Statistics and Applications {ISSN 2454-7395 (online)}
Volume 22, No. 2, 2024 (New Series), pp 91–108
http://www.ssca.org.in/journal

A New Lifetime Distribution: Statistical Inference and it's Applications

Arvind Pandey, Pawan Kumar Singh and Mahendra Saha
Department of Statistics, Central University of Rajasthan, India

Received: 14 March 2022; Revised: 16 October 2023; Accepted: 30 November 2023

Abstract

We have proposed a single parameter lifetime distribution function that has increasing and decreasing hazard rates. The proposed distribution can be used as a heavy-tailed alternative to the exponential, Weibull and gamma distributions. We discuss the different statistical properties, survival characteristics and stress-strength reliability. Different estimation procedures are used to estimate the parameter by using different methods which are given as estimation based on percentiles, least squares estimators, weighted least squares estimators, maximum likelihood estimators, Cramer-von-mises method of estimation and maximum product of the spacing method of estimation. We have compared the performance of these estimators with the help of simulation study. Also for different values of parameter we have found the estimates of hazard rate function and survival function. The proposed distribution is fitted to real data sets and it is observed that the proposed distribution is fitting quite well to the data.

Key words: Exponential distribution; Least squares estimation; Lifetime distribution; Maximum likelihood estimation; Maximum product spacing; Weighted least squares estimation

AMS Subject Classifications: 62K05, 05B05

1. Introduction

Lifetime distributions are common statistical tools used for the modeling and analysis of lifetime phenomena for different characteristics of lifetime data sets. The statistical literature contains very sophisticated multi-parameter distributions to analyze different kinds of data sets. Johnson *et al.* (1995) and Mann *et al.* (1974) discuss the importance of exponential distribution which is a single parameter distribution. The hazard rate of the exponential distribution is constant, which restricts its use in lifetime data analysis.

We are proposing a new single parameter distribution. The proposed distribution is obtained as a survival of a series system, which consists of two components, where one component follows inverse exponential and another follows Lomax distribution. Also, for the proposed distribution, we have studied the distributional properties of the series system.

Corresponding Author: Arvind Pandey
Email: arvindmzu@gmail.com

The proposed distribution has increasing hazard rate, decreasing hazard rate and first increasing then decreasing hazard rate. Also, it can be used as a heavy-tailed alternative to the exponential, Weibull and gamma distributions. This motivates us to introduce a new distribution and study some of its statistical properties. The proposed distribution can be used as a heavy-tailed alternative to the exponential, Weibull and gamma distributions.

The cumulative distribution function (CDF) of the proposed distribution is obtained by multiplying Lomax distribution and inverse exponential

$$F(x \mid \theta) = \begin{cases} \left(1 - \frac{\theta}{x+\theta}\right)\left(e^{-\theta/x}\right) & ; x \geq 0, \theta > 0 \\ 0 & ; \text{otherwise} \end{cases} \tag{1}$$

the corresponding probability density function (PDF) is given as,

$$f(x \mid \theta) = \begin{cases} \frac{\theta(2x+\theta)}{x(x+\theta)^2} e^{-\theta/x} & ; x \geq 0, \theta > 0 \\ 0 & ; \text{otherwise} \end{cases} \tag{2}$$

the hazard rate function (HRF) of the distribution for given t and it is denoted as h(t)

$$h(t) = \frac{f(t)}{S(t)} = \frac{\theta(2t+\theta)}{t(t+\theta)(te^{\theta/t} + \theta e^{\theta/t} - t)} \tag{3}$$

and survival function (SF) and it is denoted as S(t)

$$S(t) = 1 - \frac{t}{t+\theta} e^{-\theta/t} \tag{4}$$

the corresponding reversed hazard rate and it is denoted as r(t),

$$r(t) = \frac{f(t)}{F(t)} = \frac{\theta(2t+\theta)}{t^2(t+\theta)}$$

similarly, the cumulative hazard function and it is denoted as H(t)

$$H(t) = -log[S(t)] = -log\left[1 - \frac{t}{(t+\theta)} e^{-\theta/t}\right]$$

The plots of different characteristics of the proposed distribution are given in Figure (1) distribution function, Figure (2) probability density function, Figure (3) survival function and Figure (4) is hazard function. Figure (4) shows that the hazard rate first increases and then decreases for a given θ. This distribution can found its use in wide applications such as the analysis of the business failure lifetime data, income and wealth inequality, size of cities, actuarial science, medical and biological sciences, engineering, lifetime and reliability modeling.

In this article, we have proposed a new single parameter probability distribution and discuss its statistical properties. In section 2, we find the r^{th} moments and moments exist only for $r < 1$ and discuss the quantile function, skewness and kurtosis and also discuss the entropy of distribution.

