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Some Constructions of α-Resolvable Balanced Incomplete Block Designs

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Abstract

Some construction methods of α-resolvable balanced incomplete block (BIB) designs are proposed with illustrations by using known resolvable/affine resolvable balanced incomplete block designs and group divisible (GD) designs having 2k treatments.

Key words: Balanced Incomplete Block Designs, Resolvable Designs, α-Resolvable Designs, Partially Balanced Incomplete Block Designs, Group Divisible Designs, t-Design

1. Introduction

The combinatorial study of resolvability in block designs goes back at least as far as the well-known Kirkman's school girl problem formulated in 1850. The notion entered the statistical lexicon with Yates' work on square lattice designs (1936, 1940), although the term "resolvable design" was firstly introduced by Bose in 1942. He defined this for a balanced incomplete block design.

A balanced incomplete block design is an arrangement of v symbols (treatment) into b sets (blocks) such that (i) each block contains k ($< v$) distinct treatments; (ii) each treatment appears in r ($> \lambda$) different blocks and (iii) every pair of distinct treatments appears together in exactly λ blocks. Here, the parameters of balanced incomplete block design (v, b, r, k, λ) are related by the parametric relations $vr = bk$, $r\,(k\text{-}1) = \lambda\,(v\text{-}1)$ and $b \geq v$ (Fisher's inequality).

A block design is said to be resolvable if the b blocks each of size k can be grouped into r resolution sets of b/r blocks each such that in each resolution set every treatment is replicated exactly once. Bose (1942) proved that necessary condition for the resolvability of a balanced incomplete block design is

$$b \geq v+r-1 \tag{1}$$

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Further, Khan (2000) improved this inequality by

$$b \geq \left[\frac{(v-k)^2}{v-1}\right] + 2r - \lambda \tag{2}$$

and conjectured that this inequality is more stringent than given in (1)

A resolvable block design is said to be affine resolvable if and only if b = $v + r - 1$ and any two blocks belonging to same resolution set have no treatment in common, say, $q_1=0$ whereas any two blocks belonging to different resolution sets intersect in the same number, say, $q_2 = k^2/v$ of treatments.

The concept of resolvability and affine resolvability was generalized by Shirkhande and Raghavarao (1964) to α - resolvability and affine α- resolvability. An incomplete block design with parameters v, $b = \beta\, t$, $r = \alpha\, t$, k is said to be α-resolvable if the b blocks can be divided into t resolution sets of β blocks each, such that each treatment occurs α times in each resolution set. Further, α-resolvable incomplete block design is said to be affine α-resolvable if every two distinct blocks from the same resolution set intersect in the same number, say, q_1, of treatments, whereas every two blocks belonging to different α-resolution sets intersect in the same number, say, q_2, of treatments. The necessary and sufficient condition for the α - resolvable balanced incomplete block design to be affine α - resolvable with the block intersection numbers q_1 and q_2 is $q_1 = k(\alpha-1)/(\beta-1)$ and $q_2 = \alpha k/\beta = k^2/v$. There has been a very rapid development in this area of experimental designs. Some of the prominent work has been seen in Bailey et al. (1995), Banerjee et al. (1990), Caliński et al. (2008), Kageyama (1972, 1973, 1977), Kageyama et al.(1983, 2001), Rai et al. (2004), Rudra et al. (2005), Agrawal et al. (2016).

A group divisible (GD) design is a 2-associates partially balanced incomplete block design based on group divisible association scheme, i.e. an arrangement of $v = mn$ treatments in b blocks such that each block contains k ($< v$) treatments, each replicated r times, and the mn treatments can be divided into m groups of n treatments each, such that any two treatments occur together in λ_1 blocks if they belong to the same group and in λ_2 blocks if they belong to different groups. Furthermore, a group divisible (GD) design is said to be Singular (S) if $r - \lambda_1=0$; Semi regular (SR) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 = 0$; Regular (R) if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$. For the definitions of partially balanced incomplete block design with their combinatorial properties, refer, Raghavarao (1971).

A t-(v, k, λ_t) design (or simply a t-design) is a system with v points (or treatments) and b blocks containing k distinct points, each point appearing in r different blocks and every set of t distinct points are appearing in exactly λ_t blocks. Then it follows [Hedayat and Kageyama (1980), Ragavarao (1971)] that $\lambda_t \binom{v}{t} = b \binom{k}{t}$ and for each $0 \leq s \leq t$ every t-(v, k, λ_t) design is an s-(v, k, λ_s) design with $\lambda_s = \lambda_t \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s}$.

