Statistics and Applications {ISSN 2452-7395(online)}
Volume 18, No. 1, 2020 (New Series), pp 63–83

A Reward-Earning Quaternary Random Walk on a Parity Dial

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Received: 26 March 2020; Revised: 15 April 2020; Accepted: 20 April 2020

Abstract

A casino offers a game which involves a symmetric quaternary random walk on a parity dial with twelve nodes labeled as $(1, 11, 3, 9, 5, 7, 6, 8, 4, 10, 2, 0)$, reading clockwise. A player begins at Node 0; she tosses a copper coin to decide whether to move clockwise (if heads) or counterclockwise (if tails); simultaneously she tosses a silver coin to decide whether she will move one step (if tails) or two steps (if heads) in the direction determined by the copper coin. Whenever she lands at a new node she is said to have 'captured' it. If a player intends to capture c nodes and she wishes to toss the coins k times, then her admission fee is $(25 + 25c + k)$ cents (one quarter to play, one quarter per node to capture and one penny per toss). The game ends as soon as either c nodes (other than Node 0) are captured or k tosses are over, whichever event happens earlier; and the player earns as many nickels as the sum of the labels of the captured nodes. How should the player determine c and k?

The player's optimal choices can be derived from the theory of stochastic processes. Alternatively, optimal choices can be anticipated through a computer simulation. Lessons learned from the game empower entrepreneurs and consumers behave optimally to determine when and how to intervene to benefit from an opportunity and/or to prevent a catastrophe.

Key words: Probability mass function; Stopping time; Optimal strategy; Central limit theorem; Law of large numbers.

AMS Subject Classifications: 60G50, 05C81

1. Introduction

When you agree to play a game of chance offered by a casino, you should expect to lose money on average. You accept this loss in anticipation of some entertainment, and a rare possibility of winning big. When millions of players play the game multiple times, the casino makes a positive profit even after paying occasional windfalls, administrative costs, staff salaries, discounts and government taxes. When a game appears to be favorable to the player, it attracts many participants. Of course, if a game were truly favorable to the player,

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the casino would stop offering the game. But if a game only *appears* to favor the player, the casino can entice more players play it more often, and earn more profit for itself. The casino must know ahead of time the exact long-run performance of each game it offers, while the player is oftentimes attracted by the lure of apparent gain. Sarkar (2020 a) introduced such a game of a random walk on a parity dial, and proposed a wide variety of modifications to the game. In this paper, we change the rules of the random moves—from binary walk to quaternary walk—and find the optimal decision for the player.

The game serves as a model for decisions made by entrepreneurs and customers—both parties maximize their gains while abiding by some rules and coping with inherent uncertainty. The optimal decisions for each party may be derived using the theory of stochastic processes. See Ross (1996) for the general theory, and see Sarkar (2006) and Maiti and Sarkar (2019) for random walks on a circle. However, the theory being generally inaccessible to the common person, one can take recourse to a computer simulation involving repeated plays of the game. Lessons learned from the game equip all parties engaged in the marketplace to determine when and how to participate to benefit from an opportunity and/or to prevent a catastrophe. For an optimization problem of a different flavor (investing the smallest amount of input to extract a desired quality of output), see Sarkar (2020 b).

In Section 2, we describe the game of quaternary walk on the parity dial. In Section 3, for $c \leq 3$, we discover the optimal number of tosses k using exact probability distributions. In Section 4, for $4 \leq c \leq 11$, we find the optimal k via simulation. In Section 5, we give some theoretical results and beckon the reader to discover more. Section 6 compares the game of quaternary walk with that of binary walk. In Section 7, we pose some modified games and invite the reader to discover new optimal decisions.

All computations are done using the freeware R. Some codes are given in the Appendix.

2. Rules of the Game

Consider a network of twelve nodes arranged in a circle. The nodes are labeled $\mathbf{l} = (1, 11, 3, 9, 5, 7, 6, 8, 4, 10, 2, 0)$ reading clockwise. See Fig. 1. Note that the labels l_i $(1 \leq i \leq 12)$ are distinct non-negative integers, obtained from the usual dial of a clock by changing the top node from 12 to 0; and interchanging nodes within pairs $(2, 11)$, $(4, 9)$, and $(6, 7)$. Note that all odd values are on the right half (going clockwise from the top), while all even values are on the left half (going counterclockwise from the top) of the dial. Therefore, we call this network the parity dial.

Sarkar (2020 a) studied the following: "A player pays an admission price to play a game of random walks on the parity dial by repeatedly tossing a fair coin. Starting from Node 0, after each toss the player moves one position clockwise (if heads) or one position counterclockwise (if tails); and she captures the visited node. The player has total liberty to determine c, the number of nodes she intends to capture, and k, the number of times she wishes to toss. The game ends as soon as either c nodes (other than Node 0) are captured or k tosses are over. The player pays an admission price of $(25c + k)$ cents, and earns as reward as many nickels as the sum of the labels on the captured nodes. How should the player determine (c, k) to maximize her expected net reward?"

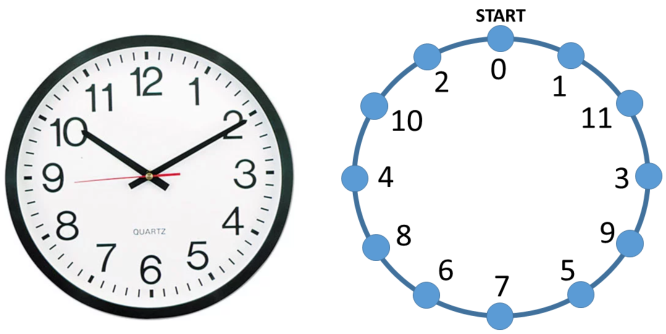

Figure 1: The usual dial of a clock and the parity dial

The answer to that problem is $(6, 28)$ for which the player pays 178 cents and earns 160.7961 cents on average (with a standard deviation (SD) of 0.0243 cents). Therefore, she loses on average 9.66% of her wager. With any other (c, k) game, she will lose even more. In this optimal game, the player tosses on average 16.2 times.

There are many possible modifications to the game. In this paper, we change the nature of the random walk: Instead of going one step clockwise or one step counterclockwise according as the outcome of a toss of a fair coin is heads or tails, we allow each move to be one or two steps clockwise, or one or two steps counterclockwise according to the outcome of tossing two coins simultaneously. We call this modified game the quaternary random walk game and the original game the binary random walk game. Since in the quaternary walk game the player has more opportunities of capturing a new node in each move than in the binary walk game, the admission fee of the quaternary walk game is one quarter more than the admission fee of the binary walk game. The stopping rule and the reward amount remain the same as before. We still ask the same question: How should the player determine (c, k) to maximize her expected net reward?"

