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Designs for asymmetrical factorial experiment through confounded symmetricals

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Abstract

A general method of obtaining block designs for asymmetrical confounded factorial experiments using the block designs for symmetrical factorial experiments is proposed. The effect of the confounded interactions of the symmetrical factorial, in the context of the association scheme(s), on the connectivity of the asymmetrical factorial is discussed. The partitioned incidence matrices for the estimation of the confounded but recoverable interactions are found to be made up of those of disconnected designs and/or Cyclic Designs. The use of fractional symmetrical factorial to get fractional designs for asymmetrical factorial experiment is also discussed.

Key words: Block design; Symmetrical factorial; Asymmetrical factorial; Cyclic design

1 Introduction

Asymmetrical factorial designs were first introduced by Yates (1937). Since then a large number of research workers contributed to their construction. However, their use for experimentation, specially, in agricultural research has been limited due to the non-availability of suitable designs in small number of replications or experimental units. It is because, most of the efforts have been to find equally replicated designs balanced for the confounded interactions and were being addressed to get individual experiment or for getting them series-wise like q x 2^n or q x 3^n. The experimenter is therefore, resorting to the use of split- or strip- plot designs or compromising to the limiting of the levels of the factors and using the symmetrical factorial designs, in their place.

One of the techniques used was taking the help of the symmetrical factorial design, and obtaining the asymmetrical ones as their fractional replications to accommodate one or two factors of asymmetry (Das, 1960). Use of incomplete block design in combination with a symmetrical factorial design to obtain asymmetrical factorial designs with one factor of asymmetry is another technique. However, all these required a large number of replications, since balancing was sought for the interactions that were affected in the design. Repetition of some levels of the factor of asymmetry, forming their equi-sized groups equal to the number of levels of the factors of symmetry and use these groups as the levels instead to get the design in 2 or 3 replications was another attempt. Another technique proposed was to use some suitably chosen linear function to replace the combinations of 2 or more factors in a 2^n symmetrical factorial design and thus introduce the factor of asymmetry (Das and Rao, 1967). Using a one to one or one to many, in terms of fractional replication, association scheme for replacement of combinations of factors of a 2^n or 3^n symmetrical factorial design with the levels of the factor of asymmetrical factorial (Banerjee, 1970, Malhotra, 1989, Handa, 1990) was another technique studied for their construction. It is also known that Extended Group Divisible (EGD) designs, whenever existent, have orthogonal factorial structure with balance. Several methods of construction of EGD designs are available in the literature. For details one may refer to Parsad et al. (2007); Gupta et al. (2011) and references cited therein.

In this paper, we propose to provide a general method for the obtaining the asymmetrical factorial designs from the confounded symmetrical designs along with the study of the effects of confounding.

2 Preliminaries

Let F_i for i = 1, 2, … , k be the k-factors of the asymmetrical factorial experiment and the design be **D**, with F_i at p_i levels denoted as 0,1, … , (p_i-1), where $s^{n_i-1} < p_i \le s^{n_i}$, s is a prime number, n_i is a positive integer. Corresponding to the factor F_i, we introduce the i-th set of n_i pseudo-factors of the symmetrical factorial design **d***, viz., X_{ij} for j = 1, 2, …, n_i each at s levels denoted as 0, 1, …, (s-1).

We note that the main effects and interactions of the pseudo factors can be denoted as $X^\alpha = \prod_{i.j} X_{ij}^{\alpha_{ij}}$ for different values of α_{ij} [= 0, 1, …, (s-1)], with the restrictions that not all of them are simultaneously zero and the first occurring non-zero α_{ij} is one, each with (s-1) degrees of freedom. The generalized interaction between X^α and $X^{\alpha'}$ is obtained as X^β, where $\beta = c(\alpha + \alpha')$ (mod s), i.e., $\beta_{ij} = c(\alpha_{ij} + \alpha'_{ij})$ (mod s) and c, the constant is chosen to make the first occurring non-zero β_{ij} as one. It is known that when X^α is the confounded interaction in a replication of **d*,** the blocks of that replication will be s equal sized groups of treatment combinations depending on the value of the linear function $\sum_{ij} \alpha_{ij} x_{ij}$ (mod s), where x_{ij} is the level of the factor X_{ij} in the treatment combination. We refer these groups as $(X^\alpha)_u$ for u = 0, 1, … , (s-1), u being the value of the linear function. Further, each one of these interactions can themselves be considered as main effect of an imaginary factor $\underline{X}^\alpha$.