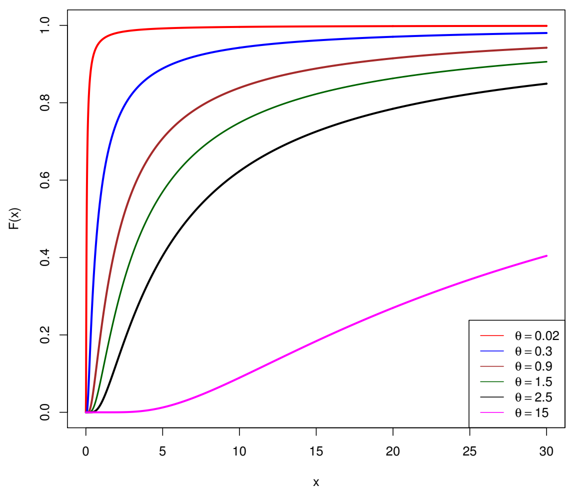


Figure 1: Various CDF forms of proposed distribution

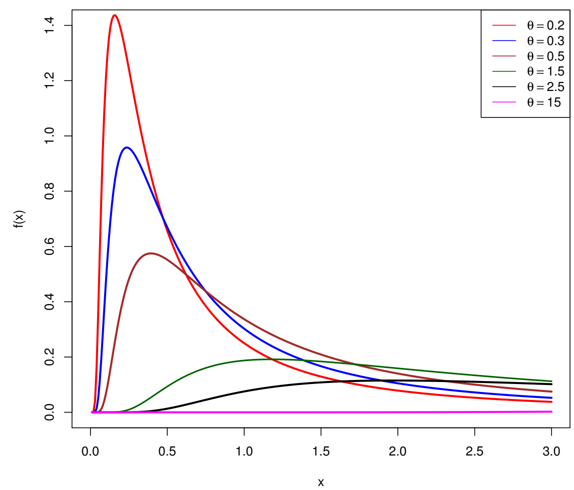


Figure 2: Various PDF forms of proposed distribution

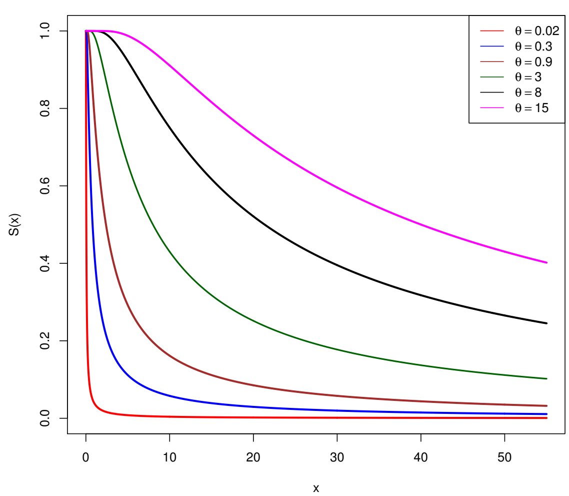


Figure 3: Various SF forms of proposed distribution

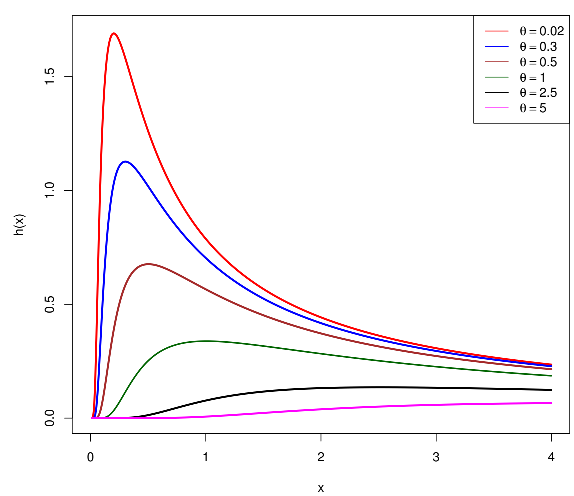


Figure 4: Various HRF forms of proposed distribution

In section (3) and section (4) we discuss the order statistics and stress strength reliability respectively. Similarly, in section (5) we discuss the different methods of estimation to estimate the parameter. In section (6), we mainly compare the different methods of estimations such as maximum likelihood estimators (MLE), estimators based on percentiles (PCE), least squares estimators (LSE), Weighted least squares estimators (WLSE), Cramer-von-Mises method of estimation (CME) and the maximum product of spacings method of estimations (MPSE), by mean squared errors (MSE) using extensive simulation techniques, Similarly, we estimate the HRF and SF. Real-life data applications are presented and discussed in section (7) and Concluding remarks can be found in section (8)

2. Some statistical properties

In this section, we discuss the different statistical properties, viz, moments, quantile function, skewness, kurtosis and entropy.

2.1. Moments

The r^{th} moments about origin is

$$E[X^r] = \int_0^\infty x^r \frac{\theta(2x+\theta)}{x(x+\theta)^2} e^{-\theta/x}\, dx \tag{5}$$

the moments exits only for $r < 1$ and $r \geq 0$ does not exits (see Appendix[1])

$$E[X^r] = \theta^r \left[\sum_{k=0}^{r} \binom{r}{k} (-1)^{(r-k)} \Gamma(k,1) + \sum_{k=1}^{r-1} \binom{r-1}{k} (-1)^{(r-k)} \Gamma(k,1) \right]$$

where, Γ(k,α) is the upper incomplete gamma function defined by

$$\Gamma(k,\alpha) = \int_\alpha^\infty e^{-x} x^{k-1} dx \qquad \alpha \geq 0$$

An integral in equation (5) is evaluated by mathematically (see Fisher and Kılıcman (2012)) and also computational method using the Monte Carlo method. First, we draw a sample from the proposed distribution with parameter $\theta = 1.5$ with sample size n. We calculate the sample mean $E[x^r] = \frac{1}{n}\sum_{i=1}^{n} x_i^r$. The sample mean values are plotted with respect to different sample sizes. For $r = 0.2$, it is convergent which is shown in the Figure (5).

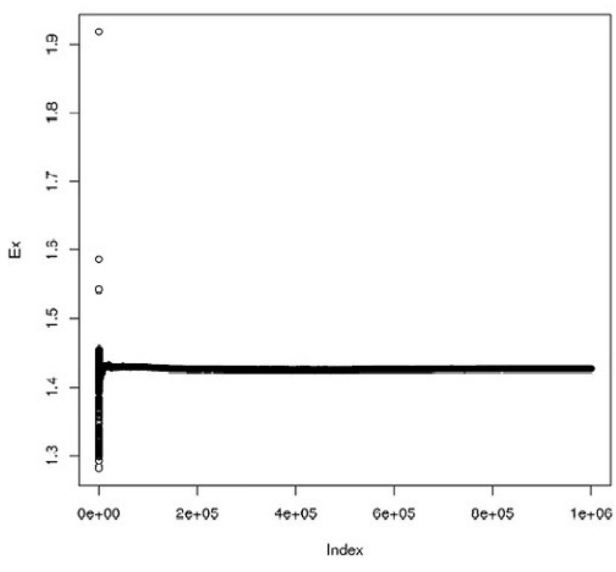


Figure 5: Moments at r=0.2

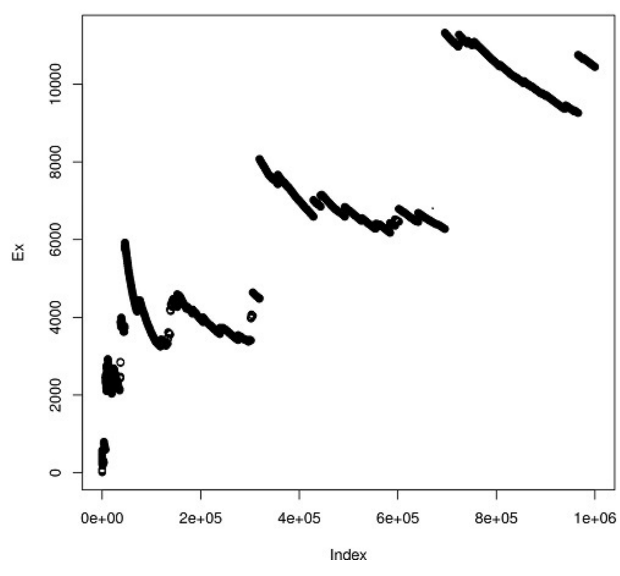


Figure 6: Moments at r=1.58

The value of the integration for $\theta = 1.5$ is 1.44. Which can be seen in Figure (5). The above integral equation (5) is not convergent for $r \geq 1$. This integral is evaluated by using the Monte Carlo method for $\theta = 1.5$ and $r = 1.5$. Observing the plot in Figure (6) the integration is not convergent, as can be seen from Figure (6).

2.2. Quantile function

The quantile function $Q(p)$ can be obtained by using the equation (1)

$$p = \frac{Q(p)}{Q(p)+\theta} e^{-\theta/Q(p)} \tag{6}$$

where Q(p) is the quantile of order p and $0 < p < 1$. If we put p=$\frac{1}{2}$ in equation (6) then we get the value of the median. Since moments of the proposed distribution does not exist, we can use the Moor measures of kurtosis (see Moors (1988)) and the Bowley measures of skewness(see Bowley (1920)) based on quantile and corresponding expressions are given in equation (7) and (8), respectively

$$K = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{3})} \tag{7}$$

$$S = \frac{Q(\frac{3}{4}) - 2Q(\frac{1}{2}) + Q(\frac{1}{4})}{Q(\frac{3}{4}) - Q(\frac{1}{4})} \tag{8}$$

2.3. Entropy

The concept of information entropy was introduced by Shannon (1948). Entropy measures the expected amount of information or "uncertainty" inherent in the possible outcomes of the variable. If the entropy is high then it indicates higher uncertainty.

2.4. Shannon entropy

Shannon's Entropy is simply the amount of information contained in a variable. It is defined as $H(f) = E\left[-log f(x)\right]$,

$$H(f) = E\left[-log\left[\frac{\theta(2x+\theta)}{x(x+\theta)^2} e^{-\theta/x}\right]\right] \tag{9}$$

To calculate the Shannon entropy by equation (9), solving by Monte Carlo integration method. The generate the x_i from proposed distribution and calculating,
$\frac{-1}{n}\sum_{i=1}^{n} log\left[\frac{\theta(2x_i+\theta)}{x_i(x_i+\theta)^2} e^{-\theta/x_i}\right]$, we see that above Figure (7) value converges at 3.24 for given value of $\theta = 1.5$.

2.5. Renyi entropy

In the information theory, if X is a random variable with density function f(x), the Renyi entropy is a measure of uncertainty of the random variable defined and it is denoted as $H(\gamma)$

$$H(\gamma) = \frac{1}{1-\gamma} log\left(\int_0^\infty \left[f(x)\right]^\gamma dx\right)$$

$$H(\gamma) = \frac{1}{1-\gamma} log\left(\int_0^\infty \left[\frac{\theta(2x+\theta)}{x(x+\theta)^2} e^{-\theta/x}\right]^\gamma dx\right) \tag{10}$$

Similarly, we can calculate the integral of equation (10) by the same method we see that the integral does not converge because in the Figure (8) plot does not converge at any point and mathematically when moments does not exits for $r \geq 1$ then Renyi entropy does not exits.