In the present paper we have proposed two construction methods of α-resolvable balanced incomplete block designs by using resolvable/affine resolvable balanced incomplete block design and group divisible design having $2k$ treatments. The constructed α-resolvable balanced incomplete block designs can be used in error correcting codes.

The following result is from Dey (1986)

Lemma 1.1 : The existence of a resolvable/affine resolvable balanced incomplete block design D having the parameters $v=2k$, $b=2(2k-1)$, $r=2k-1$, k, $\lambda=k-1$, implies the existence of column-wise BIB design D^c with the parameters $V^c=2k$, $B^c=2(2k-1)$, $R^c=2k-1$, $K^c=k$, $\Lambda^c=k-1$ and row-wise BIB design D^r with the parameters $V^r=2k$, $B^r=k(2k-1)$, $R^r=2k-1$, $K^r=2$, $\Lambda^r=1$ by proper rearrangement of treatments .

The above Lemma is illustrated in the following examples

Example 1.2: Let us consider an affine resolvable BIB design D with parameters $v=8$, $b=14$, $r=7$, $k=4$, $\lambda=3$ given by the blocks [(1,2,4,7), (0,3,5,6)], [(2,3,5,7), (1,4,6,0)], [(3,4,6,7), (2,5,0,1)], [(4,5,0,7), (3,6,1,2)], [(5,6,1,7), (4,0,2,3)], [(6,0,2,7), (5,1,3,4)], [(0,1,3,7), (6,2,4,5)]. After rearrangement of treatments of the design D as

$$D=\begin{bmatrix} 1 & 0 & 2 & 1 & 3 & 1 & 4 & 6 & 5 & 0 & 6 & 3 & 0 & 2 \\ 2 & 3 & 3 & 4 & 4 & 2 & 5 & 1 & 6 & 2 & 0 & 4 & 1 & 6 \\ 4 & 5 & 5 & 6 & 6 & 0 & 0 & 3 & 1 & 4 & 2 & 5 & 3 & 5 \\ 7 & 6 & 7 & 0 & 7 & 5 & 7 & 2 & 7 & 3 & 7 & 1 & 7 & 4 \end{bmatrix} \quad (3)$$

gives column-wise BIB design D^c with the parameters $V^c=8$, $B^c=14$, $R^c=7$, $K^c=4$, $\Lambda^c=3$ and row-wise BIB design D^r with parameters $V^r=8$, $B^r=28$, $R^r=7$, $K^r=2$, $\Lambda^r=1$ by proper rearrangement of treatments.

Example 1.3: Let us consider an affine resolvable BIB design D with parameters $v=12$, $b=22$, $r=11$, $k=6$, $\lambda=5$ given by the blocks [(1,3,4,5,9,11), (2,6,7,8,10,12)], [(2,4,5,6,10,1), (3,7, 8,9,11,12)], [(3,5,6,7,11,2), (4,8,9,10,1,12)], [(4,6,7,8,1,3), (5,9,10,11,2,12)], [(5,7,8,9,2,4), (6,10,11,1,3, 12)], [(6,8,9,10,3,5), (7,11,1,2,4,12)], [(7,9,10,11,4,6), (8,1,2,3,5,12)], [(8,10,11, 1,5,7), (9,2,3,4,6, 12)], [(9,11,1,2,6,8), (10,3,4,5,7,12)], [(10,1,2,3,7,9), (11,4,5,6,8,12)], [(11, 2,3,4,8,10), (1,5,6,7,9,12)]. After rearrangement of treatments of the design D as

$$D=\left[\begin{array}{cc|cc|cc|cc|cc|cc|cc|cc|cc|cc|cc} 1 & 2 & 2 & 7 & 3 & 8 & 4 & 11 & 5 & 12 & 6 & 4 & 7 & 5 & 8 & 9 & 6 & 7 & 1 & 6 & 3 & 7 \\ 3 & 6 & 4 & 3 & 5 & 4 & 6 & 5 & 9 & 6 & 8 & 7 & 9 & 3 & 10 & 3 & 2 & 10 & 2 & 4 & 2 & 6 \\ 4 & 7 & 5 & 9 & 6 & 10 & 7 & 10 & 8 & 11 & 9 & 2 & 10 & 1 & 11 & 6 & 1 & 3 & 3 & 5 & 4 & 12 \\ 5 & 8 & 6 & 8 & 7 & 9 & 8 & 2 & 7 & 1 & 10 & 12 & 11 & 2 & 1 & 4 & 8 & 12 & 7 & 11 & 8 & 1 \\ 9 & 10 & 10 & 11 & 11 & 1 & 1 & 9 & 2 & 3 & 3 & 11 & 4 & 8 & 5 & 2 & 9 & 4 & 9 & 12 & 10 & 5 \\ 11 & 12 & 1 & 12 & 2 & 12 & 3 & 12 & 4 & 10 & 5 & 1 & 6 & 12 & 7 & 12 & 11 & 5 & 10 & 8 & 11 & 9 \end{array}\right] \quad (4)$$