3 The method of construction

The method of construction involves in first preparing a scheme of (many to one) association scheme between the combinations of the i-th set of pseudo factors and the levels of the i-th factor of **D**, obtaining a suitable confounded symmetrical factorial design **d*** with the $n = \sum_i n_i$ pseudo factors X_{ij} in one or more replications and then replacing the pseudo factor combinations in **d*** as per the scheme of association to obtain the desired design **D** for the asymmetrical factorial experiment. In the sequel, we discuss the case for one replication of **d*** only and it can be extended, if need be, to more replications case, following the principles of the confounded symmetrical factorial designs.

Scheme of Association: Let $s^{n_i} = r_i (p_i - q_i) + (r_i + 1) q_i$. Obviously, $1 \leq r_i \leq (s-1)$. Each of the $(p_i - q_i)$ levels of F_i are to be associated with r_i treatment combinations of the i-th set of pseudo factors and each of the remaining q_i levels with $(r_i + 1)$ combinations. The combinations to be associated with a level of F_i will be chosen such that within their paired or larger groups they differ only in the level of one of the pseudo factors X_{ij} or the imaginary factor of their interaction. In other words, the combinations to be associated with the level f_i of F_i will be of the form $C_{i(u)}x_{iu}$ where $C_{i(u)}$ is a certain fixed combination of (n_i-1) factors other than X_{iu}, the chosen factor for differing levels for association and x_{iu} denote one of the differing levels of X_{iu} . Such chosen pseudo factors will be termed as AS-pseudo factors from the i-th set. In case the imaginary factor is chosen as the AS-pseudo factor, the combinations within the group could differ in their levels for more than one factor. The number of such AS-pseudo factors from a set will depend on the scheme of association chosen for that factor. For example in the case of $p_i=5$ and $s=2$ we will have $n_i=3$, $q_i=3$, and $r_i=1$. The eight combinations of the 3 pseudo factors X_{i1}, X_{i2}, X_{i3} viz., 000, 001, 010, 011, 100, 101, 110 and 111 can be associated with the 5 levels of the factor F_i respectively as (a) 0, 1, 2, 2, 3, 3, 4, and 4 or as (b) 2, 0, 2, 1, 3, 3, 4 and 4 or as (c) 2, 3, 2, 0, 1, 3, 4 and 4 when the AS-pseudo factors will be correspondingly (a) X_{i3} alone or (b) X_{i2} and X_{i3} or (c) all the three factors X_{i1}, X_{i2} and X_{i3}. Some such association schemes for the values of p_i ranging from 3 to 16 when $s=2$ and ranging from 4 to 26 when $s=3$ are given in Tables 1 and 2 in Appendix 1. These are only indicative and several other schemes are possible. We have for this presentation assumed that the higher levels will be repeated more often than the lower levels and have kept the number of AS-pseudo factors (imaginary factors representing their interactions not considered) to the maximum possible.

The next step involves choosing a suitable confounded symmetrical factorial design **d*** with the n pseudo factors. The choice will be in the block size and the interactions to be confounded between the blocks of the replication(s). The confounded interactions in **d*** in turn will decide those confounded or requiring adjustment for the blocks in the required design **D**. We, therefore, study the effects of confounding different interactions in **d*** on those of **D**.

Effects of confounding in **d***:
The contrasts for the main effects and interactions between the pseudo factors of i-th set, when considered in terms of the associated levels of the factor F_i, will not be always meaningful. However, those for the interactions free of the AS-pseudo factors will represent the orthogonal and independent contrasts between disjoint groups of the levels of the Factor F_i. The remaining of the (p_i-1) contrasts of the main effect of F_i along with the $(s^{n_i}-p_i)$ error contrasts, being the comparisons between the combinations used to associate with the same level of F_i in the association scheme, are obtained by recasting the contrasts for the interactions involving the AS-pseudo factors of the i-th set. This can be extended to other interactions involving pseudo factors from 2 or more sets. Accordingly, an interaction in terms of the pseudo factors of **d*** will represent an interaction of **D** obtained by replacing the pseudo factors of i-th set by F_i and/or error.