More specifically, the player begins at Node 0. She tosses a copper coin and a silver coin simultaneously. The outcome of the copper coin determines whether to move clockwise (if heads) or counterclockwise (if tails). The outcome of the silver coin determines whether she will move one step (if tails) or two steps (if heads) in the direction determined by the copper coin. Whenever she lands at a new node for the first time she is said to have 'captured' it. She does not capture the node she skips over. If a player intends to capture c nodes and she wishes to toss the coins k times, then she must pay an admission fee of $(25 + 25c + k)$ cents (one quarter to avail the quaternary walk game, one quarter per node she wishes to capture and one penny per toss or the pair of coins). At the random time T, when either c nodes (other than Node 0) are captured or k tosses are over, whichever event happens earlier, the game stops; and the player earns as many nickels as the sum of the labels of the captured nodes.

Here's how the game is played: The player begins at Node 0; and after each toss of the copper and the silver coins, the player moves one step clockwise if the outcome is HT, two steps clockwise if HH, or one step counterclockwise if the outcome is TT or two steps counterclockwise if TH; that is, each of the four moves happens with probability $1/4$. She is said to have captured a node on the first visit to it. She does not visit or capture the node she skips over. When the c nodes (other than Node 0) are captured or when the k tosses are over, whichever event happens earlier, the player must stop. Here is a simple way to think about the stopping time T: Toss the coin k times; let Z_t denote the number of nodes captured (other than Node 0) after t tosses (for $t = 1, 2, \ldots, k$). Either the game stops after k tosses, if fewer than c nodes are captured; or it stops as soon as c nodes are captured. That is,

$$T = \begin{cases} \min\{1 \leq t \leq k : Z_t = c\}, & \text{if } Z_k \geq c; \\ k, & \text{if } Z_k < c. \end{cases} \tag{1}$$

Let S_T denote the random set of nodes visited and captured by the random walk on the parity dial when the game ends. The player will earn as many nickels N as the sum of the labels in S_T. Thus, her reward is $N = \sum_{i \in S_T} l_i$ nickels or $5N$ cents, where l_i is the label of Node i. How should the player choose (c, k) to maximize her expected net reward?

3. Analyzing the (c, k) Games for $c = 1, 2, 3$

In this section we study the (c, k) games for $c = 1, 2, 3$, using the exact probability mass function (PMF) of N, the number of nickels earned when the game ends.

3.1 The $(1, 1)$ game

Obviously, $c = 1$ is a terrible choice for the player. For then, she must also choose $k = 1$ toss (since there is no opportunity to toss after capturing one node with the first toss); and she will earn 10, 2, 1, 11 nickels with probability 1/4 each. Therefore, per play she will pay 51 cents; she will earn, on average, $5(10 + 2 + 1 + 11)/4 = 30$ cents; and lose 21 cents—a whopping 41.2% loss!

3.2 Prospects of the $(2, k)$ games

How about choosing $c = 2$? Surely, in this case $k \geq 2$, since with only one toss, it is not possible to capture two nodes. But with only two tosses, there is 3/4 chance of capturing two distinct nodes and a pretty high chance of 1/4 that the player will return to 0 after capturing just one node. With three tosses there is a 1/16 or 6.25% chance of revisiting the already captured node and earn no additional reward. Consider a simple-minded player, Amber, who is contemplating tossing $k = 4$ tosses. She reasons as follows:

> "I will toss the coin $k = 4$ times. There is a very small chance (less than 2%) that I will capture only one node from $\{1, 2, 10, 11\}$. With a high chance I will capture two nodes out of $\{1, 2, 3, 4, 8, 9, 10, 11\}$. Since this set has an average of 6, the two nodes I will capture are worth 12 nickels or 60 cents, on average. Since I have to pay $25 + 25(2) + 4 = 79$ cents, my loss will be about 24.05%. The game is highly unfair! I won't play."

Later that day Amber wondered: "Why is the average sum of all possible pairs 12?" She listed the $\binom{8}{2} = 28$ pairs, computed the within-pair sums and averaged them. Indeed, the average turned out to be 12. (We encourage the reader to verify the same.) However, Amber did not stop there. As she stared at the list, all at once it dawned on her that not all possible pairs of nodes are admissible: In fact, 16 pairs are inadmissible and only 12 pairs are admissible. More precisely, with Node 8 we can pair up only Node 10; with Node 4 we can pair up only Nodes 2 and 10. Similarly, with Node 9 we can pair up only Node 11; with Node 3 we can pair up only Nodes 1 and 11. Finally, we can have all 6 pairs from among nodes $\{1, 2, 10, 11\}$. Therefore, when we capture two nodes, the total number of nickels we will earn are

$$8+10, 4+2, 4+10;\ \ 9+11, 3+1, 3+11;\ \ 1+2, 1+10, 1+11, 2+10, 2+11, 10+11.$$

Thus the sums (after sorting) are $3, 4, 6, 11, 12, 12, 13, 14, 14, 18, 20, 21$. Hence, the average earning is $12 + 1/3$ nickels, or $61 + 2/3$ cents; and the player's loss is about 21.94%. Amber was fascinated with her finding. "What can I do with my discovery?" she asked herself while tossing and turning that night.

> Next day, Amber went to the casino and told the manager: "The $(c = 2, k = 4)$ game allows me to earn 12 nickels on average. So it will be a fair game if you charge 60 cents, instead of 79 cents."
>
> The manager said: "Look, we don't let players dictate games. But I will make an exception for you, Amber, and only for today. Tell you what? I will even give you one free toss. Go ahead, and play the $(c = 2, k = 5)$ game on payment of 60 cents."
>
> Amber was ecstatic! She said to herself: "This is my lucky day! I can earn about 5/3 of a penny per game; or about \$1.66 per 100 games."

Amber jumped to action. However, after playing 100 times, she lost about 5 dollars! What went wrong? Did the casino tamper with the random walk? Amber quit the game; and visited her statistician friend, Staci, for an explanation.

Staci explained that Amber was correct in thinking she will capture two nodes with a high probability. In fact, with 4 tosses the chance of not capturing two nodes is only 1/64; and with 5 tosses it is 1/256. She was also correct in identifying the admissible pairs, whether she tossed 4 times or 5. However, she had blundered in assuming that all admissible pairs are equally likely. They are not! To demystify the reason for her loss, Staci must help Amber understand the exact probability distribution of N, the number of nickels captured.

We already noted that when the copper and the silver coins are tossed for the first time, then N is equally likely to be in $\{10, 2, 1, 11\}$. If the coins are tossed twice, then enumerating all $4^2 = 16$ possible outcomes we see that N takes values

$$18, 14, 12, 10;\quad 6, 12, 2, 3;\quad 3, 1, 12, 4;\quad 11, 12, 14, 20.$$

For $k \geq 3$ tosses, manually enumerating all 4^k outcomes becomes tedious. However, one can write a small program (see the Appendix for a code in our favorite software R) to do the job efficiently, and tabulate the values of N in Table 1. Some theoretical properties of the frequencies in Table 1 are discussed in Section 5. Table 1 shows that the possible values of N are not equally likely, as Amber was prone to assume.

