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Statistical Inference for a One Unit System With Dependent Structure

V.S. Vaidyanathan
Department of Statistics
Pondicherry University, Puducherry, India

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Abstract

Under the assumption that the lifetime and repair time of a one unit system is bivariate exponential, measures of system performance such as system reliability, MTBF, point availability and steady state availability are obtained. Further, a $100(1-\alpha)\%$ asymptotic confidence interval for steady state availability of the system is derived.

Key words: Bivariate exponential distribution; CAN estimator; One unit system; Slutsky theorem; Steady state availability.

AMS Subject Classifications: 60K10, 60F05

1. Introduction

Analysis of one unit repairable system has received considerable attention and has been extensively studied by several researchers in the past. A system is said to be one unit system if it is made up of only one component or it has a single crucial component whose failure causes the system to fail (Barlow and Hunter, 1961). If the lifetime density and repair time density of the unit are assumed to be arbitrary, then one may obtain highly formal expressions for the probability distributions and other quantities of interest. These expressions are rarely suitable for numerical computations. In most of the cases, analytically explicit expressions are obtained only under negative exponential distributional assumptions. PH distribution introduced by Neuts (1975) is more general in the sense that Erlang distribution and negative exponential distribution are only particular cases of continuous PH distribution and this distribution can be used to describe the lifetimes or repair times of a unit when it is non-exponential. Chandrasekhar and Natarajan (2000) have obtained several measures of system performance of a complex one unit system assuming that the lifetime and repair time of the unit has PH distribution with different representations.

Generally speaking, the failure time and repair time are assumed to be independent random variables. However, this assumption of independence need not hold good always. For example, a component that fails frequently has to be thoroughly examined for defects before it is put back into the system so as to prevent future failures, thereby increasing its

Correponding Author: V.S. Vaidyanathan
Email: vaidya.stats@gmail.com

repair time. Thus the repair time and lifetime (failure time) are dependent on each other. Hence, both from theoretical and application perspective, analysing reliability models with dependent structure will be of much use. In the past, a number of bivariate exponential distributions have been proposed and studied well in the literature to describe the lifetimes of two unit systems. But the bivariate exponential distribution proposed by Marshall and Olkin (1967) is widely used among researchers because of its properties. An attempt is made in this paper to derive performance measures of one unit system under the assumption that the lifetime and repair time of the unit are governed by Marshall-Olkin bivariate exponential distribution. Also, point and interval estimation of steady state availability of the system is carried out.

2. Model (One Unit System With Dependent Structure)

The system description and assumptions involved are given below:

The system under consideration consists of only one unit with dependent structure and when it fails, it is taken up for repair instantaneously. Let T and R denote respectively the lifetime and repair time of the failed unit. Since the lifetime and repair time is assumed to be dependent, Marshall-Olkin bivariate exponential distribution for T and R with the survival function given by

$$\bar{F}(t,r) = e^{-[\lambda_1 t + \lambda_2 r + \lambda_3 max(t,r)]}, t, r > 0; \lambda_1, \lambda_2 > 0, \lambda_3 \geq 0 \tag{1}$$

is considered. see Marshall and Olkin (1967).

The stochastic process underlying the behaviour of the system is an alternating renewal process. At time $t = 0$, the unit just begins to operate. It should be noted that

- The lifetime T and repair time R are exponential random variables each with parameters $(\lambda_1 + \lambda_3)$ and $(\lambda_2 + \lambda_3)$ respectively.
- $E(T) = \frac{1}{(\lambda_1+\lambda_3)}$ and $E(R) = \frac{1}{(\lambda_2+\lambda_3)}$.
- $V(T) = \frac{1}{(\lambda_1+\lambda_3)^2}$ and $V(R) = \frac{1}{(\lambda_2+\lambda_3)^2}$.
- The covariance between T and R is given by $Cov(T,R) = \frac{\lambda_3}{(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}$.
- T and R are independent if and only if $\lambda_3 = 0$.

3. Operating Characteristics of The System

In this section, various measures of performance that describe the operating characteristics of the system are discussed. To analyse the system, we note that at any given time t, the system will be found in any one of the following mutually exclusive and exhaustive states namely, state 0: the unit is operating (online) and state 1: the unit is under repair.