gives column-wise BIB design D^c with the parameters $V^c=12$, $B^c=22$, $R^c=11$, $K^c=6$, $\Lambda^c=5$ and row-wise BIB design D^r with parameters $V^r=12$, $B^r=66$, $R^r=11$, $K^r=2$, $\Lambda^r=1$ by proper rearrangement of treatments.

Now, in the next section we have given two construction methods of α-resolvable BIB designs (i) By considering rows of each resolution set of BIB design as association schemes of the GD design, and (ii) By considering columns of each resolution set of BIB design as association schemes of the GD design.

2. Method of Construction-I

In a row-wise BIB design having the parameters V^r=2k, B^r=k(2k-1), R^r=2k-1, K^r=2, Λ^r=1, there are r resolution sets with 2 blocks each. Considering the rows of each resolution set of this design as association schemes, clearly there will be r such association schemes with k groups of size 2. The association scheme for each of these resolution sets is given as follows:

$$\begin{array}{cccc} G_1 & G_2 & \cdots & G_k \\ \hline \theta_1 & \theta_2 & \cdots & \theta_k \\ \theta_{(k+1)} & \theta_{(k+2)} & \cdots & \theta_{2k} \end{array} \tag{5}$$

Let the chosen GD design have the parameters v^*=2k, b^*, r^*, k^*, λ_1^*, λ_2^*, m^*=k, n^*=2. Now, using each of these association schemes given in (5) for each $r_{i,}$ (i=1,2,…r) and constructing GD designs with parameters v^*=2k, b^*, r^*, k^*, λ_1^*, λ_2^*, m^*=k, n^*=2. Denote these designs by $D_{i,}$ (i=1,2,….,r) and their juxtaposition given in following structure

$$D_1^* = [D_1 \vdots D_2 \vdots D_3 \vdots \cdots \vdots D_r] \tag{6}$$

gives r*- resolvable balanced incomplete block designs with the parameters V_1=v^*, B_1=b^*r, R_1=r^*r, K_1=k^*, Λ_1= λ_1^*+ λ_2^*(r-1).

Theorem 2.1: The existence of a row-wise BIB design and group divisible design with parameters V^r=2k, B^r=k(2k-1), R^r=2k-1, K^r=2, Λ^r=1 and v^*=2k, b^*, r^*, k^*, λ_1^*, λ_2^*, m^*=k, n^*=2, respectively, implies the existence of r*-resolvable balanced incomplete block design with the parameters V_1=v^*, B_1=b^*r, R_1=r^*r, K_1=k^*, Λ_1= λ_1^*+ λ_2^*(r-1).

Proof: In a row-wise BIB design with the parameters (2k, k(2k-1), 2k-1, 2, 1), considering rows of each resolution set as an association scheme. Here there are 2k treatments, which are arranged in m^*=k groups of size n^*=2 in each resolution set, in such way that any two treatments in the same group are first associates and any two treatments from different groups are second associates. The association scheme for the chosen GD design is describe in (5).

The GD design used here have the parameters v^*=2k, b^*, r^*, k^*, λ_1^*, λ_2^*, m^*=k, n^*=2. Under the present construction method, the required structure D_1^* defined in (6) yields the parameters V_1=v^*, B_1=b^*r, R_1=r^*r, K_1=k^*, which are obvious from the construction. Further, any pair, say (θ, ϕ), in row-wise BIB design is occurring only once, therefore in one resolution set of D_1^*; it will be first associate and will occurs λ_1^* times and in the remaining (r-1) resolution sets of D_1^*; it will be second associate, so it will occurs λ_2^* times. Thus, the frequency of pair (θ,ϕ) in the resultant design D_1^* can be calculated as

$$\Lambda_1 = \lambda_1^* + \lambda_2^*(r\text{-}1)$$

Also we can see that in each $D_{i,}$ (i=1,2,….,r), every treatment is replicated r^* times. Hence the resultant design D_1^* is r^*- resolvable design. This completes the proof.