Based on the exact distribution of N, the number of nickels earned, under the $(2, k)$ game, we can compute the expected loss under the publicly available admission price of $75 + k$ cents. This is shown in Table 2. When $c = 2$, the optimal number of tosses where

Table 1: The distribution of number of nickels earned when $c = 2$ and $k \geq 1$

	deno-	N nickels												
k	minator	1	2	3	4	6	10	11	12	13	14	18	20	21
1	4	1	1	0	0	0	1	1	0	0	0	0	0	0
2	16	1	1	2	1	1	1	1	4	0	2	1	1	0
3	64	1	1	10	4	4	1	3	20	2	8	4	4	2
4	256	1	1	42	17	17	1	9	84	8	34	17	17	8
5	1024	1	1	170	68	68	1	35	340	34	136	68	68	34
6	4096	1	1	682	273	273	1	137	1364	136	546	273	273	136

the percentage loss is minimized is $k_* = 4$. Moreover, in Section 5 we will show that as $k \to \infty$, the expected reward increases monotonically (but at a progressively slower rate), until it approaches an asymptotic value of $55 + 1/3$ cents, but the price keeps on increasing linearly. Hence, as k increases, the percentage loss initially decreases steadily (though at a progressively slower rate), and later it monotonically increases until it approaches the asymptotic value of one. As we shall see in Section 4, this property holds for all $c \geq 2$.

Table 2: Expected loss when $c = 2$ and $k \geq 1$

	nickels		cents			
k	mean	SD	E[Reward]	E[Loss]	Price	E[% Loss]
1	6.00	5.52	30.000	46.000	76	60.52
2	9.63	5.81	48.125	28.875	77	37.50
3	10.75	5.54	53.75	24.25	78	31.09
4	10.98	5.44	54.90	24.10	79	30.51
5	11.05	5.42	55.25	24.75	80	30.94
6	11.06	5.41	55.30	25.70	81	31.73
15	11.07	5.40	55.33	34.66	90	38.51

When the casino manager offered the $(c = 2, k = 5)$ game to Amber for an admission fee of 60 cents, he knew quite well that Amber's expected loss will be $60 - 5(11.047) = 4.77$ cents, or 7.95%. Hence, in view of the central limit theorem [see Dudewicz and Mishra (1988), for example], after playing the $(c = 2, k = 5)$ game 100 times, Amber should have expected an approximately normally distributed net loss with a mean of 4.77 dollars and a standard deviation of 0.542 dollars. Amber's actual experience seems to be less than half a standard deviation below the expected value. There is no reason to suspect any foul play on part of the casino. With Staci's expert guidance and some self-study using Wikipedia (2020), Amber learned a whole lot about the central limit theorem. (Readers will act wisely to do the same.)

Let us return to the publicly available $(2, k)$ game with an admission fee of $75 + k$ cents. Regarding the optimality of $k = 4$, we have two additional messages for our simple-minded gambler friend Amber.

First, we should explain to her that $k = 4$ is better than $k = 5$. If after 4 tosses she already captures two nodes, she cannot use her fifth toss at all. The only time she can make use of the fifth toss, is when she captures only one node after four tosses. This means after four tosses she has earned 1, 2, 10, or 11 nickels with probability 4^{-4} each, and returned to Node 0. Using the fifth toss, she can capture a new node with an average node label of $23/4, 22/4, 14/4, 13/4$ respectively for the above four cases. Thus, over and above what she has earned with four tosses, the additional expected earning with the fifth toss is only $4^{-4} \times (23 + 22 + 14 + 13)/4 = 0.07032$ nickels, or 0.3516 cents. This is exactly the amount the casino manager had offered Amber when permitting a free fifth toss. Why should any other player (and Amber on any other day) pay an extra penny at the beginning of the game knowing that on average they will earn about one-third of a penny more?

Second, consider a make-belief scenario to convince Amber why she should not pay for any more than 4 tosses. Suppose that after 4 tosses Amber has captured only one node (and returned to Node 0); and she has earned 1, 2, 10, or 11 nickels, which events happen with probability 4^{-4} each. Suppose also that the casino very generously offers her *at no cost* an *unlimited* number of tosses until she captures a new node! Then Amber is expected to earn an additional $11.07 - 10.98 = .09$ nickels, or 0.45 cents. See a more detailed reason in Section 5. If Amber had to pay even one penny more for these infinitely-many tosses she would certainly lose even more than 7.95% of her wager. If this make-belief scenario is too incredible to be true, we can transform it into a more realistic scenario: At the outset when Amber agrees to pay 60 cents, the casino makes this offer: "Should you fail to capture two nodes with your 4 tosses, we will let you toss an unlimited number of times (until you capture a second node) if you will pay just one penny more right now." However, we have already reasoned that accepting this offer is more disadvantageous to the player than to simply toss 4 times. For there is a high chance that she will forfeit her unlimited number of tosses anyways!

When our gambler friend Amber learns all these truth, having chosen $c = 2$, she should pay for exactly 4 tosses and be prepared to lose roughly 8% of her wager. On any other day, her admission price will be 79 cents, just like for any other player. But will she have the appetite to lose 30.5% per play? Although choosing $(c = 2, k = 4)$ is surely better than choosing $(c = 1, k = 1)$, which had an expected loss of 41.2%, it is not an attractive offer to a gambler. Games that are so unattractive to the gambler are not conducive to the casino's business prospect either. The gambler must inspect other choices.

3.3 Prospects of the $(3, k)$ games

Amber, slightly more enlightened by now, continues to investigate other alternative choices of (c, k) games. She has learned that it is not enough to simply list the possible number of nickels she will earn. It is important to know the associated probabilities also. Correcting her flawed logic in case of $c = 2$ and paying heed to our above messages, Amber might reason as follows:

> "For $c = 1$, I choose $k = 1$ toss; I pay 51 cents, earn 30 cents on average and lose about 41%. For $c = 2$, I just learned that I should choose $k = 4$ tosses, or pay a total of 79 cents. I expect to earn 55 cents on average; and so I lose about 30%. Perhaps for $c = 3$, I ought to choose $k = 3^2 = 9$ tosses, or pay \$1.09. But how can I calculate the actual probability distribution of the reward? I know, I will imitate the R codes for

the ($c = 2, k = 4$) game that Staci gave me and write the codes for the ($c = 3, k = 9$) game."

Amber modified our R codes for the ($c = 2, k = 4$) game and constructed the exact distribution of N for the ($c = 3, k = 9$) game involving $4^9 = 262144$ possible sequences of outcomes shown in Table 3. With PMF so constructed, Amber computed the mean and the SD of the expected reward. (We urge the reader to do the same.)