When an interaction free of any of the AS-pseudo factors is confounded in **d***, the values of the linear function for the treatment combinations of the pseudo factors that are associated with a level of the factor F_i will be same and will, therefore, be in the same group of blocks. This will result in disjoint groups of blocks and the corresponding interaction component of (s-1) contrasts will be confounded in **D** also.

In case the confounded interaction involves pseudo factors of a single set, say i-th, including the AS- pseudo factor, say X_{iu}, the groups formed will be connected in **D** through the r_i (if > 1) and/or (r_i+1) combinations $C_{i(u)}x_{iu}$ for different levels x_{iu} associated with the same level of F_i as they will be occurring in different groups in **d***. However, not all the sets need be connected. The design **D** can be connected if (r_i+1) = s for at least one of the levels. For other values of r_i+1 it may be possible to suitably choose the association scheme so that different such sets together make the design **D** connected. Since a block design (v, b, k) can be connected if $v \leq b(k-1)+1$, the design **D** can be connected even if the minimum value 1 of r_i is satisfied for a maximum of p_i-s+1 of the levels of F_i,. In the later case, it would be necessary that the association scheme is suitably chosen after taking into account the confounded interaction. Let $Z_i = \prod_{.j} X_{ij}^{\alpha_{ij}}$ be the interaction, involving only factors from i-th set, confounded in **d***. If, in Z_i, X is a factor X_{ij} with corresponding $\alpha_{ij} \neq 0$, it can be chosen as AS- pseudo factor and find treatment combinations, one from each of any two groups of the interaction i.e., $(Z_i)_0$, $(Z_i)_1$, … , $(Z_i)_{s-1}$ such that they differ only in the level of the factor X. Associating these two treatment combinations with a level of F_i provides connectivity of the corresponding blocks of the design **D**. Repeating the processes, with the same or different AS- pseudo factors, we can the get design **D** connected for this confounding. Only (s-1) such pairs will be required for this purpose and in this case for $p_i \leq (s^{n_i}$ - s +1). This is illustrated with the example of p_i=7, s=3. The pseudo factor combinations viz., 00, 01, 02, 10, 11, 12, 20, 21 and 22 may be associated a) with 0, 1, 2, 3, 5, 5, 6, 4 and 6 respectively if the confounded interaction is $X_{i1}X_{i2}$; b) with 0, 1, 2, 3, 5, 5, 6, 6 and 4 respectively if it is $X_{i1}X^2_{i2}$; and c) with 0, 1, 2, 5, 6, 6, 5, 3 and 4 respectively if it is either for getting connectivity. The AS- pseudo factors respectively are a) X_{i2}, b) X_{i2}, and c) both X_{i1} and X_{i2}.

However, when the interaction confounded involves pseudo factors from two or more sets and includes AS- pseudo factor(s), the sets formed will always be connected. We prove it below taking the case of interaction involving the 1st and the 2nd sets of the pseudo factors in **d*** and equivalently a 2-factor interaction F_1F_2 in **D**. The case of confounded interactions involving pseudo factors from more than 2 sets in **d*** or equivalently component of interaction involving more than 2-factors in **D** can be similarly shown to result in connected design **D**.