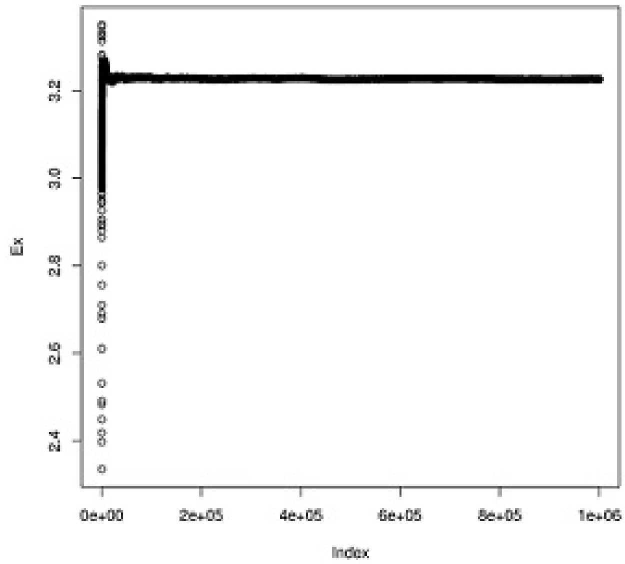


Figure 7: Shannon Entropy

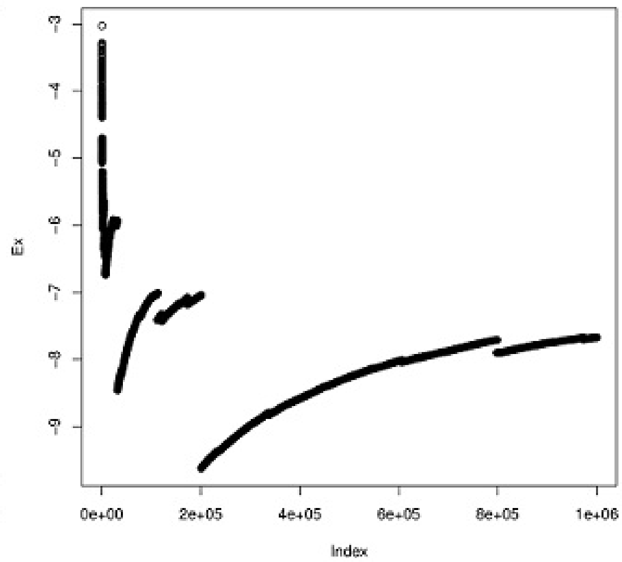


Figure 8: Renyi Entropy

2.6. Regularly varying tail behavior of the proposed distribution

A distribution function F with survival function (SF), $S(y) = 1 - F(y)$, is said to be heavy-tailed if for every t ≥ 0, $\lim_{y\to\infty} \frac{1-F(y)}{e^{-ty}} = \infty$ (see section 2.4 of Rolski *et al.* (2009)). A distribution function F is said to belong to the regularly varying class if

$$\lim_{y\to\infty} \frac{1 - \frac{y}{y+\theta} e^{-\theta/y}}{e^{-ty}}$$

$$\lim_{y\to\infty} \frac{e^{tx}}{(y+\theta)^2} \lim_{y\to\infty} \frac{\theta(2 + \frac{\theta}{y}) e^{\frac{-\theta}{y}}}{2t\theta}$$

$$\lim_{y\to\infty} \theta t e^{ty} = \infty \qquad \forall \;\; t \geq 0$$

3. Distributions of order statistics and ordering property

First, we know the PDF, CDF and the moment of the i^{th} order statistics $x_{(i)}$. Let $X_1, X_2, ..., X_n$ are random samples of size n from the proposed continuous distribution, then the PDF of order statistics is given by the following formula

$$f_{(i,n)}(x \mid \theta) = \frac{n!}{(n-i)!(i-1)!} \left[\frac{x}{x+\theta} e^{-\theta/x} \right]^{i-1} \left[1 - \frac{x}{x+\theta} e^{-\theta/x} \right]^{n-i} \frac{\theta(2x+\theta)}{x(x+\theta)^2} e^{-\theta/x}$$

The PDF of maximum of $X_1, X_2, ..., X_n$ is

$$f(x_{(n)} \mid \theta) = n\,[F(x)]^{n-1}\,f(x) = n\left[\frac{x}{x+\theta}e^{-\theta/x}\right]^{n-1}\frac{\theta(2x+\theta)}{x(x+\theta)^2}e^{-\theta/x}$$

The PDF of minimum of $X_1, X_2, ..., X_n$ is

$$f(x_{(1)} \mid \theta) = n\,[1-F(x)]^{n-1}\,f(x) = n\left[1-\frac{x}{x+\theta}e^{-\theta/x}\right]^{n-1}\frac{\theta(2x+\theta)}{x(x+\theta)^2}e^{-\theta/x}$$

The joint CDF of minimum and maximum distribution of $X_1, X_2, ..., X_n$ is

$$F_{x_{(1)},x_{(n)}}(x,y) = P[x_{(1)} \le x, x_{(n)} \le y] = [F(y)]^n - [F(y)-F(x)]^n$$

$$F_{x_{(1)},x_{(n)}}(x,y) = \left[\frac{y}{y+\theta}e^{-\theta/y}\right]^n - \left[\frac{y}{y+\theta}e^{-\theta/y} - \frac{x}{x+\theta}e^{-\theta/x}\right]^n$$

The joint PDF of minimum and maximum of $X_1, X_2, ..., X_n$ is

$$f_{x_{(1)},x_{(n)}}(x,y) = n(n-1)[F(y)-F(x)]^{n-2}f(x)f(y)$$

$$f_{x_{(1)},x_{(n)}}(x,y) = n(n-1)\left[\frac{y}{y+\theta}e^{-\theta/y} - \frac{x}{x+\theta}e^{-\theta/x}\right]^{n-2}\frac{\theta^2(2x+\theta)(2y+\theta)}{xy(x+\theta)^2(y+\theta)^2}e^{-\theta(\frac{x+y}{xy})}$$

4. Stress-strength reliability

Stress-strength reliability (SSR) describe the life of a component having a random strength X that is subject to a random stress Y. The component fails at an instant, when the stress applied to it exceeds the strength and the component function satisfactorily whenever $X > Y$. Let X and Y follows the proposed distribution with parameters θ_1 and θ_2 then,

$$R = P[X > Y] = \int_0^\infty P[X > Y|Y = y]f_Y(y)dy$$

$$P[X > Y] = \int_0^\infty \frac{\theta_2(2y+\theta_2)}{(y+\theta_1)(y+\theta_2)^2}e^{-\left(\frac{\theta_1+\theta_2}{y}\right)}dy$$

Table 1: True value of SSR of given value of θ_1 and θ_2

θ_2 \ θ_1	0.2	0.8	1.0	1.5	2.0	2.5	5.0
0.2	0.5000	0.2156	0.1820	0.1313	0.1027	0.0845	0.0005
0.8	0.7844	0.5000	0.4486	0.3584	0.2994	0.2575	0.1524
1.0	0.8179	0.5514	0.5000	0.4073	0.3447	0.2994	0.1820
1.5	0.8687	0.6415	0.5926	0.5000	0.4338	0.3839	0.2461
2.0	0.8972	0.7005	0.6552	0.5661	0.5000	0.4485	0.2994
2.5	0.9155	0.7424	0.7005	0.6160	0.5514	0.5000	0.3447
5.0	0.9552	0.8475	0.8179	0.7539	0.7005	0.6552	0.5000

5. Statistical inference

In this section, we discuss six classical methods of estimation, viz. method of MLE, method of LSE, method of WLSE, method of CME, method of PCE and methods of MPSE.