Table 3: Exact reward distribution, mean and SD for the ($c = 3, k = 9$) game

	N nickels									
value	1	2	3	4	5	6	7	8	9	10
freq	1	1	10	15	0	10223	10208	0	4680	1
value	11	12	13	14	15	16	17	18	19	20
freq	31	5700	33694	35249	31331	31331	1555	15	6235	14903
value	21	22	23	24	25	26	27	28	29	30
freq	10238	21419	19864	6235	10915	0	4680	0	1555	1555

mean(N)=16.68917 SD(N)=5.285686 Total frequency=$4^9 = 262144$

After hanging out with her statistician friend, Staci, long enough, Amber has learned to ask some critical questions. This is what she asked next:

> "Why am I using $k = 9$ tosses? Am I being duped to thinking $k = c^2$ because the formula held true for $c = 1, 2$? But the optimal k may be different. If I choose a higher k, computing the PMF is time consuming. But if I choose a smaller k, I can compute the PMF quickly. In this latter case, although I may earn less, I will also pay less. Maybe I will reduce my percentage loss! Let me try various values of $5 \le k \le 8$ for $c = 3$."

What she found is documented in Table 4.

Table 4: Expected loss when $c = 3$ and $5 \le k \le 9$

	nickels		cents			
k	mean(N)	SD(N)	E[Reward]	E[Loss]	Price	E[% Loss]
5	15.99121	5.698083	79.96	25.04	105	23.85
6	16.36841	5.499275	81.84	24.16	106	22.79
7	16.56256	5.376299	82.81	24.19	107	22.60
8	16.64687	5.317425	83.23	24.77	108	22.93
9	16.68917	5.285686	83.45	25.55	109	23.44

Thus, Amber discovered that for $c = 3$, the optimal choice for k (that minimizes the percentage loss) is not 3^2, rather it is $k = 7$. For the ($c = 3, k = 7$) game, we show the exact PMF in Table 5. In particular, the mean reward is 82.81 cents with a SD of 26.88 cents. Moreover, she wins (earns over \$1.09) with probability $P\{N \ge 22\} = .2487$; she never wins more that half-a-dollar; but she loses over half-a-dollar with probability $P\{N \leq 11\} = .1031$. The player wins over a quarter with probability only $P\{N \geq 27\} = .0294$; but she loses over a quarter with a high probability of $P\{N \leq 16\} = .6266$.

Table 5: Exact reward distribution, mean and SD for the $(c = 3, k = 7)$ game

	N nickels									
value	1	2	3	4	5	6	7	8	9	10
freq	1	1	126	7	0	627	620	0	292	1
value	11	12	13	14	15	16	17	18	19	20
freq	15	544	2050	2145	1919	1919	95	7	387	919
value	21	22	23	24	25	26	27	28	29	30
freq	634	1311	1216	387	679	0	292	0	95	95

mean(N)=16.56256 SD(N)=5.376299 Total freq. $=4^7 = 16384$

4. The Best Choice of (c, k) for $c \geq 4$

Amber has learned the inevitable truth that the reward random walk game on the parity dial is unfavorable to her if she chooses $c = 1, 2, 3$. Perhaps she has already resigned to accepting that every (c, k) game will be unfavorable to her. Consequently, she is willing to tolerate a 10% loss per play, in exchange for the entertainment and thrill she experiences during the game. What if she chooses $c \geq 4$? Amber dived deeper into her thoughts.

> "I see my percentage loss keeps on reducing as I try higher values of c. I must try other values $4 \leq c \leq 11$. Perhaps I can reduce my loss further; maybe I can even earn a positive return on investment! I won't bet on it though; I will be happy if my loss is under 10%. Nonetheless, for each c, finding the corresponding optimal k requires computing the exact PMF of the number of nickels captured in the (c, k) game. However, such computations become exceedingly time consuming as $k \geq 10$ becomes large, since 4^k increases exponentially. What can I do? I suppose I need some help from superwoman. Let me visit Staci once more."

And so she did. Incredibly, Staci had another trick up her sleeve. She said:

> "Amber, you do not need to know the exact PMF. You only need to know the mean (and perhaps the SD) of the reward you will earn. There is a law of large numbers [see Dudewicz and Mishra (1988), for example] that says: "If from any distribution (with a finite expectation), you take many, many (independent) observations, then the long-run sample mean will be close to the expected value of the distribution." So simply replicate the game many times (say, 10^{2m} times); and then compute the mean and the SD of the rewards earned in these plays. That mean will approximate the long run expected reward, with the precision of approximation given by the standard error (SE), which equals SD/10^m. You can learn more about this law of large numbers by reading Wikipedia (2020)."

Amber was so overjoyed to learn about this wonderful trick that she forgot to ask Staci anything about the optimal choice of k. Left to her own devices, she reasoned: "For $c = 2$ and 3, the corresponding optimal k were respectively 4 and 7. I think $k = 1 + c(c+1)/2$ may hold true, in general." Amber modified the codes again to play the game 10^4 times, where each game consists of tossing a pair of fair copper and silver coins (or equivalently, choosing from a discrete uniform random variable which takes values $-2, -1, 1, 2$ equally likely) until either c nodes are captured or $k = 1 + c(c + 1)/2$ tosses are over, whichever event happens first. See the codes for simulation in the Appendix. The average reward and the SD of the rewards earned in these repeated games suffice to approximate the expected loss and the associated SE. Thereafter, it is easy to calculate the expected percentage loss as a fraction of the admission fee. The results of her simulation study are summarized in Table 6.

Table 6: Expected reward and loss to capture $c \geq 3$ nodes in $k = 1 + c(c + 1)/2$ tosses, via simulation based on 10^4 iterations

		cents				
c	k	Price	E[Reward]	SE	E[Loss]	E[% Loss]
2	4	79	54.97	0.27	24.03	30.42
3	7	107	82.80	0.27	24.20	22.61
4	11	136	112.90	0.26	23.10	16.99
5	16	166	143.63	0.26	22.37	13.48
6	22	197	174.83	0.25	22.17	11.25
7	29	229	205.74	0.25	23.26	10.16
8	37	262	236.93	0.24	25.07	†9.57
9	46	296	267.74	0.22	28.24	†9.55
10	56	331	297.44	0.18	33.56	10.14
11	67	367	326.01	0.14	40.99	11.17

Amber concluded that all (c, k) games are unfavorable to her. Only two games were within her tolerance limit of 10% loss—$(8, 37)$, $(9, 46)$—with the latter being slightly preferable. Are Amber's above reasoning justified?

Amazingly, our friend Amber has reasoned very wisely. We applaud her quick understanding of the law of large numbers and her smart implementation of the simulation. Nonetheless, she could have done a little better: Corresponding to each c, instead of relying on her conjecture $k = 1 + c(c + 1)/2$, she should have searched for the optimal k, again via a more thorough simulation study. Then she could discover the best available choice.