Corollary 2.2: The complementary design of D_1^* also produces (β-α)-resolvable balanced incomplete block design with parameters $V_1^c = v^*$, $B_1^c= b^* r$, $R_1^c=r\ (b^*- r^*)$, $K_1^c= v^*- k^*$, $\Lambda_1^c = r\ (b^*\text{-}2\ r^*+ \lambda_2^*) + \lambda_1^*- \lambda_2^*$ (here $\beta = b^*$ and $\alpha = r^*$).

Proof: The α-resolvable balanced incomplete block design D_1^*, constructed in Theorem 2.1 has the parameters $V_1=v^*$, $B_1=b^*r$, $R_1=r^*r$, $K_1=k^*$, $\Lambda_1=\lambda_1^*+\lambda_2^*(r\text{-}1)$. Then, the complementary design D_1^{*c} of the design D_1^* is obtained by replacing each block by another block containing those treatments which are not included in the original block. Thus the parameters $V_1^c = v^*$, $B_1^c= b^*r$, $R_1^c=r\ (b^*- r^*)$, $K_1^c= v^*- k^*$ and $\Lambda_1^c = r\ (b^*\text{-}2\ r^*+ \lambda_2^*) + \lambda_1^*- \lambda_2^*$ are obvious from the construction.

In D_1^*, every treatment is replicated α times in each resolution sets. So, in the resultant design D_1^{*c} after complementing, every treatment will be replicated (β-α) times in each resolution set. Hence, the resultant design D_1^{*c} is (β-α)-resolvable balanced incomplete block design (here $\beta = b^*$ and $\alpha = r^*$). This completes the proof.

Example 2.3: Considering rows of BIB design D given in (3) with parameters v=8, b= 14, r=7, k=4, λ=3 as association schemes of GD Design

G_1	G_2	G_3	G_4	G_1	G_2	G_3	G_4	G_1	G_2	G_3	G_4	G_1	G_2	G_3	G_4	G_1	G_2	G_3	G_4	G_1	G_2	G_3	G_4	G_1	G_2	G_3	G_4
1	2	4	7	2	3	5	7	3	4	6	7	4	5	0	7	5	6	1	7	6	0	2	7	0	1	3	7
0	3	5	6	1	4	6	0	1	2	0	5	6	1	3	2	0	2	4	3	3	4	5	1	2	6	5	4

Now, construct GD designs with parameters $v^*=b^*$=8, $r^*= k^*$=3, λ_1^*=0, λ_2^*=1, m^*=4, n^*=2 with each of these association schemes. Then the construction method yields a 3-resolvable balanced incomplete block design D_1^* with parameters V_1=8, B_1=56, R_1=21, K_1=3, Λ_{12}=6, Λ_{13}=1; whose blocks are given as follows

[(1,2,7), (2,4,0), (4,7,3), (7,0,5), (0,3,6), (3,5,1), (5,6,2), (6,1,4)]

[(2,3,7), (3,5,1), (5,7,4), (7,1,6), (1,4,0), (4,6,2), (6,0,3), (0,2,5)]

[(3,4,7), (4,6,1), (6,7,2), (7,1,0), (1,2,5), (2,0,3), (0,5,4), (5,3,6)]

[(4,5,7), (5,0,6), (0,7,1), (7,6,3), (6,1,2), (1,3,4), (3,2,5), (2,4,0)]

[(5,6,7), (6,1,0), (1,7,2), (7,0,4), (0,2,3), (2,4,5), (4,3,6), (3,5,1)]

[(6,0,7), (0,2,3), (2,7,4), (7,3,5), (3,4,1), (4,5,6), (5,1,0), (1,6,2)]

[(0,1,7), (1,3,2), (3,7,6), (7,2,5), (2,6,4), (6,5,0), (5,4,1), (4,0,3)]

Here, in the present illustration $b=\binom{8}{3}$=56. Thus, the obtained design is irreducible design and hence it is a 3-design as well.