5.1. Estimation based on percentiles

Among the most easily obtained estimators of the parameters of the Weibull distribution are the graphical approximation to the best linear unbiased estimators. It can be obtained by fitting a straight line to the theoretical points obtained from the distribution function and the sample percentile points. This method was originally explored by Kao (1959), see also Mann *et al.* (1974) and Johnson *et al.* (1995). It is possible for the Weibull case because of the nature of its distribution function. In the case of a proposed distribution also it is possible to use the same concept to obtain the estimator of θ based on the percentiles, because of the structure of its distribution function.
Now

$$F(x \mid \theta) = \frac{x}{x+\theta} e^{-\theta/x}$$

If p_i denotes some estimate of $F(x \mid \theta)$ then the estimate of θ can be obtained by minimizing

$$\sum_{i=1}^{n} [p_i(x_i + \theta) - x_i e^{-\theta/x_i}]^2 \tag{11}$$

Partially differentiate equation (11) with respect to θ, we get

$$\sum_{i=1}^{n} [(p_i((x_i + \theta) - x_i e^{-\theta/x_i})(p_i + e^{-\theta/x_i})] = 0 \tag{12}$$

Solving equation (12) using non-linear method and find the value of θ for different value of p_i, for example $p_i = (i/(n+1))$ is the most used estimator of $F(x_{(i)})$, as $(i/(n+1))$ is the expected value of $F(x_{(i)})$. We have also used this p_i here. Some of the other choices of p_i's are $p_i = ((i-3/8)/(n+1/4))$ or $p_i = ((i-1/2)/n)$ (see Mann *et al.* (1974)) although they have not pursued here. We get the value of θ is know as $\hat{\theta}_{PCE}$ substituting the $\hat{\theta}_{PCE}$ in equation (3) and (4), we can get the estimators of HRF estimate h(x) SF S(x) given as

$$\hat{h}(x)_{PCE} = \frac{\hat{\theta}_{PCE}(2x + \hat{\theta}_{PCE})}{x(x+\hat{\theta}_{PCE})(xe^{\hat{\theta}_{PCE}/x} + \hat{\theta}_{PCE}e^{\hat{\theta}_{PCE}/x} - x)} \tag{13}$$

and

$$\hat{S}(x)_{PCE} = 1 - \frac{x}{x+\hat{\theta}_{PCE}} e^{-\hat{\theta}_{PCE}/x} \tag{14}$$

5.2. Least squares estimators

In this method we provide the regression based method estimators of the unknown parameters, which was originally suggested by Swain *et al.* (1988) to estimate the parameters of Beta distributions. It can be using some other cases also. Suppose $X_1, ..., X_n$ is a random sample of size n from distribution function $G(.)$ and suppose $x_{(i)}; i = 1, ..., n$ denotes the ordered sample. The proposed method uses the distribution of $G(X_{(i)})$. For a sample of size n, we have

$$E(G(X_{(i)})) = \frac{i}{n+1}$$
$$V(G(X_{(i)})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$$

See Johnson *et al.* (1995). Using the expectations and the variances, two variants of the least squares methods can be used. The least square estimate can be obtained by minimizing the,

$$LS(\theta) = \sum_{i=1}^{n} \left[\frac{x_i}{x_i+\theta} e^{-\theta/x_i} - \frac{i}{n+1} \right]^2 \tag{15}$$

When differentiated equation (15) with respect to θ and equated to 0. Then we get the value of θ which is called $\hat{\theta}_{LSE}$. Substituting these $\hat{\theta}_{LSE}$ in equation (3) and (4), we can get the estimators of HRF estimate h(x), SF S(x) given as,

$$\hat{h}(x)_{LSE} = \frac{\hat{\theta}_{LSE}(2x+\hat{\theta}_{LSE})}{x(x+\hat{\theta}_{LSE})(xe^{\hat{\theta}_{LSE}/x} + \hat{\theta}_{LSE} e^{\hat{\theta}_{LSE}/x} - x)} \tag{16}$$

and

$$\hat{S}(x)_{LSE} = 1 - \frac{x}{x+\hat{\theta}_{LSE}} e^{-\hat{\theta}_{LSE}/x} \tag{17}$$

5.3. Weighted least squares estimators

The weighted least squares estimation minimizes the equation given below,

$$\sum_{i=1}^{n} W_i \left(G(X_i) - \frac{i}{n+1} \right)^2$$

with respect to the unknown parameters, where

$$W_i = \frac{1}{V(G(X_{(i)}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$$

Therefore, in case of distribution the weighted least squares of θ can be obtained by minimizing

$$W(\theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_i \mid \theta) - \frac{i}{n+1} \right]^2$$
$$W(\theta) = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[\frac{x_i}{x_i+\theta} e^{-\theta/x_i} - \frac{i}{n+1} \right]^2 \tag{18}$$

Differentiating the above equation (18) with respect to θ and equating to 0 we get the estimate of θ and the estimated value of θ is denoted as $\hat{\theta}_{WLSE}$. Substituting the $\hat{\theta}_{WLSE}$, in equation (3) and (4) , we can get the estimate $\hat{h}(x)$ and $\hat{S}(x)$ given below,

$$\hat{h}(x)_{WLSE} = \frac{\hat{\theta}_{WLSE}(2x+\hat{\theta}_{WLSE})}{x(x+\hat{\theta}_{WLSE})(xe^{\hat{\theta}_{WLSE}/x} + \hat{\theta}_{WLSE} e^{\hat{\theta}_{WLSE}/x} - x)} \tag{19}$$

and

$$\hat{S}(x)_{WLSE} = 1 - \frac{x}{x+\hat{\theta}_{WLSE}} e^{-\hat{\theta}_{WLSE}/x} \tag{20}$$

5.4. Maximum likelihood estimator

In this section, the MLE of the distribution function is considered. If $X_1, X_2..., X_n$ is a random sample from proposed distribution with parameter θ, then the likelihood function, L(θ), is

$$L = \prod_{i=1}^{n} \frac{\theta(2x_i+\theta)}{x_i(x_i+\theta)^2} e^{-\theta/x_i}$$

$$log(L(\theta)) = nlog(\theta) + \sum_{i=1}^{n} log(2x_i+\theta) - \sum_{i=1}^{n} log(x_i) - \sum_{i=1}^{n} 2log(x_i+\theta) - \sum_{i=1}^{n} \frac{\theta}{x_i} \tag{21}$$

Differentiating the above equation (21) with respect to θ and $\frac{\partial logL(\theta)}{\partial \theta} = 0$ we get

$$\frac{n}{\theta} + \sum_{i=1}^{n} \frac{1}{(2x_i+\theta)} - \sum_{i=1}^{n} \frac{2}{(x_i+\theta)} - \sum_{i=1}^{n} \frac{1}{x_i} = 0 \tag{22}$$

The above equation is a non-linear equation which can be solved by a simple iterative procedure can be used to find a solution and we can estimate the value of θ and the estimated value is called as $\hat{\theta}_{MLE}$ substituting the $\hat{\theta}_{MLE}$, in equation (3) and (4), we can get the estimators of HRF estimate h(x), SF S(x) given as,

$$\hat{h}(x)_{MLE} = \frac{\hat{\theta}_{MLE}(2x+\hat{\theta}_{MLE})}{x(x+\hat{\theta}_{MLE})(xe^{\hat{\theta}_{MLE}/x} + \hat{\theta}_{MLE}e^{\hat{\theta}_{MLE}/x} - x)} \tag{23}$$

and

$$\hat{S}(x)_{MLE} = 1 - \frac{x}{x+\hat{\theta}_{MLE}} e^{-\hat{\theta}_{MLE}/x} \tag{24}$$

5.5. Cramer-von-Mises method of estimation

To motivate our choice of CME type minimum distance estimators, Macdonald (1971) provided empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators. Thus, the proposed estimators are based on the Cramer-von Mises statistics given by,

$$C(\theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F(x,\theta) - \frac{2i-1}{2n} \right]^2$$

$$C(\theta) = \frac{1}{12n} + \sum_{i=1}^{n} \left[\frac{x_i}{x_i+\theta} e^{-\theta/x_i} - \frac{2i-1}{2n} \right]^2 \tag{25}$$