While we could simulate the game for all values of $k \geq c$, we follow a *smart search algorithm*. For $c = 2$, we already know the optimal choice is $k = 4$. For $c = 3$, we successively tried $k = 5, 6, 7, 8, 9$. Since the optimal value turns out to be $k = 7$, for the next choice $c = 4$, we should successively try $k \geq 8$. When the optimal value for k is found (by continuing as long as expected percentage loss decreases, and as soon as it begins to increase, by trying out one more value of k to verify that the increasing trend continues), we stop the search. Then we repeat the process for the next value of c starting with the value of k greater than the optimal value for the previous c. In Table 7, we report the performance of the optimal k for each $c \geq 2$, together with the performance of two values of k below and and two values of k above the optimal as demonstration. In summary, the optimal values are

$k_*(c) = (1, 4, 7, 10, 14, 18, 22, 27, 32, 39, 48)$ for $c = 1, 2, \ldots, 11$.

Table 7: Expected reward and expected loss for $2 \le c \le 11$ and associated optimal k_* together with two values below and two values above it

			cents						cents		
c	k	price	E[rew]	E[loss]	E[%loss]	c	k	price	E[rew]	E[loss]	E[%loss]
	2	77	48.10	28.90	37.53		5	105	79.88	25.12	23.92
	3	78	53.73	24.27	31.12		6	106	81.86	24.14	22.77
2	4	79	54.88	24.12	*30.53	3	7	107	82.79	24.21	*22.62
	5	80	55.26	24.74	30.92		8	108	83.26	24.74	22.91
	6	81	55.31	25.69	31.71		9	109	83.42	25.58	23.47
	8	133	109.37	23.63	17.77		12	162	140.13	21.87	13.50
	9	134	111.10	22.90	17.09		13	163	141.40	21.60	13.25
4	10	135	112.18	22.82	*16.90	5	14	164	142.34	21.66	*13.21
	11	136	112.79	23.21	17.07		15	165	142.99	22.01	13.34
	12	137	113.22	23.78	17.36		16	166	143.46	22.54	13.58
	16	191	170.29	20.71	10.84		20	220	199.89	20.11	9.14
	17	192	171.47	20.53	10.69		21	221	201.15	19.85	8.98
6	18	193	172.43	20.57	*10.66	7	22	222	202.10	19.90	*8.96
	19	194	173.19	20.81	10.73		23	223	202.96	20.04	8.98
	20	195	173.79	20.21	10.88		24	224	203.70	20.30	9.06
	25	250	229.90	20.10	8.04		30	280	258.78	21.22	7.58
	26	251	230.98	20.02	7.98		31	281	259.90	21.10	7.51
8	27	252	232.01	19.99	*7.93	†9	32	282	260.96	21.04	†*7.46
	28	253	232.90	20.10	7.94		33	283	261.85	21.15	7.47
	29	254	233.70	20.30	7.99		34	284	262.66	21.34	7.51
	37	312	288.02	23.98	7.69		46	346	314.63	31.37	9.07
	38	313	288.98	24.02	7.67		47	347	315.56	31.44	9.06
10	39	314	289.98	24.02	*7.65	11	48	348	316.51	31.49	*9.05
	40	315	290.82	24.18	7.67		49	349	317.40	31.60	9.05
	41	316	291.66	24.34	7.70		50	350	318.15	31.85	9.10

Based on the results of Table 7, we learn that Amber's conjecture for the optimal k_*, as a function of $c \ge 4$, was wrong. Moreover, we learn that the best choice game is $(c = 9, k = 32)$; and with this choice, a gambler faces a 7.47% expected loss (instead of a 9.55% loss as Amber had anticipated based on Table 6 where she misjudged k to be 46 when $c = 9$). Amazingly, Amber was right in choosing $c = 9$. But she was acting suboptimally by spending $46 - 32 = 14$ cents more to increase her expected reward by only $267.71 - 260.96 = 6.75$ cents. Having discovered the optimal choice, the gambler must stick to playing only the $(c = 9, k = 32)$ game, for any other game would cause her to lose a higher percentage of her wager.

Our increasingly wiser, inquisitive friend Amber cannot stop asking more questions. Here is a sample of questions she asked. (Readers will do well to ask more questions.)

(1) "How many nodes will I actually capture (when I set out to capture 9 nodes)?"

(2) "How many times will I actually toss (for more tosses mean more entertainment)? Equivalently, what is the distribution of the stopping time T defined in Eq. (1)?"

(3) "How much reward will I collect?"

The answers to Amber's questions are not numbers, rather they are random variables that can be described by their PMFs. These PMFs need not be exact; it suffices to estimate them based on simulation. These are reported in Table 8 and Figures 2 and 3 based on simulation involving 10^6 iterations.

Table 8: Frequencies of the number of nodes the gambler captured when playing the $(c = 9, k = 32)$ game 10^6 times

c	k	3	4	5	6	7	8	9	total
9	32	2	245	3590	18864	49448	91004	836847	10^6

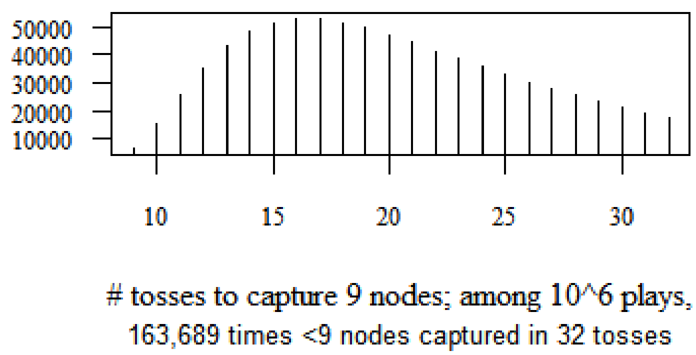

Figure 2: When playing the $(c = 9, k = 32)$ game 10^6 times, the stopping time T has a mean 21.98, SD 7.20, three quartiles 16, 21, 28. Moreover, about 16.4% of times fewer than 9 nodes are captured in all 32 tosses.

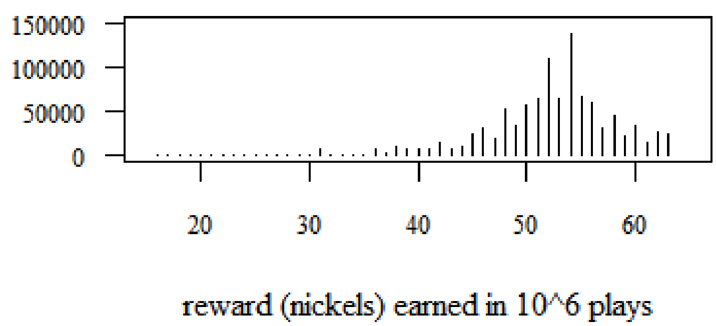

Figure 3: When playing the $(c = 9, k = 32)$ game 10^6 times, the number of nickels the gambler earned has a mean 52.19, SD 5.98 and three quartiles 50, 53, 56.