Let X_{2u} be an AS-pseudo factor in the confounded interaction $X_1^{\alpha}X_2^{\beta}$ of **d*** between factors from the 1st and the 2nd sets. Let also the interaction X_2^{β} be not confounded int ehe design. Further let $C_{2(u)}x_{2u}$ and $C_{2(u)}x'_{2u}$ be two of the combinations used to represent the same level f_2 of the factor F_2 and $D_iC_{2(u)}x_{2u}$ be the set of combinations occurring in the ith block of **d***, where D_i denotes certain combinations d_{i1}, d_{i2},… etc. of the pseudo factors of the 1st set and may be associated with the levels, say $f_{(1,1)}$, $f_{(1,.2)}$, … etc, of F_1, not all of them may be different. In this group the combination $C_{2(u)}x'_{2u}$ of the 2nd set will also occur but with different set of combinations, say D'_I, of the 1st set of factors, associated with the levels $f'_{(1,1)}$, $f'_{(1,2)}$, … etc. of F_1, i.e., as $D'_iC_{2(u)}x'_{2u}$. The combination $D_0C_{2(u)}x'2u$ will occur in the tth block where $\beta_{2u}(x'_{2u}-x_{2u}) = t$ (mod s). Thus the 0th and the tth blocks of **D** will be connected through the common combinations. In fact, the sets D_i and $D'_{(i+t)}$ will be the same. Further, D'_i and $D_{(i+s-t)}$ will also be same. Since s is prime, the 0th block will be connected with the tth, through it with (2t)th , then with $(3t)^{th}$, … , $(st)^{th}$ and hence the design **D** will be a connected one for this confounding. This is similar to the one in the case of cyclic designs with a block size 2 with x_{2u} and x'_{2u} in the initial blocks and the other blocks generated over the levels of X_{2u}. Contrasts of the corresponding interaction component of F_1F_2 in **D** will be estimable.

We will state the above results in the following theorems.

Theorem 1: Given an interaction $\prod_{i,j} X_{ij}^{\alpha_{ij}}$ of **d***, its corresponding interaction in **D** will be a component of $\prod_i F_i^{\delta_i}$, where δ_i=0 if α_{ij}=0 for all j; and =1 otherwise. In other words it is that of the one obtained by replacing all the factors occurring from the i-th set of pseudo factors in the interaction of **d*** by the factor F_i of **D**.

Theorem 2: If the interaction confounded in **d*** involves none of the AS-pseudo factors, the corresponding component interaction will also be confounded with blocks in **D**.

Theorem 3: If the interaction confounded in **d*** involves factors, including at least one of the AS-pseudo factors, from a single set of pseudo factors, the corresponding main effect component of **D** will not be completely confounded between the blocks and some of its contrasts will be estimable. It is however possible to choose the association scheme suitably for $p_i \leq (s^{n_i}$ - s +1) so that all the contrasts of the corresponding interaction component of **D** are estimable.

Theorem 4: If the interaction confounded in **d*** involves factors from more than one set of pseudo factors and include at least one of the AS-pseudo factors X_{iu}, and further if no interaction between factors of this i-th set only is confounded, the corresponding interaction component of **D** will not be confounded in the sense that all the corresponding interaction contrasts will be estimable.

We have in the theorem 4 above placed the restriction that if X_{iu} is considered as the AS-pseudo factor, then no interaction involving only factors from the i-th set can be confounded for connectivity of the design **D**. Consider the case where the requirements mentioned in theorem 3 for the connectivity of **D** are satisfied when such an interaction of factors of the i-th set only is confounded. Let A denote the AS-pseudo factor and A$\underline{B}$ be the confounded interaction, $\underline{B}$ being the imaginary factor for the interaction, involving other i-th set factors. We denote a treatment combination of the i-th set as C_i ab, where C_i is a combination of levels of the other n_i-2 factors and a and b of the factors A and $\underline{B}$ .When A$\underline{B}$ is confounded, the combinations C_i have no role to play in deciding the placement of the combinations into the groups $(A\underline{B})_r$ and hence we ignore them in our further discussion, without any loss of generality. Let a_rb_r be the combination from the r-th group and $a'_{r+1}b'_{r+1}$ from the(r+1)–th group that are associated with the same level of Fi for r = 0, 1,…, (s-2). This will ensure the connectivity of **D** when A$\underline{B}$ is confounded. We now extended it to the case r = (s-1) when r+1 = s = 0 (mod s) and this will result in connectivity similar to that in case of cyclic designs. We differentiate these two types of connectivity as linear in the former case and circular in the later case,

Now, let further an interaction involving factors from 2 or more sets is confounded in **d*** which includes the AS-pseudo factor A. We denote it as A$\underline{Y}$, ignoring other factors of i-th set, where $\underline{Y}$ is an imaginary factor for an interaction involving factors from sets other than i-th. The confounding of $A\underline{B}$ and $A\underline{Y}$ result in forming s^2 groups of combinations and we need their corresponding ones in $\underline{\mathbf{D}}$ to be connected. We can denote these groups as (r, t) which are common to $(A\underline{B})_r$ and $(A\underline{Y})_t$.