Then the Cramer-von-Mises estimator are obtained by minimizing the above equation (25) with respect to θ, $\frac{\partial C(\theta)}{\partial \theta} = 0$. This estimator can also be obtained by solving the following non-linear equation. We get the estimated value of θ and the estimated value is called as $\hat{\theta}_{CME}$

$$\sum_{i=1}^{n} \left[\left(\frac{x_i}{x_i+\theta} e^{-\theta/x_i} - \frac{2i-1}{n} \right) \left(-\frac{2x_i+\theta}{(x_i+\theta)^2} e^{-\theta/x_i} \right) \right] = 0 \tag{26}$$

Substituting the $\hat{\theta}_{CME}$, in equation (3) and (4), we can get the estimators of HRF estimate h(x), SF S(x) given as,

$$\hat{h}(x)_{CME} = \frac{\hat{\theta}_{CME}(2x+\hat{\theta}_{CME})}{x(x+\hat{\theta}_{CME})(xe^{\hat{\theta}_{CME}/x} + \hat{\theta}_{CME}e^{\hat{\theta}_{CME}/x} - x)} \tag{27}$$

and

$$\hat{S}(x)_{CME} = 1 - \frac{x}{x + \hat{\theta}_{CME}} e^{-\hat{\theta}_{CME}/x} \tag{28}$$

5.6. Maximum product of spacing method of estimation

The Maximum Product of Spacing method of estimation (MPSE) is an alternative to MLE for the estimation of the unknown parameters of continuous uni-variate distributions. This method is used for estimating parameter, In this method estimation of the parameter θ is obtained by maximizing the geometric mean of the spacing with respect to θ

$$G(\theta) = \left[\prod_{i=1}^{n+1} \left[F(x_{(i,n)}|\theta) - F(x_{(i-1,n)}|\theta)\right]\right]^{\frac{1}{n+1}}$$

Taking log on both sides the above equation we get,

$$logG(\theta) = \frac{1}{n+1} \sum_{1}^{n+1} log\left[F(x_{(i,n)}|\theta) - F(x_{(i-1,n)}|\theta)\right] \tag{29}$$

To maximize the above equation (29) we differentiate with respect to θ, $\frac{\partial logG(\theta)}{\partial \theta} = 0$, we get the estimated value of θ and the estimated value is called as $\hat{\theta}_{MPSE}$ shown in equation (30), substituting the $\hat{\theta}_{MPSE}$, in equation (3) and (4), we can get the estimators of HRF estimate h(x), SF S(x) given as,

$$\frac{1}{n+1} \frac{\sum_{i=1}^{n} \left(\frac{(2x_i+\theta)}{(x_i+\theta)^2} e^{-\theta/x_i} - \frac{(2x_{i-1}+\theta)}{(x_{i-1}+\theta)^2} e^{-\theta/x_{i-1}}\right)}{\sum_{i=1}^{n} \left(\frac{x_i}{x_i+\theta} e^{-\theta/x_i} - \frac{x_{i-1}}{x_{i-1}+\theta} e^{-\theta/x_{i-1}}\right)} = 0 \tag{30}$$

$$\hat{h}(x)_{MPSE} = \frac{\hat{\theta}_{MPSE}(2x + \hat{\theta}_{MPSE})}{x(x + \hat{\theta}_{MPSE})(xe^{\hat{\theta}_{MPSE}/x} + \hat{\theta}_{MPSE} e^{\hat{\theta}_{MPSE}/x} - x)} \tag{31}$$

and

$$\hat{S}(x)_{MPSE} = 1 - \frac{x}{x + \hat{\theta}_{MPSE}} e^{-\hat{\theta}_{MPSE}/x} \tag{32}$$

6. Simulation study

In this section, we generate random sample from the proposed distribution by using the inverse transformation method. We used the Monto Carlo simulation study to assess the performance of the proposed estimators (PSE, LSE, WLSE, MLE, CME, MPSE) of the parameter θ for the proposed distribution. We used the particular values of θ= 0.2, 1.05, 2.5, 5 and the corresponding sample size is n= 5, 15, 30, 50, 100, 500 for each design we draw the sample of size n from the original sample and it is replicated 10,000 times. We calculate the mean of parameters θ using PSE, LSE, WLSE, MLE, CME and MPSE, and their corresponding MSEs, the results are reported in Table (6). It is observed that all the methods follow the same pattern, by increasing sample size the corresponding MSEs are decreasing in all the methods of estimation. On basis of MSEs we find the Maximum Product of Spacing method of estimation is the best method of estimation among the method used.

Similarly, the simulation for hazard estimations and survival estimations for same sample sizes and for a given value of t= 0.75, 1, 1.5 and given $\hat{\theta}_{MLE}$ we calculate the hazard and survival estimations and corresponding calculate MSEs for 10,000 times. The calculated average estimate of hazard and survival estimation with corresponding MSE and the results are reported in Table (7) and Table (8). It is observed that the HRF estimation and SF estimation also follow the same pattern as with, the increase in the sample size corresponding MSEs decreases.

7. Real data applications

In this Section, we use two data sets and comparing the existing model. For more details we see in Shukla (2019) and Shanker *et al.* (2015).

7.1. Data set 1

We use the survival times of a group of patients suffering from head and neck cancer disease and treated using a combination of radiotherapy and chemotherapy which is reported by Shukla (2019) and Shanker *et al.* (2015).
432.00 140.00 119.00 47.38 58.36 195.00 155.00 339.00 209.00 112.00 194.00 519.00 68.46 25.87 179.00 78.26 159.00 84.00 31.98 110.00 1776.00 725.00 173.00 41.35 94.00 74.47 633.00 319.00 127.00 146.00 281.00 23.56 92.00 249.00 37.00 23.74 133.00 130.00 12.20 63.47 81.43 55.46 817.00 469.00

Table 2: Values of the estimate of parameter for given real data set 1 and corresponding AIC, AICc, BIC, KS and *p*-value

Model	$\hat{\theta}$	-2ln L	AIC	AICc	BIC	KS	p-value
Lindley	0.00891	579.16	581.16	581.26	582.95	0.219	0.0243
Exponential	0.00447	564.02	566.02	566.11	567.80	0.145	0.2838
PD_{MLE}	45.33775	558.77	560.77	560.86	562.55	0.089	0.8509
PD_{MPSE}	41.95271	558.98	560.98	561.07	562.76	0.074	0.9562

Table 3: Value of hazard estimate and survival estimate for real data set 1 for given value of time t

t	θ_{MPSE}	$\hat{h(t)}_{MPSE}$	$\hat{S(t)}_{MPSE}$	θ_{MLE}	$\hat{h(t)}_{MLE}$	$\hat{S(t)}_{MLE}$
t=223.48(mean)	41.9527	0.00357	0.30216	45.33775	0.00351	0.32131
t=128.50(median)		0.00531	0.45611		0.00516	0.48057

7.2. Data set 2

We use the times between successive failures of air conditioning equipment in a Boeing 720 airplane data set which is reported by Shukla (2019) and Shanker *et al.* (2015) come from data set (13). 386 70 57 12 59 29 74 27 153 48 326 21 26 29 502.