Here state 0 denotes the system upstate and state 1 denotes the system down state. Let $X(t)$ denote the state of the system (unit) at time t. Clearly, $\{X(t), t \geq 0\}$ is a continuous time Markov process with the state space given by $E = \{0, 1\}$. Since bivariate Marshall and Olkin distribution satisfy the bivariate lack of memory property and its marginal distributions are exponential, it follows that $\{X(t), t \geq 0\}$ is a Markov process with infinitesimal generator Q given by

$$Q = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} \quad 0 & \qquad\qquad 1 \end{array} \\ \begin{pmatrix} -(\lambda_1+\lambda_3) & (\lambda_1+\lambda_3) \\ (\lambda_2+\lambda_3) & -(\lambda_2+\lambda_3) \end{pmatrix} \end{array}. \tag{2}$$

Let $p_i(t) = P[X(t) = i], i = 0, 1$ represent the probability that the system is in state i at time t with the initial condition $p_0(0) = 1$.

3.1. System reliability

Since system reliability $R(t)$ is the probability of failure free operation of the system in the time interval $(0, t]$ and the marginal distribution of T is exponential with mean $(\lambda_1+\lambda_3)$, an expression for $R(t)$ can be readily obtained as

$$R(t) = e^{-(\lambda_1+\lambda_3)}. \tag{3}$$

Also, the system Mean Time Before Failure (MTBF) is given by

$$MTBF = \int_0^\infty R(t)dt = \frac{1}{(\lambda_1+\lambda_3)}. \tag{4}$$

3.2. Point and steady state availability of the system

The system availability $A(t)$ is the probability that the system operates within the tolerances at a given instant of time t and is obtained as follows:

From the infinitesimal generator given in (2), the following system of differential – difference equations are obtained.

$$dp_0(t)dt = -(\lambda_1+\lambda_3)p_0(t) + (\lambda_2+\lambda_3)p_1(t) \tag{5}$$

$$dp_1(t)dt = (\lambda_1+\lambda_3)p_0(t) - (\lambda_2+\lambda_3)p_1(t) \tag{6}$$

Let $L_i(s)$ be the Laplace transformation of $p_i(t), i = 0, 1$. Taking Laplace transforms on both the sides of the above differential-difference equations and solving for $L_i(s), i = 0, 1$ and inverting, we get $p_i(t), i = 0, 1$ as follows:

$$P_0(t) = \frac{(\lambda_2+\lambda_3)}{[(\lambda_2+\lambda_3)+(\lambda_1+\lambda_3)]} + \frac{(\lambda_1+\lambda_3)}{[(\lambda_2+\lambda_3)+(\lambda_1+\lambda_3)]}e^{-[(\lambda_2+\lambda_3)+(\lambda_1+\lambda_3)]t} \tag{7}$$

$$P_1(t) = \frac{(\lambda_1+\lambda_3)}{[(\lambda_2+\lambda_3)+(\lambda_1+\lambda_3)]} - \frac{(\lambda_1+\lambda_3)}{[(\lambda_2+\lambda_3)+(\lambda_1+\lambda_3)]}e^{-[(\lambda_2+\lambda_3)+(\lambda_1+\lambda_3)]t} \tag{8}$$

Hence, the point availability of the system is given by $A(t) = p_0(t)$. The system steady state availability is the expected fractional amount of time in a continuum of operating time that the system is in upstate and is given by

$$A_\infty = \lim_{t\to\infty} A(t) = \frac{(\lambda_2+\lambda_3)}{[(\lambda_1+\lambda_3)+(\lambda_2+\lambda_3)]}. \tag{9}$$

In the next section, point and interval estimation of the steady state availability of the system is carried out.

4. Point and Interval Estimation of A_∞

Let $(Y_{1i}, Y_{2i}), i = 1, 2, ..., n$ be a random sample of size n from a bivariate exponential population with the survival function given by (1). Here Y_1 denote the lifetime of the unit and Y_2 denote the repair time of the unit upon failure. Let $\bar{Y}_1$ and $\bar{Y}_2$ denote the corresponding sample means.

4.1. Point estimator of A_∞

It can be established that $\bar{Y}_1$ and $\bar{Y}_2$ are the moment estimators of $\frac{1}{(\lambda_1+\lambda_3)}$ and $\frac{1}{(\lambda_2+\lambda_3)}$ respectively. Writing $\theta_1 = \frac{1}{(\lambda_1+\lambda_3)}$ and $\theta_2 = \frac{1}{(\lambda_2+\lambda_3)}$, A_∞ of the system given in (9) reduces to

$$A_\infty = \frac{\theta_1}{\theta_1+\theta_2}. \tag{10}$$

Hence, a point (moment) estimator of the steady state availability A_∞ of the system is given by