Example 2.4: Considering rows of BIB design D given in (4) with parameters v=12, b= 22, r=11, k=6, λ=5 as association scheme of GD design

G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6
1	3	4	5	9	11	2	4	5	6	10	1	3	5	6	7	11	2	4	6	7	8	1	3
2	6	7	8	10	12	7	3	9	8	11	12	8	4	10	9	1	12	11	5	10	2	9	12

G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6
5	9	8	7	2	4	6	8	9	10	3	5	7	9	10	11	4	6	8	10	11	1	5	7
12	6	11	1	3	10	4	7	2	12	11	1	5	3	1	2	8	12	9	3	6	4	2	12

G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6	G_1	G_2	G_3	G_4	G_5	G_6
6	2	1	8	9	11	1	2	3	7	9	10	3	2	4	8	10	11
7	10	3	12	4	5	6	4	5	11	12	8	7	6	12	1	5	9

Now, construct GD designs with parameters $v^*=b^*=12$, $r^*=k^*=7$, $\lambda_1^*=2$, $\lambda_2^*=4$, $m^*=6$, $n^*=2$ with each of these association schemes. Then the construction method yields a 7-resolvable balanced incomplete block design D_1^* with parameters $V_1=12$, $B_1=132$, $R_1=77$, $K_1=7$, $\Lambda_{12}=42$.

The replication number of the resultant design D_1^* is very high. Therefore, by corollary 2.2 we consider complementary design D_1^{*c} of the resultant design with parameters $V_1^c = 12$, $B_1^c= 132$, $R_1^c= 55$, $K_1^c= 5$, $\Lambda_1^c = 20$; which is a 5-resolvable BIBD, whose blocks are given as follows

[(6,7,8,9,12), (1,7,8,10,11), (2,3,8,10,12), (1,4,6,10,12), (1,3,5,7,12),(1,3,4,8,9), (3,4,5,10,11), (2,4,5,9,12), (1,5,6,9,11), (2,3,7,9,12), (2,4,6,8,11), (2,5,6,7,10)]

[(3,8,9,10,12), (1,2,8,9,11), (4,7,8,11,12), (2,3,5,11,12), (2,4,6,9,12),(2,4,5,8,10), (1,4,5,6,11), (5,6,7,10,12), (1,2,3,6,10), (1,4,7,9,10), (1,3,5,7,8), (3,6,7,9,11)]

[(4,9,10,11,12), (1,2,3,9,10), (1,5,8,9,12), (1,3,4,6,12), (3,5,7,10,12), (3,5,6,9,11), (1,3,5,6,7), (6,7,8,11,12), (2,3,4,7,11), (2,5,8,10,11), (2,4,6,8,9), (1,4,7,8,10)]

[(1,2,5,10,12), (2,3,4,9,10), (2,6,9,11,12), (4,5,7,9,12), (4,6,8,10,12), (1,2,4,6,7), (3,6,7,8,9), (1,7,8,11,12), (1,3,4,5,8), (1,3,6,10,11), (2,3,5,7,11), (5,8,9,10,11)]

[(1,2,6,10,11), (1,3,4,5,11), (1,3,9,10,12), (3,5,6,8,10), (5,6,7,9,10), (1,2,5,8,9), (3,4,7,8,9), (2,7,8,10,12), (2,4,5,6,7), (2,4,9,11,12), (1,4,6,8,12), (3,6,7,11,12)]

[(1,2,3,7,12), (2,5,6,11,12), (1,4,8,11,12), (1,6,7,9,11), (1,2,6,8,10),(3,6,8,9,12), (5,8,9,10,11), (1,3,4,9,10), (3,5,6,7,10), (2,3,4,5,8), (4,5,7,9,12), (2,4,7,10,11)]

[(1,2,3,4,12), (1,2,6,7,8), (2,5,8,9,12), (3,7,8,9,12), (1,7,9,11,12), (2,4,7,9,10), (6,8,9,10,11), (4,5,10,11,12), (3,4,6,7,11), (1,4,5,6,9), (2,3,5,6,10), (1,3,5,8,11)]

[(3,4,5,6,12),(2,4,6,7,8),(2,4,9,10,12), (2,3,8,11,12), (1,6,8,10,12),(4,5,8,10,11), (1,2,7,10,11), (1,5,9,11,12), (1,3,5,7,8), (5,6,7,9,10), (3,4,7,9,11), (1,2,3,6,9)]

[(3,5,9,10,12), (3,4,6,11,12), (2,4,5,7,8), (1,4,5,6,10), (2,3,5,6,8), (1,2,6,9,12), (1,2,4,8,11), (1,5,7,8,9), (6,8,9,10,11), (2,3,5,7,9,11), (1,7,10,11,12), (3,4,7,8,10)]

[(4,5,8,9,11), (1,5,10,11,12), (2,6,8,11,12), (1,3,4,8,12),(1,2,5,7,8), (1,2,3,9,11), (2,3,7,10,12), (3,6,7,8,9), (1,4,7,9,10), (2,5,6,9,10), (3,4,6,10,11), (4,5,6,7,12)]
[(1,6,9,10,12), (1,3,5,11,12), (1,2,5,7,9), (3,4,5,6,9), (2,3,8,9,12), (1,2,3,4,10), (2,4,5,8,11), (4,7,8,9,10), (3,6,8,10,11), (2,7,10,11,12), (1,4,6,7,11), (5,6,7,8,12)]

Here every treatment is replicated 5 times in each resolution set. Hence, the design constructed above is 5-reolvable balanced incomplete block design.