As we satisfy Amber's curiosity, additional features of the optimal game $(c = 9, k = 32)$ are revealed. For this game, the gambler has about 83.7% chance of capturing all 9 nodes she intended to capture and an additional 9.1% chance of capturing 8 nodes. Thus, there is a 7.2% chance of capturing 7 or fewer nodes when all 32 tosses are over. The mean number of tosses until 9 nodes are captured (or 32 tosses are over) is 21.98, with a standard deviation of 7.20. About 82.0% of times the gambler will have to forgo one or more tosses (for she has already captured 9 nodes), and 16.3% of times she will wish she could toss more often (for she has not captured 9 nodes with 32 tosses). Also, for the same game, while she wagers 282 cents, she has a 19.7% chance of coming out a winner with 57 nickels or more (that is, earning more than her wager). She has a 15% chance of losing over half-a-dollar and a 2.2% chance of losing over a dollar.

5. Theoretical Results

For want of space we refrain from studying at length the manifold theoretical issues. We illustrate only a few theoretical queries to whet the readers' appetite; but we leave many other interesting questions for the readers to pursue on their own.

5.1 Theory when $c = 2$

In Subsection 3.2, we learned that for $c = 2$, the optimal number of tosses is $k = 4$, with a 30.51% expected percentage loss. In particular, this means that the additional expected reward earned by the fifth toss over and above that earned in the first four tosses must be less than 0.695 cents. For otherwise, paying for five tosses would be better than paying for four. In fact, this additional expected reward was shown in Table 2 to be $55.25 - 54.90 = 0.35$. The reason was explained to the gambler as a first message regarding the optimality of $k = 4$. Here we continue the argument by computing the additional expected reward earned in each successive toss, shown in Table 9. There we see that between the fourth and the fifth tosses the exact additional expected reward is $72/4^5 = 9/128$ nickels, or 0.3516 cents. Moreover, the additional expected reward earned by an infinite number of tosses after the first four tosses until two nodes are captured is

$$\left(\frac{72}{4^5} + \frac{58}{4^6}\right) + \left(\frac{72}{4^7} + \frac{58}{4^8}\right) + \ldots = \left(\frac{144}{2^{11}} + \frac{29}{2^{11}}\right) \left[1 + \frac{1}{16} + \frac{1}{16^2} + \ldots\right] = \frac{173}{2^{11}}\,\frac{16}{15} = 0.0901$$

nickels, or 0.4505 cents.

Here is another way to derive the expected reward until the gambler captures two nodes (using an unlimited number of tosses). We consider two mutually exclusive, exhaustive cases.

Case 1: With the first two tosses she captures two distinct nodes (with probability 3/4), and the number of nickels she earns on average is

$$\frac{1}{4}\left[\left(1 + \frac{16}{4}\right) + \left(2 + \frac{15}{4}\right) + \left(10 + \frac{14}{4}\right) + \left(11 + \frac{13}{4}\right)\right] = \frac{1}{4}\left[24 + \frac{58}{4}\right] = 9 + \frac{5}{8}. \tag{2}$$

Case 2: With the remaining probability 1/4, the gambler does not capture two distinct nodes with the first two tosses. In fact, with probability 1/16, she captures Node 1 and then returned to Node 0. Thereafter, how many nickels will she earn on average, if she is allowed an unlimited number of additional tosses until she captures a second node? Call this average number of additional nickels μ_1. Then by conditioning on the next two moves, we see that μ_1 satisfies the following recursive relation

$$\mu_1 = \frac{10 + 2 + 11}{4} + \frac{1}{4}\left[\frac{2 + 11 + 3 + \mu_1}{4}\right].$$

Table 9: Additional reward earned in each successive toss when $c = 2$

k	nickels					cents
1	10	2	1	11	$24/4 = 6$	30
	8	4	2	0		
	4	10	0	1		
	2	0	11	3		
	0	1	3	9		
2	14	15	16	13	$58/4^2 = 29/8$	145/8
	0	10	10	10		
	2	0	2	2		
	1	1	0	1		
	11	11	11	0		
3	14	22	23	13	$72/4^3 = 9/8$	45/8
	8	4	2	0		
	4	10	0	1		
	2	0	11	3		
	0	1	3	9		
4	14	15	16	13	$58/4^4$	145/128
	0	10	10	10		
	2	0	2	2		
	1	1	0	1		
	11	11	11	0		
5	14	22	23	13	$72/4^5$	45/128
6	14	15	16	13	$58/4^6$	145/1024
7	14	22	23	13	$72/4^7$	45/1024
$\vdots$					$\vdots$	$\vdots$
sum					$6 + \frac{29+9}{8} * \frac{16}{15} = 11\frac{1}{15}$	$55\frac{1}{3}$

Solving the above recursive relation, we obtain $\mu_1 = 108/15$ nickels. Likewise, with probability $1/16$, she captures Node 2 and then returns to Node 0; and thereafter, she will eventually earn on average $\mu_2 = 103/15$ nickels. With probability $1/16$, she captures Node 10 and then returns to Node 0; thereafter, she will eventually earn on average $\mu_{10} = 70/15$ nickels. Finally, with probability $1/16$, she captures Node 11 and then returns to Node 0; thereafter, she will eventually earn on average $\mu_{11} = 65/15$ nickels. Thus, in Case 2, the number of nickels the gambler earns on average is

$$\frac{1}{16}\left[\mu_1 + \mu_2 + \mu_{10} + \mu_{11}\right] = \frac{1}{16}\left[\frac{108 + 103 + 70 + 65}{15}\right] = \frac{173}{120}. \tag{3}$$

Adding Eq. (2) and Eq. (3) together, on average the gambler earns

$$9 + \frac{5}{8} + \frac{173}{120} = 9 + \frac{248}{120} = 11\frac{1}{15}$$

nickels, or $55\frac{1}{3}$ cents.

5.1.1 Formula for the PMF of N for the $(c = 2, k)$ game

In Table 1, we documented the distribution of N, the number of nickels earned, for $1 \leq k \leq 6$ using a computer software code. Here we shall discover a pattern among these frequencies and hence write down the formulas in general, so that we can construct the exact distribution of N for larger values of k without having to use the codes. First, from rows corresponding to odd $k = 2i - 1 \geq 3$, we subtract Row 1; and from rows corresponding to even $k = 2i \geq 4$, we subtract Row 2. See Table 10.

Table 10: To discover patterns in the frequencies of N for various $k \geq 1$, subtract the first row from all odd rows 3 or higher and the second row from all even rows 4 or higher.