In Table 2 and Table 4 estimated values of parameter, -2lnL, AIC, AICc, BIC, KS statistics and p value are given and the basis of these value our proposed distribution is better

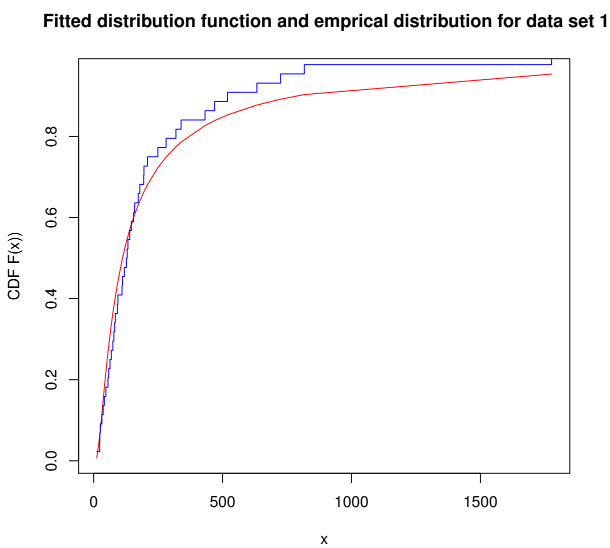
Table 4: Values of the estimate of parameter for given real data set 2 and corresponding AIC, AICc, BIC, KS and p-value

Model	Estimate	-2ln L	AIC	AICc	BIC	KS	p-value
Lindley	0.01636	181.34	183.34	183.65	184.05	0.386	0.0159
Exponential	0.00824	173.94	175.94	176.25	176.65	0.277	0.1662
PD_{MLE}	23.12604	169.25	171.25	171.56	171.96	0.176	0.6789
PD_{MPSE}	21.82070	169.29	171.29	171.60	172.00	0.158	0.7923

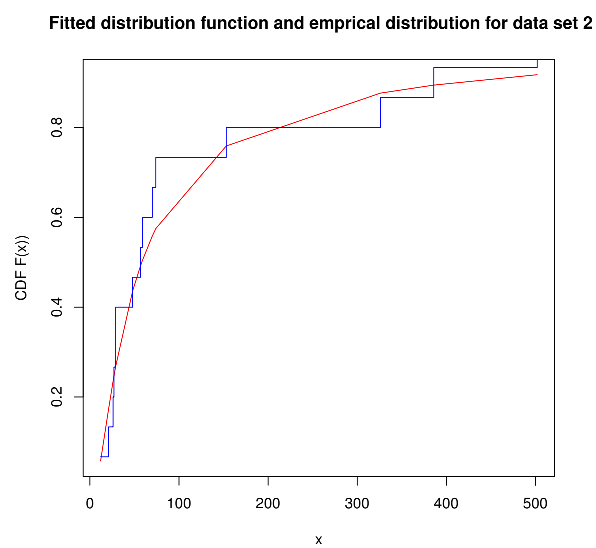
Table 5: Value of hazard estimate and survival estimate for real data set 2 for given value of time t

t	θ_{MPSE}	$\hat{h(t)}_{MPSE}$	$\hat{S(t)}_{MPSE}$	θ_{MLE}	$\hat{h(t)}_{MLE}$	$\hat{S(t)}_{MLE}$
t=121.2667(mean)	21.8207	0.00664	0.29206	23.12604	0.00656	0.30598
t=57(median)		0.01126	0.50685		0.01098	0.52587

than exponential distribution and one parameter Lindley distribution. It is also observed that estimate by using MPSE value provide better fit for both data sets 1 and 2 on basis of KS statistics and p value but there is no difference of -2lnL, AIC, AICc, BIC. It is same line supports the results for simulation study for our proposed distribution MPSE is better results than other methods.



(a) data set-1



(b) data set-2

8. Conclusions

In this article, we have proposed a new probability distribution, which is having an increasing hazard rate and decreasing hazard rate, heavy-tailed properties hold. Moments exist only for $r < 1$ and moments for $r \geq 1$ is divergent. Oder statistics and stress strength reliability properties are obtained and also find the quantile function, skewness and kurtosis. Simulation is conducted for different estimation methods (MLE, PSE, LSE, WLSE, CME, and MPSE) are used to estimate used to obtain the estimate of $\theta, h(t)$ and $S(t)$. Discuss the different statistical properties of the proposed distribution. In the simulation, it found that MPSE performed best among the different methods of estimation. We used two data sets to find the suitability proposed distribution and use the MPSE for this purpose. It is observed that our proposed probability distribution is fitted well to the data sets. The new proposed probability distribution performed well as compared to competitor models like exponential, Lindley, etc in terms of various model selection criteria like AIC, AICc, BIC, KS, and p value. The above comparison is tabulated in Tables 2, 3, 4, and 5. Several inferential aspects of the proposed model are yet to study, which we may be adjust in further another publication.

References

Bowley, A. L. (1920). *Elements of Statistics, 4th Edn.* Charles Scribner, New York.

Fisher, B. and Kılıcman, A. (2012). Some results on the gamma function for negative integers. *Applied Mathematics and Information Sciences*, **6**, 173–176.

Johnson, N. L., Kotz, S., and Balakrishnan, N. (1995). *Continuous Univariate Distributions.* John Wiley and Sons.

Kao, J. H. (1959). A graphical estimation of mixed weibull parameters in life-testing of electron tubes. *Technometrics*, **1**, 389–407.

Macdonald, P. (1971). Comments and queries comment on "an estimation procedure for mixtures of distributions" by choi and bulgren. *Journal of the Royal Statistical Society: Series B (Methodological)*, **33**, 326–329.

Mann, N. R., Schafer, R. E., and Singpurwalla, N. D. (1974). *Methods for Statistical Analysis of Reliability and Life Data.* John Wiley and Sons, Inc.

Moors, J. (1988). A quantile alternative for kurtosis. *Journal of the Royal Statistical Society: Series D (The Statistician)*, **37**, 25–32.

Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. L. (2009). *Stochastic Processes for Insurance and Finance.* John Wiley and Sons.

Shanker, R., Hagos, F., and Sujatha, S. (2015). On modeling of lifetimes data using exponential and lindley distributions. *Biometrics and Biostatistics International Journal*, **2**, 1–9.

Shannon, C. E. (1948). Claude elwood shannon. *Bell System Technical Journal*, **27**, 379–423.

Shukla, K. (2019). A comparative study of one parameter lifetime distributions. *Biometrics and Biostatistics International Journal*, **8**, 111–123.

Swain, J. J., Venkatraman, S., and Wilson, J. R. (1988). Least-squares estimation of distribution functions in johnson's translation system. *Journal of Statistical Computation and Simulation*, **29**, 271–297.

Appendix 1

Table 6: True values of θ and estimates (MLE, LSE, WLSE, PCE, CME, MPSE) and corresponding MSEs.