3. Method of Construction-II

In a column-wise BIB design D^c with parameters $V^c=2k$, $B^c=2(2k-1)$, $R^c=2k-1$, $K^c=k$, $\Lambda^c=k-1$, there are r resolution sets with 2 blocks each. Considering the columns of each resolution set of this design as association schemes, clearly there will be r such association schemes with 2 groups of size k. The association scheme for each of these resolution sets is given as follows:

$$\begin{array}{cc} G_1 & G_2 \\ \hline \theta_1 & \theta_{(k+1)} \\ \theta_2 & \theta_{(k+2)} \\ \vdots & \vdots \\ \vdots & \vdots \\ \theta_k & \theta_{2k} \end{array} \tag{7}$$

Let the chosen GD design have the parameters $v^{**}=2k$, b^{**}, r^{**}, k^{**}, λ_1^{**}, λ_2^{**}, $m^{**}=2$, $n^{**}=k$. Now, using these association schemes given in (7) for each $r_{i,}$ (i=1,2,…r) and constructing GD designs with parameters $v^{**}=2k$, b^{**}, r^{**}, k^{**}, λ_1^{**}, λ_2^{**}, $m^{**}=2$, $n^{**}=k$. Denote these designs by $D_{i,}$ (i=1,2,….,r) and their juxtaposition given in the following structure

$$D_2^* = \left[D_1 \vdots D_2 \vdots D_3 \vdots \cdots \vdots D_r\right] \tag{8}$$

gives r^{**}- resolvable balanced incomplete block designs with the parameters $V_2=v^{**}$, $B_2=b^{**}r$, $R_2=r^{**}r$, $K_2=k^{**}$, $\Lambda_2=\lambda(\lambda_1^{**}-\lambda_2^{**})+r\,\lambda_2^{**}$.

Theorem 3.1: The existence of a column-wise BIB design and GD design with parameters $V^c=2k$, $B^c=2(2k-1)$, $R^c=2k-1$, $K^c=k$, $\Lambda^c=k-1$ and $v^{**}=2k$, b^{**}, r^{**}, k^{**}, λ_1^{**}, λ_2^{**}, $m^{**}=2$, $n^{**}=k$, respectively, implies the existence of r^{**}- resolvable balanced incomplete block design with the parameters $V_2=v^{**}$, $B_2=b^{**}r$, $R_2=r^{**}r$, $K_2=k^{**}$, $\Lambda_2=\lambda(\lambda_1^{**}-\lambda_2^{**})+r\,\lambda_2^{**}$.

Proof: In a column-wise BIB design with the parameters ($2k$, $2(2k-1)$, $2k-1$, k, $k-1$)), considering columns of each resolution set as an association scheme. Here there are 2k treatments, which are arranged in $m^{**}=2$ groups of size $n^{**}=k$ in each resolution set, in such a way that any two treatments in the same group are first associates and any two treatments from different groups are second associates. The association scheme for the chosen GD design is described in (7).

The GD design used here have the parameters v^{**}=2k, b^{**}, r^{**}, k^{**}, λ_1^{**}, λ_2^{**}, m^{**}=2, n^{**}=k. Under the present construction method, the required structure D_2^* defined in (8) yields the parameters V_2=v^{**}, B_2=$b^{**}r$, R_2=$r^{**}r$, K_2=k^{**}, which are obvious from the construction. Further, any pair, say (θ, ϕ), in column-wise BIB design is occurring λ times, therefore in λ resolution sets of D_2^*; it will be first associate and will occurs λ_1^{**} times and in the remaining (r-λ) resolution sets of D_2^*; it will be second associate, so it will occurs λ_2^{**} times. Thus, the frequency of pair (θ, ϕ) in the resultant design D_2^*can be calculated as

$$\Lambda_2= \lambda\lambda_1^{**} + (r - \lambda)\, \lambda_2^{**} = \lambda\, (\lambda_1^{**} - \lambda_2^{**}) + r\, \lambda_2^{**}$$

Also we can see that in each D_i, (i=1,2,….,r), every treatment is replicated r^{**} times. Hence the resultant design D_2^* is r^{**}- resolvable design. This completes the proof.