	N nickels													
k	1	2	3	4	6	10	11	12	13	14	18	20	21	
1	1	1	0	0	0	1	1	0	0	0	0	0	0	
2	1	1	2	1	1	1	1	4	0	2	1	1	0	
3	0	0	10	4	4	0	2	20	2	8	4	4	2	
4	0	0	40	16	16	0	8	80	8	32	16	16	8	
5	0	0	170	68	68	0	34	340	34	134	68	68	34	
6	0	0	680	272	272	0	136	1360	136	544	272	272	136	
⋮							⋮							
⋮							⋮							
$2i-1$	(0	0	5	2	2	0	1	10	1	4	2	2	1	$)\times f_{2i-1}$
$2i$	(0	0	5	2	2	0	1	10	1	4	2	2	1	$)\times f_{2i}$

Then we divide each row 3 or higher by the entry f_i in the last column (under $N = 21$) to see that the quotient vector is a constant! It remains to find a formula for f_i, for then we can reverse the steps (multiply the constant quotient vector by f_i, and add Row 1 or Row 2 according as i is odd or even) to reconstruct all frequencies in each row.

The entry in the last column in any even position is four times the entry in the immediately previous odd position; that is, $f_{2i} = 4f_{2i-1}$. The entries in the odd positions are: $(0, 2, 34, 546, \ldots)$, which satisfy the recursive relation $f_{2i+1} = 16\, f_{2i-1} + 2$, and hence the formula

$$f_{2i-1} = \frac{2}{15}(16^{i-1} - 1). \tag{4}$$

Thereafter, using Eq. (4), we can obtain the limiting probabilities as

$$P\{N = 21 | k = 2i - 1\} = \frac{f_{2i-1}}{4^{2i-1}} = \frac{2}{15}\frac{16^{i-1} - 1}{4^{2i-1}} \to \frac{2}{15}16^{-1/2} = \frac{1}{30};$$

and

$$P\{N = 21 | k = 2i\} = \frac{f_{2i}}{4^{2i}} = \frac{4f_{2i-1}}{4\, 4^{2i-1}} = \frac{f_{2i-1}}{4^{2i-1}} \ldots \to \frac{2}{15}16^{-1/2} = \frac{1}{30}.$$

In particular, as $k \to \infty$, either through even values or through odd values, in the limit the number N of nickels earned takes on values $(3, 4, 6, 11, 12, 13, 14, 18, 20, 21)$ with associated probabilities $(5, 2, 2, 1, 10, 1, 4, 2, 2, 1)/30$. Hence, the limiting mean and SD of N are respectively $11 + 1/15$ and 5.41. Equivalently, the expected reward is $55 + 1/3$ cents with a SD of 27.05 cents.

5.2 Theory when $c = 3$

In Subsection 3.3, we learned that for $c = 3$, the optimal number of tosses is $k = 7$, with a 22.60% expected percentage loss. In particular, this means that the additional expected reward earned by the eighth toss over and above that earned in the first 7 tosses must be less than 0.774 cents. In fact, this additional expected reward was shown in Table 2 to be $83.23 - 82.81 = 0.42$. Likewise, the additional expected reward earned by an infinite number of tosses after the first 7 tosses until three nodes are captured is about $83.64 - 82.81 = 0.83 \pm 0.0263 > 0.774$, obtained by simulating the $(c = 3, k = 20)$ game 10^6 times. This means that to a player who agrees to play the $(c = 3, k = 7)$ game on payment of \$1.07, if the casino offers an unlimited number of tosses until three nodes are captured on payment of just one penny more, then the gambler should take it. But if the charge is two pennies or higher, the gambler should decline the offer.

The exact probability distribution of N, the number of nickels earned until three nodes are captured or k tosses are over, are documented in Table 11 using a modified code along the lines of that used to construct Table 1.

As we demonstrated in Subsection 5.1 for the case of $c = 2$, we now invite the reader to find a formula for the frequencies in Table 11. If the game must stop as soon as the player captures three nodes (using as many tosses as needed), then we conjecture that the PMF of the number of nickels the player will earn is as given in Table 12. If our conjecture holds, then the player will earn on average 16.79 nickels (SD 5.30), or 83.95 cents (SD 26.48 cents).

5.3 Theory when $c = 9$

To a player willing to play the overall optimal game $(c = 9, k = 32)$ with an admission fee of \$2.82 at a 7.46% expected loss, if the casino offers an unlimited number of tosses until nine nodes are captured on payment of 8 cents or less, then the gambler should take it. But if the charge is 9 cents or higher, then the gambler should decline the offer. How did we discover this threshold? We simply estimated the expected rewards of the $(c = 9, k = 100)$ game and the $(c = 9, k = 32)$ game via simulation based on 10^6 plays of each game, and then we computed their difference $269.19 - 260.96 = 8.23$ cents with a SD of .04 cents. It sufficed to consider $k = 100$ because in all but 108 cases of the 10^6 plays of the $(c = 9, k = 100)$ game, all 9 nodes were captured. Among the other 108 cases, 99 times 8 nodes are captured and the remaining 7 times 7 nodes are captured.

6. Binary Vs. Quaternary Random Walks

In this paper, we have studied a game that allows a symmetric quaternary random walk on the parity dial. How does this game compare with the original game of a symmetric binary random walk on the parity dial studied by Sarkar (2020 a)? In Table 13, we summarize the expected performance of the (c, k_*) game for $1 \leq c \leq 11$ and the associated optimal number of tosses k_* side by side for the two types of random walks.

Table 11: Frequencies of N for $c = 3$ and $1 \leq k \leq 9$

row	N\k->	1	2	3	4	5	6	7	8	9
[1,]	1	1	1	1	1	1	1	1	1	1
[2,]	2	1	1	1	1	1	1	1	1	1
[3,]	3	0	2	6	14	30	62	126	254	510
[4,]	4	0	1	1	3	3	7	7	15	15
[5,]	6	0	1	2	10	37	156	627	2544	10223
[6,]	7	0	0	1	7	34	149	620	2529	10208
[7,]	9	0	0	1	4	18	72	292	1168	4680
[8,]	10	1	1	1	1	1	1	1	1	1
[9,]	11	1	1	3	3	7	7	15	15	31
[10,]	12	0	4	13	32	78	196	544	1676	5700
[11,]	13	0	0	5	24	116	492	2050	8340	33694
[12,]	14	0	2	5	29	121	523	2145	8743	35249
[13,]	15	0	0	4	23	109	465	1919	7781	31331
[14,]	16	0	0	4	23	109	465	1919	7781	31331
[15,]	17	0	0	0	1	5	23	95	387	1555
[16,]	18	0	1	1	3	3	7	7	15	15
[17,]	19	0	0	1	5	23	95	387	1555	6235
[18,]	20	0	1	3	14	55	228	919	3712	14903
[19,]	21	0	0	3	9	40	155	634	2543	10238
[20,]	22	0	0	2	15	73	317	1311	5321	21419
[21,]	23	0	0	2	14	68	294	1216	4934	19864
[22,]	24	0	0	1	5	23	95	387	1555	6235
[23,]	25	0	0	2	9	41	167	679	2723	10915
[24,]	27	0	0	1	4	18	72	292	1168	4680
[25,]	29	0	0	0	1	5	23	95	387	1555
[26,]	30	0	0	0	1	5	23	95	387	1555
[All]	sum	4	4^2	4^3	4^4	4^5	4^6	4^7	4^8	4^9