$$\hat{A}_\infty = \frac{\bar{Y}_1}{\bar{Y}_1+\bar{Y}_2}. \tag{11}$$

It may be noted that $\hat{A}_\infty$ given in (11) is a real valued differentiable function in $\bar{Y}_1$ and $\bar{Y}_2$. Applying multivariate central limit theorem, it is readily seen that
$\sqrt{n}[(\bar{Y}_1, \bar{Y}_2) - (\theta_1, \theta_2)] \xrightarrow{d} N_2(0, \Sigma)$ as $n \to \infty$, where the dispersion matrix Σ is given by

$$\Sigma = \begin{array}{c} \\ \bar{Y}_1 \\ \bar{Y}_2 \end{array} \begin{array}{c} \begin{array}{cc} \bar{Y}_1 & \bar{Y}_2 \end{array} \\ \begin{pmatrix} \theta_1^2 & \frac{\lambda_3\theta_1^2\theta_2^2}{\theta_1+\theta_2-\lambda_3\theta_1\theta_2} \\ \frac{\lambda_3\theta_1^2\theta_2^2}{\theta_1+\theta_2-\lambda_3\theta_1\theta_2} & \theta_2^2 \end{pmatrix} \end{array} \tag{12}$$

Applying the results in Chapter 6 of Rao (2009), we have

$\sqrt{n}(\hat{A}_\infty - A_\infty) \xrightarrow{d} N(0, \sigma^2(\theta))$, where $\theta = (\theta_1, \theta_2)$ and

$$\sigma^2(\theta) = \sum_{i=1}^{2} \theta_i^2 \left(\frac{\partial A_\infty}{\partial \theta_i}\right)^2 + \frac{2\lambda_3\theta_1^2\theta_2^2}{\theta_1+\theta_2-\lambda_3\theta_1\theta_2}\left(\frac{\partial A_\infty}{\partial \theta_1}\right)\left(\frac{\partial A_\infty}{\partial \theta_2}\right). \tag{13}$$

The expressions for the partial derivatives are obtained as

$$\frac{\partial A_\infty}{\partial \theta_1} = \frac{\theta_2}{(\theta_1+\theta_2)^2}$$

and

$$\frac{\partial A_\infty}{\partial \theta_2} = \frac{-\theta_1}{(\theta_1+\theta_2)^2}.$$

By substituting the partial derivatives in (13) and simplifying, we get

$$\sigma^2(\theta) = \frac{2\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4} - \frac{2\lambda_3\theta_1^2\theta_2^2}{(\theta_1+\theta_2)^4(\theta_1+\theta_2-\lambda_3\theta_1\theta_2)}. \tag{14}$$

Thus, $\hat{A}_\infty$ is a consistent and asymptotic normal (CAN) estimator of A_∞.

4.2. Interval estimation of A_∞

Let $\hat{\sigma^2}$ be an estimator of $\sigma^2(\theta)$ obtained by replacing θ by its consistent estimator namely, $\hat{\theta} = (\bar{Y_1}, \bar{Y_2})$. Since $\sigma^2(\theta)$ is a continuous function of θ, $\hat{\sigma^2}$ is a consistent estimator of $\sigma^2(\theta)$ i.e., $\hat{\sigma^2} \xrightarrow{p} \sigma^2(\theta)$ as $n \to \infty$. Thus, applying Slutsky theorem, we get,

$$\frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\sqrt{\hat{\sigma^2}}} \xrightarrow{d} N(0,1).$$

Hence, for $\alpha \in (0,1)$, we have

$$P\left(-Z_{\frac{\alpha}{2}} < \frac{\sqrt{n}(\hat{A}_\infty - A_\infty)}{\sqrt{\hat{\sigma^2}}} < Z_{\frac{\alpha}{2}}\right) = (1-\alpha)$$

where $Z_{\frac{\alpha}{2}}$ denote the upper $(\frac{\alpha}{2})^{th}$ percentile point of the standard normal distribution. Thus, a $100(1-\alpha)\%$ confidence interval for the steady state availability A_∞ of the system is given by $\hat{A}_\infty \pm Z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{\sigma^2}}{n}}$, where $\hat{\sigma^2}$ is obtained from (14).

5. Conclusion

Inferential aspects of performance measures within the framework of reliability models with dependent structure is less addressed in statistical literature. The present article has focused on point and interval estimation of system reliability and steady state availability of a one unit system assuming the lifetime and repair time to be modelled by bivariate Marshall-Olkin distribution. Such models can be applied in reliability studies involving electrical devices like, for example, water pumps used in commercial, agricultural and domestic activities, ceiling fans, wherein lifetime and repair time of the motor are dependant. The methodology adapted can be applied to models involving more than one component by using multivariate Marshall-Olkin distribution for modelling the dependency. Interested readers are encouraged to go through Yadavalli *et al.* (2017), Vaidyanathan and Chandrasekhar (2018) and the references cited therein.

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