Example 3.2: Considering columns of BIB design D given in (3) with parameters v =8, b =14, r =7, k =4, λ =3 as association scheme of GD design

G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2	G_1	G_2
1	0	2	1	3	1	4	6	5	0	6	3	0	2
2	3	3	4	4	2	5	1	6	2	0	4	1	6
4	5	5	6	6	0	0	3	1	4	2	5	3	5
7	6	7	0	7	5	7	2	7	3	7	1	7	4

Now, construct GD designs having parameters v^{**}= 8, b^{**}=12, r^{**}=6, k^{**}=4, λ_1^{**}=2, λ_2^{**}=3, m^{**}=2, n^{**}=4 with each of these association schemes. Then the construction method yields a 6-resolvable balanced incomplete block design D_2^* with parameters V_2=8, B_2=84, R_2=42, K_2=4, Λ_{22}=18, Λ_{23}=6; whose blocks are given as follows

[(1,2,0,3), (4,7,5,6), (1,2,5,6), (4,7,0,3), (1,4,0,5), (2,7,3,6), (3,6,1,4), (0,5,2,7), (0,6,1,7), (3,5,2,4), (3,5,1,7), (0,6,2,4)]

[(2,3,1,4), (5,7,6,0), (2,3,6,0), (5,7,1,4), (2,5,1,6), (3,7,4,0), (4,0,2,5), (1,6,3,7), (1,0,2,7), (4,6,3,5), (4,6,2,7), (1,0,3,5)]

[(3,4,1,2), (6,7,0,5), (3,4,0,5), (6,7,1,2), (3,6,1,0), (4,7,2,5), (2,5,3,6), (1,0,4,7), (1,5,3,7), (2,0,4,6), (2,0,3,7), (1,5,4,6)]

[(4,5,6,1), (0,7,3,2), (4,5,3,2), (0,7,6,1), (4,0,6,3), (5,7,1,2), (1,2,4,0), (6,3,5,7), (6,2,4,7), (1,3,5,0), (1,3,4,7), (6,2,5,0)]

[(5,6,0,2), (1,7,4,3), (5,6,4,3), (1,7,0,2), (5,1,0,4), (6,7,2,3), (2,3,5,1), (0,4,6,7), (0,3,5,7), (2,4,6,1), (2,4,5,7), (0,3,6,1)]

[(6,0,3,4), (2,7,5,1), (6,0,5,1), (2,7,3,4), (6,2,3,5), (0,7,4,1), (4,1,6,2), (3,5,0,7), (3,1,6,7), (4,5,0,2), (4,5,6,7), (3,1,0,2)]

[(0,1,2,6), (3,7,5,4), (0,1,5,4), (3,7,2,6), (0,3,2,5), (1,7,6,4), (6,4,0,3), (2,5,1,7), (2,4,0,7), (6,5,1,3), (6,5,0,7), (2,4,1,3)]

The design constructed above is 6-resolvable balanced incomplete block design.

Since in the present illustration, the resultant design D_2^* yields 84 blocks out of which 70 blocks are distinct and remaining 14 blocks are repeated blocks which are from original design D. Thus D_2^* is a combination of two designs

(i) v=8, b= $\binom{8}{4}$ = 70, r=$\binom{7}{3}$ = 35, k=4, λ_2=$\binom{6}{2}$=15, λ_3=5, λ_4=1

(ii) v =8, b =14, r =7, k =4, λ_2 =3, λ_3 =1

Also in D_2^* Λ_{23}=6, which is obtained as follows

By definition of t-design, we have

$$\lambda_s = \lambda_t \binom{v-s}{t-s} \Big/ \binom{k-s}{t-s}$$

In design (i) given above, for t=4 and s=3, we get

$$\Rightarrow \lambda_3 = \lambda_4 \binom{5}{1} \Big/ \binom{1}{1}$$

$$\Rightarrow \lambda_3 = 1 * 5 = 5$$

and in design (ii) given above , λ_3 =1.

Hence, in the resultant design D_2^*; Λ_{23}=5+1=6 and it is 3-design as well.