Table 12: A conjecture regarding the limiting distribution, mean and SD of N, the number of nickels earned, for the $(c = 3, k = \infty)$ game

value	6	7	9	13	14	15	16	17	19
freq	7	7	3	22	23	20	20	1	4
value	20	21	22	23	24	25	27	29	30
freq	10	7	14	13	4	7	3	1	1

mean(N)=16.79 SD(N)=5.30 Total freq=167

For symmetric quaternary walk games, the optimal choice is the $(9, 32)$ game with an admission fee of \$2.82 and a 21.04 cents (or 7.46%) expected loss. The same for symmetric

Table 13: Expected percentage loss for $1 \le c \le 11$ and associated optimal k_*

	symmetric binary walk					symmetric quaternary walk				
			cents		%			cents		
c	k_*	price	E[rew]	E[loss]	E[loss]	k_*	price	E[rew]	E[loss]	E[% loss]
1	1	26	7.50	18.50	71.15	1	51	30.00	21.00	41.17
2	6	56	44.31	11.69	20.88	4	79	54.88	24.12	30.53
3	10	85	70.13	14.87	17.50	7	107	82.79	24.21	22.62
4	16	116	102.27	13.73	11.84	10	135	112.18	22.82	16.90
5	22	147	131.53	15.47	10.53	14	164	142.34	21.66	13.21
6	28	178	160.76	17.24	†9.69	18	193	172.43	20.57	10.66
7	36	211	189.72	21.28	10.09	22	222	202.10	19.90	8.96
8	44	244	220.29	23.71	9.72	27	252	232.01	19.99	7.93
9	54	279	248.65	30.35	10.88	32	282	260.96	21.04	†7.46
10	64	314	279.96	34.04	10.84	39	314	289.98	24.02	7.65
11	72	347	304.72	42.28	12.18	48	348	316.51	31.49	9.05

binary walk games is the $(6, 28)$ game with an admission fee of \$1.78 and a 17.20 cents (or 9.66%) expected loss. However, in the optimum quaternary walk game, the gambler tosses the pair of coins on average 21.98 times (with an SD of 7.20), which can be calculated from Table 8, and in the optimum binary walk game, she tosses the coin on average 16.2 times (with an SD of 6.23). Thus, the gambler loses just under a penny per toss in the optimum quaternary walk game, and just over a penny per toss in the optimum binary walk game. The entertainment value (proportional to the number of tosses) of the optimum quaternary walk game is only marginally higher than that of the optimum binary walk game.

7. Modifications to the Game

The reward random walk (binary or quaternary) on the parity dial is designed to educate gamblers make optimal decisions when the casino offers a game. Recognizing that different gamblers may respond differently, the casino may offer modifications to the initial offer—creating new decision-making opportunities. Sarkar (2020 a) proposed four modifications to the binary walk game: Should the player

(a) interchange nodes within any of the pairs $(1, 2), (3, 4), (5, 6), (8, 9), (10, 11)$?
(b) permute nodes $(8, 9, 10, 11)$?
(c) permute nodes $(5, 6, 7)$?
(d) pay an extra fee of $\lceil k/10 \rceil$ cents for the option to sell back at any time the remaining tosses at half-a-penny each?

Here we pose those same modifications to the quaternary walk game. Let us also pose a couple of new modifications:

(1) For the ternary random walk (which goes from any node to its two neighboring nodes and the node diametrically opposite it with probability 1/3 each), how much admission fee (of the form $a_0 + a_1 c + k$, where a_0, a_1 are constants) should the casino charge so that even after making the optimal choice of (c, k), the gambler will lose between 5% and 10% of her wager?

(2) Change the usual dial of a 24-hour clock into a parity dial by replacing 24 by 0, and interchanging the pairs $(2,23),(4,21),(6,19),(8,17),(10,15),(12,13)$. Advice the casino how much admission fee they should charge in order to construct a reasonably attractive game (which is still profitable to the casino) involving either a binary, a ternary or a quaternary random walk on this new parity dial.

We also encourage interested readers to construct new games out of other random walks, such as those in Sarkar (2020 a) and Barhoumi, *et al.* (2020), and study business and economics lessons drawn from them.

Acknowledgments

I thank my student Ms. D. Chatterjee and my colleague Dr. M. Rashid for some discussions and technical assistance. Sincere thanks are due to the Chair Editor for his guidance and encouragement.

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APPENDIX: R Codes

1. Compute exact distribution of N for the $(2,4)$ game

```
### (c=2, k=4)
rf=c(1,11,3,9,5,7,6,8,4,10,2,0)
rew=rep(0,4^4); j=1; toss=c(-2,-1,1,2)
for (x1 in toss){
  for (x2 in toss){
    for (x3 in toss){
      for (x4 in toss){
        u=c(y1,y2,y3,y4)
        cu=cumsum(u); cu=cu-12*floor(cu/12); cu=cu[cu>0]
        f1=min(length(unique(cu)),2); cu=unique(cu)[1:f1]
        rew[j]=sum(rf[cu])
        j=j+1
        } } } }
table(rew)
mean(rew)
sd(rew)
```

2. Simulate random reward earned in any (c,k) game

```
### random reward earned when 4 neighbors are equally likely
rw4=function(f,k){ # f=vertices to capture, k=tosses allowed
  rf=c(1,11,3,9,5,7,6,8,4,10,2,0); toss=c(-2,-1,1,2)
  step=sample(toss,k,replace=T)
  cs=cumsum(step); cs=cs-12*floor(cs/12); cs=cs[cs>0]
  f1=min(length(unique(cs)),f)
  cs=unique(cs)[1:f1]
  sum(rf[cs]) }
data=replicate(10^6, rw4(6, 28)) # vary
mean(data)
sd(data)
table(data)

k=2 # initialize the number of tosses
for (c in 2:11){
  k=k+c; pay=25*(c+1)+k
  data=replicate(10^4,rw4(c,k))
  me=5*mean(data); se=5*sd(data)/10^2
  print( round(c(c, k, pay, me, se, 100*(1-me/pay)), 2) )
  }
```

3. Document and verify optimal k, for each c, via simulation

```
### optimal (c,k) in rw4
k0=c(1, 4, 7, 10, 14, 18, 22, 27, 32, 39, 48)
for (f in 2:11){
  k1=k0[f]-2; k2=k0[f]+2
  for (k in k1:k2){
    data=replicate(10^6,rw4(f,k))
    price=25+25*f+k
    reward=5*mean(data)
    se=5*sd(data)/10^3
    print( round(c(k, price, reward, se, 100*(1-reward/price)),2) ) }
}
```