	Methods	$n = 5$	$n = 15$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$\theta = 0.2$	MLE	0.25281	0.21462	0.20761	0.20496	0.20222	0.20047
		(0.02638)	(0.00446)	(0.00198)	(0.00115)	(0.00053)	(0.00010)
	LSE	0.47376	0.21083	0.20579	0.20364	0.20134	0.20042
		(24.1709)	(0.0056)	(0.0025)	(0.0014)	(0.00065)	(0.00013)
	WLSE	0.49936	0.20992	0.20543	0.20346	0.20141	0.20042
		(24.7952)	(0.0052)	(0.0023)	(0.0013)	(0.00059)	(0.00012)
	PCE	0.12575	0.17549	0.18882	0.19144	0.19647	0.19927
		(0.0211)	(0.0241)	(0.0097)	(0.0032)	(0.0018)	(0.00031)
	CME	0.34324	0.21302	0.20692	0.20433	0.20169	0.20049
		(5.8857)	(0.0057)	(0.0025)	(0.0014)	(0.00065)	(0.00013)
	MPSE	0.16493	0.18012	0.18796	0.19204	0.19501	0.19866
		(0.0114)	(0.0034)	(0.0017)	(0.0010)	(0.00052)	(0.00009)
$\theta = 1.05$	MLE	1.28750	1.13026	1.08756	1.07410	1.06000	1.05201
		(0.6111)	(0.1270)	(0.0539)	(0.0304)	(0.0145)	(0.0028)
	LSE	1.33088	1.11065	1.07748	1.06844	1.05637	1.05169
		(1.5951)	(0.1558)	(0.0672)	(0.0375)	(0.0179)	(0.0035)
	WLSE	1.31587	1.10524	1.07561	1.06750	1.05628	1.05153
		(1.5474)	(0.1450)	(0.0624)	(0.0344)	(0.0164)	(0.0032)
	PCE	0.67763	0.91507	0.99103	1.01397	1.03069	1.04593
		(0.8187)	(0.4546)	(0.2164)	(0.1123)	(0.1210)	(0.00884)
	CME	1.34438	1.12206	1.08345	1.07204	1.05818	1.05205
		(1.3668)	(0.1588)	(0.0681)	(0.0378)	(0.0181)	(0.00356)
	MPSE	0.86779	0.948460	0.98447	1.00631	1.02222	1.04252
		(0.3252)	(0.0954)	(0.0474)	(0.0282)	(0.0142)	(0.00278)
$\theta = 2.5$	MLE	3.00848	2.69749	2.59059	2.55393	2.53204	2.50549
		(3.1287)	(0.7112)	(0.3106)	(0.1735)	(0.0841)	(0.0161)
	LSE	3.16709	2.64741	2.56482	2.54427	2.52676	2.50419
		(9.0752)	(0.8941)	(0.3808)	(0.2230)	(0.1057)	(0.0199)
	WLSE	3.13132	2.63481	2.55994	2.54115	2.525	2.50432
		(8.7689)	(0.8299)	(0.3514)	(0.2038)	(0.0965)	(0.0182)
	PCE	1.64480	2.18839	2.33701	2.41562	2.46184	2.4732
		(7.2999)	(2.0666)	(1.1109)	(0.6250)	(0.2943)	(0.0505)
	CME	3.21388	2.67504	2.57896	2.55280	2.53108	2.50505
		(8.5114)	(0.9103)	(0.3854)	(0.2248)	(0.1062)	(0.0199)
	MPSE	2.07919	2.6378	2.34567	2.39264	2.44134	2.48291
		(1.9694)	(0.5314)	(0.2727)	(0.1613)	(0.0802)	(0.0161)
$\theta = 5$	MLE	5.57307	5.17582	5.19181	5.10321	5.05499	5.01262
		(10.2351)	(10.1722)	(1.2233)	(0.7032)	(0.3364)	(0.0643)
	LSE	6.18395	5.29498	5.15390	5.07877	5.03796	5.01025
		(30.3224)	(3.6853)	(1.5322)	(0.8862)	(0.4194)	(0.0808)
	WLSE	6.09790	5.26990	5.14333	5.07328	5.03828	5.01056
		(28.0626)	(3.42865)	(1.4120)	(0.8149)	(0.3841)	(0.0736)
	PCE	3.54518	4.37968	4.70454	4.80893	4.91191	4.98667
		(1324.831)	(8.5011)	(5.1263)	(2.8744)	(1.1243)	(0.2073)
	CME	6.29332	5.34993	5.18218	5.09584	5.04661	5.01199
		(30.64088)	(3.7582)	(1.5522)	(0.8934)	(0.4212)	(0.0809)
	MPSE	4.11762	4.52001	4.70706	4.78218	4.87486	4.96747
		(7.5738)	(2.2041)	(1.0637)	(0.6579)	(0.3261)	(0.0641)

Table 7: True value of $h(t)$ and estimate(MLE) & corresponding MSEs for $t = 0.75$, $t = 1$, & $t = 1.5$

θ & t	Method	$n = 5$	$n = 15$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
	$\hat{\theta}$	0.25496	0.21564	0.20654	0.20479	0.20201	0.20018
$\theta = 0.2$	h(t)	0.89696	0.95072	0.96372	0.96625	0.97028	0.97295
$t = 0.75$	$\hat{h(t)}$	0.92154	0.95615	0.96617	0.96768	0.97099	0.97309
		(0.07605)	(0.01336)	(0.00475)	(0.00274)	(0.00118)	(0.00021)
	$\hat{\theta}$	1.32072	1.13019	1.08827	1.07346	1.06291	1.05226
$\theta = 1.05$	h(t)	0.22596	0.29048	0.30731	0.31433	0.31943	0.32204
$t = 0.75$	$\hat{h(t)}$	0.29245	0.29895	0.30068	0.30124	0.30149	0.30264
		(0.04675)	(0.01539)	(0.00744)	(0.00446)	(0.00221)	(0.00042)
	$\hat{\theta}$	3.15398	2.69375	2.59982	2.56007	2.52611	2.50674
$\theta = 2.5$	h(t)	0.01920	0.03520	0.03982	0.04196	0.04387	0.04500
$t = 0.75$	$\hat{h(t)}$	0.06857	0.05317	0.04951	0.04783	0.46915	0.45614
		(0.00997)	(0.00227)	(0.00107)	(0.00061)	(0.00029)	(0.00005)
	$\hat{\theta}$	6.16170	5.39945	5.20226	5.11701	5.06865	5.01182
$\theta = 5$	h(t)	0.00035	0.00098	0.00127	0.00142	0.00152	0.001642
$t = 0.75$	$\hat{h(t)}$	0.01562	0.00395	0.00281	0.00236	0.00198	0.00173
		(0.00936)	(0.00005)	(0.00001)	(0.000006)	(0.000002)	(0.0000003)
	$\hat{\theta}$	0.25237	0.21597	0.20773	0.20481	0.20250	0.20052
$\theta = 0.2$	h(t)	0.74177	0.77309	0.78042	0.78303	0.78511	0.78690
$t = 1$	$\hat{h(t)}$	0.75393	0.77579	0.78165	0.78374	0.78546	0.78697
		(0.03941)	(0.00668)	(0.00238)	(0.00122)	(0.00052)	(0.00008)
	$\hat{\theta}$	1.34594	1.13448	1.09333	1.07245	1.06215	1.05246
$\theta = 1.05$	h(t)	0.23958	0.29552	0.30791	0.31441	0.31767	0.32076
$t = 1$	$\hat{h(t)}$	0.29889	0.31263	0.31598	0.31926	0.32005	0.32123
		(0.03540)	(0.01036)	(0.00493)	(0.00293)	(0.00145)	(0.00028)
	$\hat{\theta}$	3.17655	2.69800	2.59951	2.55627	2.52797	2.50549
$\theta = 2.5$	h(t)	0.13842	0.18462	0.19426	0.19889	0.20248	0.20504
$t = 1$	$\hat{h(t)}$	0.09068	0.08216	0.07953	0.07877	0.07798	0.07737
		(0.01150)	(0.00324)	(0.00157)	(0.00092)	(0.00045)	(0.00009)
.	$\hat{\theta}$	6.08392	5.39642	5.20642	5.11668	5.05185	5.01264
$\theta = 5$	h(t)	0.002234	0.00442	0.00534	0.00584	0.00622	0.00647
$t = 1$	$\hat{h(t)}$	0.02941	0.01040	0.00853	0.00779	0.00724	0.00667
		(0.01348)	(0.00020)	(0.00007)	(0.00003)	(0.00001)	(0.000002)
	$\hat{\theta}$	0.25543	0.21631	0.20744	0.20483	0.20254	0.200619
$\theta = 0.2$	h(t)	0.54324	0.56002	0.56392	0.56507	0.56609	0.56694
$t = 1.5$	$\hat{h(t)}$	0.54795	0.56097	0.56434	0.56532	0.56621	0.56697
		(0.01839)	(0.00264)	(0.00078)	(0.00039)	(0.00015)	(0.00002)
	$\hat{\theta}$	1.34439	1.13237	1.09132	1.07585	1.06092	1.05250
$\theta = 1.05$	h(t)	0.25025	0.28903	0.29727	0.30045	0.30355	0.30531
$t = 1.5$	$\hat{h(t)}$	0.28143	0.29741	0.30125	0.30273	0.30467	0.30554
		(0.01869)	(0.00488)	(0.00226)	(0.00129)	(0.00062)	(0.00012)
	$\hat{\theta}$	3.15152	2.69424	2.59652	2.55265	2.52997	2.50530
$\theta = 2.5$	h(t)	0.07607	0.10256	0.10932	0.11251	0.11419	0.11605
$t = 1.5$	$\hat{h(t)}$	0.11653	0.11604	0.11597	0.11643	0.11614	0.11644
		(0.00998)	(0.00313)	(0.00150)	(0.00089)	(0.00044)	(0.00008)
	$\hat{\theta}$	6.04436	5.41528	5.18005	5.1028	5.0607	5.01264
$\theta = 5$	h(t)	0.01142	0.01728	0.02017	0.02122	0.02181	0.02251
$t = 1.5$	$\hat{h(t)}$	0.040103	0.026824	0.025033	0.024219	0.02329	0.02281
		(0.00668)	(0.00062)	(0.00027)	(0.00015)	(0.00007)	(0.00001)