Also, in the present construction V_2=2K_2 i.e. every block is having its complementary block. Thus, after rearrangement of blocks, this design can further be reduces to 1-resolvable balanced incomplete block design as

[(1,2,0,3), (4,7,5,6)], [(1,2,5,6), (4,7,0,3)], [(1,4,0,5), (2,7,3,6)], [(3,6,1,4), (0,5, 2,7)],[(0,6, 1, 7), (3,5,2,4)], [(3,5,1,7), (0,6,2,4)]

[(2,3,1,4), (5,7,6,0)], [(2,3,6,0), (5,7,1,4)], [(2,5,1,6), (3,7,4,0)], [(4,0,2,5), (1,6,3,7)], [(1,0,2, 7), (4,6,3,5)], [(4,6,2,7), (1,0,3,5)]

[(3,4,1,2), (6,7,0,5)], [(3,4,0,5), (6,7,1,2)], [(3,6,1,0), (4,7,2,5)], [(2,5,3,6), (1,0,4,7)],[(1,5,3, 7), (2,0,4,6)], [(2,0,3,7), (1,5,4,6)]

[(4,5,6,1), (0,7,3,2)], [(4,5,3,2), (0,7,6,1)], [(4,0,6,3), (5,7,1,2)], [(1,2,4,0), (6,3,5,7)],[(6,2,4, 7), (1,3,5,0)], [(1,3,4,7), (6,2,5,0)]

[(5,6,0,2), (1,7,4,3)], [(5,6,4,3), (1,7,0,2)], [(5,1,0,4), (6,7,2,3)], [(2,3,5,1), (0,4,6,7)], [(0,3,5, 7), (2,4,6,1)], [(2,4,5,7), (0,3,6,1)]

[(6,0,3,4), (2,7,5,1)], [(6,0,5,1), (2,7,3,4)], [(6,2,3,5), (0,7,4,1)], [(4,1,6,2), (3,5,0,7)],[(3,1,6, 7), (4,5,0,2)],[(4,5,6,7), (3,1,0,2)]

[(0,1,2,6), (3,7,5,4)], [(0,1,5,4), (3,7,2,6)], [(0,3,2,5), (1,7,6,4)], [(6,4,0,3), (2,5,1,7)],[(2,4,0, 7), (6,5,1,3)], [(6,5,0,7), (2,4,1,3)]

4. Results and Discussion

The following Tables I and II provide the list of α-resolvable balanced incomplete block designs for the Methods I and II; which are obtained by using certain known resolvable / affine resolvable BIB designs having 2*k* treatments and GD designs from Clatworthy table (1973).

Table I: For Method I

S. No.	Designs used				Resultant design						
	BIBD (when rows are taken as a association scheme)			GD design	α-Resolvable BIBD*						
	v	k	λ		V_1	B_1	R_1	K_1	Λ_{12}	Λ_{13}	α
1	8	4	3	R54	8	56	21	3	6	-	3
2	12	6	5	R175	12	132	77	7	42	-	7
3	12	6	5	SR69	12	176	88	6	40	16	8
4	12	6	5	S58	12	165	55	4	15	-	5
5	16	8	7	SR91	16	180	90	8	42	18	6

Table II: For Method II

S. No.	Designs used				Resultant design						
	BIBD (when columns are taken as a association scheme			GD design	α-Resolvable BIBD*						
	v	k	λ		V_2	B_2	R_2	K_2	Λ_{22}	Λ_{23}	α
1	8	4	3	SR38	8	84	42	4	18	6	6

*Since, the replication numbers of other designs are very high. Therefore, we have ignored those designs.

Khan (2000) gave the following conjecture in the form of an open problem "If *D* is a nontrivial balanced incomplete block design with parameters v, b, r, k, λ then

$$b \geq \left[\frac{(v-k)^2}{v-1}\right] + 2r - \lambda. \tag{9}$$

In the present paper this inequality holds true for every α-resolvable balanced incomplete block design, we have constructed in Methods I and II.

The following Table III provides the list of Bose's lower bound ($b \geq v+r-1$) and bound given in (9) for the parameter sets listed in Tables I and II.

Table III: Comparison of Lower Bounds

S.No.	Parameter sets (v,b,r,k,λ) of resultant designs	Block sizes of resultant design	α	Khan's Lower Bound	Bose's Lower Bound	Reference
1	(8,56,21,3,6)	56	3	40	28	Method I
2	(8,84,42,4,18)	84	6	66	49	Method II
3	(12,132,77,7,42)	132	7	114	88	Method I
4	(12,176,88,6,40)	176	8	139	99	Method I
5	(12,165,55,4,15)	165	5	101	66	Method I
6	(16,180,90,8,42)	180	6	142	105	Method I

On comparing the lower bound given in (9) with the Bose's lower bound, we have found that the lower bound given in (9) is more nearer and strengthen than the one given by the Bose.

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