamma}{n^{\gamma-1}}\right)^j}{j!}.$$

Example 9: Let us consider two renewal processes $\{N_F(t), t \geq 0\}$ and $\{N_G(t), t \geq 0\}$ where F is Pareto (α, β) and G is exponential (μ) . Then we have the following bounds:

$$\left(1 + \frac{t}{n\beta}\right)^{-n\alpha} \sum_{j=0}^{n-1} \frac{\left(n\alpha \ln\left(1 + \frac{t}{n\beta}\right)\right)^j}{j!} \leq P(N_F(t) < n) \leq \left(1 + \frac{t}{\beta}\right)^{-\alpha} \sum_{j=0}^{n-1} \frac{\left(\alpha \ln\left(1 + \frac{t}{\beta}\right)\right)^j}{j!},$$

$t \geq 0, n \geq 1, \alpha > 0, \beta > 0, \mu > 0$.

We have $F(t) = 1 - \left(\frac{\beta}{\beta+t}\right)^\alpha, t > -\beta, G^{-1}(y) = -\frac{\ln(1-y)}{\mu}, 0 < y < 1, W(t) = G^{-1}(F(t)) =$

$\frac{\alpha}{\mu} \ln\left(1 + \frac{t}{\beta}\right), t > 0$. So, $\frac{dW}{dt} = \frac{\alpha}{\mu} \left(\frac{1}{\beta+t}\right), \frac{d^2 W}{dt^2} = -\frac{\alpha}{\mu} \left(\frac{1}{\beta+t}\right)^2 < 0$ for $t \geq 0, \alpha > 0, \beta > 0, \mu > 0$.

Therefore $W(\cdot)$ is concave and by Theorem 7, since $G \sim$ exponential (μ) , $\{N_G(t), t \geq 0\}$ is a Poisson process with rate μ and $P(N_G(W(t)) < n) = \left(1 + \frac{t}{\beta}\right)^{-\alpha} \sum_{j=0}^{n-1} \frac{\left(\alpha \ln\left(1 + \frac{t}{\beta}\right)\right)^j}{j!}$. By

Theorem 7, $P\left(N_G\left(nW\left(\frac{t}{n}\right)\right) < n\right) = \left(1 + \frac{t}{n\beta}\right)^{-n\alpha} \sum_{j=0}^{n-1} \frac{\left(n\alpha \ln\left(1 + \frac{t}{n\beta}\right)\right)^j}{j!}$.

4. Conclusions

In this paper, we have presented some bounds on the sf and df of the renewal counting variable $N(t)$ when the renewal distribution F is concave, log-concave, convex, log-convex, IFR, DFR, NBU, NWU. The bounds would be useful when the exact value for the probability distribution of a renewal process can not be obtained. The bounds also appears to be more strict than the previously obtained bounds which is illustrated with some numerical examples. Some bounds are also obtained by comparing renewal processes with respect to stochastic order relationship between the inter-arrival times. In future, we plan to develop bounds for the probability distribution of bivariate renewal process based on various forms of the baseline joint distribution function.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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Appendix

Assuming that a rv X has df $F(\cdot)$ and sf $\bar{F}(\cdot)$ with mean μ , $F(0) = 0$, we have used the following definitions (for details see Barlow and Proschan 1975, Shaked and Shanthikumar 2007, Bagnoli and Bergstrom 2005):

1. X / F is said to be new better than used (NBU) or new worse than used (NWU) accordingly as $\bar{F}(x + y) \leq (\geq) \bar{F}(x)\bar{F}(y)$ for all $x, y \geq 0$.
2. X / F is said to be new better than used in expectation (NBUE) or new worse than used in expectation (NWUE) accordingly as $\int_t^\infty \bar{F}(x)dx \leq (\geq) \mu\bar{F}(t)$ for all $t \geq 0$.

3. X / F is said to be increasing failure rate (IFR) or decreasing failure rate (DFR) if and only if $\frac{F(t+x)-F(t)}{F(t)}$ is increasing (decreasing) in t for all $x > 0$.
4. X / F is said to be increasing failure rate average (IFRA) or decreasing failure rate average (DFRA) if $(\bar{F}(x))^{\frac{1}{x}}$ decreasing (increasing) in x for $x > 0$.
5. F is said to be concave (convex) if for any $x, y \geq 0$ and for any $\alpha \in [0, 1]$

$$F((1 - \alpha)x + \alpha y) \geq (\leq) (1 - \alpha)F(x) + \alpha F(y).$$

6. F is said to be log-concave (log-convex) if $\ln F$ is concave (convex).
7. F is said to be super-additive (sub-additive) if $F(x + y) \geq (\leq) F(x) + F(y)$ for all $x, y \geq 0$.
8. For two independent rvs X and Y with respective dfs F and G , X is said to be smaller than Y is convex transform order, denoted $X \leq_c Y$, if $G^{-1}F$ is convex on the support of F .
9. For two independent rvs X and Y with respective dfs F and G , X is said to be smaller than Y is super-additive (sub-additive) order, denoted $X \leq_{su} (\leq_{sa}) Y$, if $G^{-1}F$ is super-additive (sub-additive), or equivalently, $G^{-1}F(x + y) \geq (\leq) G^{-1}F(x) + G^{-1}F(y)$ for all $x, y \geq 0$.