Table 8: True value of $S(t)$ and estimate(MLE) & corresponding MSEs for $t = 0.75$, $t = 1$, & $t = 1.5$

θ & t	Method	$n = 5$	$n = 15$	$n = 30$	$n = 50$	$n = 100$	$n = 500$
$\theta = 0.2$ $t = 0.75$	$\hat{\theta}$	0.25496	0.21564	0.20654	0.20479	0.20201	0.20018
	S(t)	0.46877	0.41739	0.40466	0.40218	0.39821	0.39557
	$\hat{S(t)}$	0.43577	0.40951	0.40106	0.40007	0.39716	0.39537
		(0.02359)	(0.00719)	(0.00349)	(0.00207)	(0.001044)	(0.00021)
$\theta = 1.05$ $t = 0.75$	$\hat{\theta}$	1.32072	1.13019	1.08827	1.07346	1.06291	1.05226
	S(t)	0.93211	0.90319	0.89505	0.89159	0.88904	0.88773
	$\hat{S(t)}$	0.88859	0.89339	0.89531	0.89619	0.89697	0.89715
		(0.01056)	(0.00358)	(0.00174)	(0.00104)	(0.00052)	(0.00010)
$\theta = 2.5$ $t = 0.75$	$\hat{\theta}$	3.15398	2.69375	2.59982	2.56007	2.52611	2.50674
	S(t)	0.99713	0.99399	0.99300	0.99253	0.99211	0.99185
	$\hat{S(t)}$	0.98202	0.98870	0.99015	0.99080	0.99121	0.99167
		(0.00133)	(0.00016)	(0.00006)	(0.00003)	(0.00001)	(0.000003)
$\theta = 5$ $t = 0.75$	$\hat{\theta}$	6.16170	5.39945	5.20226	5.11701	5.06865	5.01182
	S(t)	0.99997	0.99990	0.99987	0.99986	0.99985	0.99983
	$\hat{S(t)}$	0.99339	0.99946	0.99966	0.99973	0.99978	0.99982
		(0.00502)	(0.000001)	(0.0000003)	(0.0000001)	(0.00000003)	(0.000000004)
$\theta = 0.2$ $t = 1$	$\hat{\theta}$	0.25237	0.21597	0.20773	0.20481	0.20250	0.20052
	S(t)	0.37961	0.33735	0.32731	0.32370	0.32084	0.31837
	$\hat{S(t)}$	0.35642	0.33190	0.32479	0.32225	0.32011	0.31823
		(0.01929)	(0.00562)	(0.00274)	(0.00161)	(0.00081)	(0.00015)
$\theta = 1.05$ $t = 1$	$\hat{\theta}$	1.34594	1.13448	1.09333	1.07245	1.06215	1.05246
	S(t)	0.88904	0.84933	0.83992	0.83489	0.83235	0.82992
	$\hat{S(t)}$	0.83001	0.82987	0.83040	0.82908	0.82946	0.82935
		(0.01692)	(0.00569)	(0.00283)	(0.00175)	(0.00088)	(0.00017)
$\theta = 2.5$ $t = 1$	$\hat{\theta}$	3.17655	2.69800	2.59951	2.55627	2.52797	2.50549
	S(t)	0.94869	0.92345	0.91775	0.91496	0.91278	0.91120
	$\hat{S(t)}$	0.96381	0.97194	0.97422	0.97506	0.97579	0.97638
		(0.00329)	(0.00063)	(0.00028)	(0.00016)	(0.00007)	(0.00001)
$\theta = 5$ $t = 1$	$\hat{\theta}$	6.08392	5.39642	5.20642	5.11668	5.05185	5.01264
	S(t)	0.99967	0.99929	0.99911	0.99901	0.99894	0.99889
	$\hat{S(t)}$	0.98283	0.99773	0.99831	0.99853	0.99869	0.99884
		(0.01260)	(0.000014)	(0.000004)	(0.000002)	(0.0000008)	(0.0000001)
$\theta = 0.2$ $t = 1.5$	$\hat{\theta}$	0.25543	0.21631	0.20744	0.20483	0.20254	0.200619
	S(t)	0.27930	0.24339	0.23495	0.23245	0.23024	0.22838
	$\hat{S(t)}$	0.26506	0.24039	0.23361	0.23166	0.22985	0.22831
		(0.01376)	(0.00352)	(0.00166)	(0.00097)	(0.00048)	(0.00009)
$\theta = 1.05$ $t = 1.5$	$\hat{\theta}$	1.34439	1.13237	1.09132	1.07585	1.06092	1.05250
	S(t)	0.78479	0.73215	0.72035	0.71576	0.71124	0.70866
	$\hat{S(t)}$	0.72525	0.71401	0.71150	0.71062	0.70868	0.70815
		(0.02417)	(0.00839)	(0.00430)	(0.00254)	(0.00129)	(0.00026)
$\theta = 2.5$ $t = 1.5$	$\hat{\theta}$	3.15152	2.69424	2.59652	2.55265	2.52997	2.50530
	S(t)	0.96055	0.94065	0.93515	0.93250	0.93108	0.92951
	$\hat{S(t)}$	0.91684	0.92465	0.92706	0.92764	0.92865	0.92902
		(0.00834)	(0.00238)	(0.00111)	(0.00066)	(0.00032)	(0.00006)
$\theta = 5$ $t = 1.5$	$\hat{\theta}$	6.04436	5.41528	5.18005	5.1028	5.0607	5.01264
	S(t)	0.99646	0.99413	0.99289	0.99243	0.99216	0.99185
	$\hat{S(t)}$	0.97250	0.98850	0.99002	0.99065	0.99129	0.99167
		(0.01145)	(0.00018)	(0.00006)	(0.00003)	(0.00001)	(0.000002)

Appendix 2

1: Distribution Function

$$F(x \mid \theta) = \begin{cases} \left(1 - \frac{\theta}{x+\theta}\right)\left(e^{-\theta/x}\right) & ; x \geq 0, \theta > 0 \\ 0 & ; \text{otherwise} \end{cases} \tag{33}$$

Proof

- $F(-\infty) = 0$
- $F(+\infty) = 1$
- If $x \leq y$ then $F(x|\theta) \leq F(y|\theta)$
- $F(x|\theta)$ is right continuous.

Hence $F(x|\theta)$ is a distribution function.

2: r^{th} Moment

The r^{th} moments about origin exits only for $r < 1$

$$E[X^r] = \int_0^\infty x^r \frac{\theta(2x+\theta)}{x(x+\theta)^2} e^{-\theta/x} \, dx \tag{34}$$

when we put the $\theta/x = t$ and simplify it then we get

$$E[X^r] = \theta^r \left[\int_0^\infty \frac{1}{t^r(1+t)} e^{-t} \, dt + \int_0^\infty \frac{1}{t^r(1+t)^2} e^{-t} \, dt\right]$$

First term of integral we say I_1 and second term of integral we say that I_2
Now, put r=1 we get I_1 is

$$I_1 = \int_0^\infty \left[\frac{1}{t} - \frac{1}{1+t}\right] e^{-t} \, dt$$

First term of above integral we say I_3 and second term of integral we say that I_4
Now, The I_3 is negative integer for gamma function which is divergent (see Fisher and Kılıcman (2012)) and I_4 is incomplete gamma function which is finite, so equation (34) is divergent