Starting from the (0,0) group we shall examine when or how they are connected. Let $a_0b_0y_0$ be a combination in (0,0). Thus we have $a_0+b_0 = 0$ and $a_0+y_0=0$. We find that the combination $a'_1b'_1\, y_0$ will be occurring in $(1,t^1{}_1)$ group where $a'_1+b'_1 = 1$ and $a'_1+y_0 = t^1{}_1$ (mod s). In this group, the combination $a_1b_1y^1{}_1$ will also be occurring, where $y^1{}_0 = y_0$ and $y^1{}_1 = y^1{}_0 + a'_1 - a_1$.Thus in $\underline{\mathbf{D}}$ the groups corresponding to (0,0) and $(1,t^1{}_1)$ of **d*** will be connected. Continuing, we see that the groups in $\underline{\mathbf{D}}$ corresponding to the groups $(r,t^1{}_r)$. and $(r+1,t^1{}_{r+1})$ of **d*** will be connected for r = 0, 1, 2, … , (s-2) only in case of linear connectivity of the groups for the confounding of $A\underline{B}$ in **d*** and for r = 0, 1, 2, … , (s-1) in case of circular connectivity through the combinations $a_rb_ry^1{}_r$ and $a_{r+1}b_{r+1}y^1{}_r$ where $t^1{}_0 = 0$, $t^w{}_s = t^{w+1}{}_0$ $y^w{}_{r+1} = y^w{}_r + a'_{r+1} - a_{r+1}$ for w = 1, 2, … , s. We note that in case of linear connectivity, only s out of s^2 of the corresponding (r,t) groups will be connected in $\underline{\mathbf{D}}$. In case of circular connectivity for $A\underline{B}$, it gets reconnected to $(0,t^1{}_0)$ if in the association scheme $\sum_r (a'_r-a_r) = 0$ (mod s), when again it results in connectivity of only s out of s^2 of the corresponding (r,t) groups of $\underline{\mathbf{D}}$. However, by choosing the association scheme such that $\sum_r (a'_r-a_r) = v \neq 0$ (mod s), the connectivity continues through groups corresponding to (0,v), (0,2v), …, ending with (0, 0) of **d*** and thus $\underline{\mathbf{D}}$ will be connected for the confounding of $A\underline{B}$ and $A\underline{Y}$. We can arrive at the same result even if we started from any of the (r,t) groups instead of (0,0) group of **d***. We need to study each of the confounded interactions as also their generalized interactions in **d*** for concluding about the connectedness of $\underline{\mathbf{D}}$.

We illustrate the above with the help of an example. Consider **d*** a confounded design for a 3^3 experiment in 3 plot blocks with the pseudo factors X_{11}, X_{12} and X_{21} each at 3 levels and confounding the interactions $X_{11}X_{21}$, $X_{12}X_{21}$ and their generalized interactions $X_{11}X_{12}X_{21}{}^2$ and $X_{11}X_{12}{}^2$. The interactions corresponding to the A, $\underline{B}$ and $\underline{Y}$ mentioned above are respectively X_{11}, $X_{12}{}^2$ and X_{21}. The blocks formed as the groups (r,t) are

(r , t)	(0 ,0)	(0, 1)	(0,2)	(1, 0)	(1, 1)	(1, 2)	(2, 0)	(2, 1)	(2, 2)
Block Contents	000 112 221	001 110 222	002 111 220	020 102 211	021 100 212	022 101 210	010 122 201	011 120 202	012 121 200

We then consider the design $\underline{\mathbf{D}}$ with factors F_1 at levels 7 or 6 and F_2 at 3 levels that can be obtained. The corresponding interactions that will be affected in $\underline{\mathbf{D}}$ will be F_1F_2 due to the first three interactions and the main effect F_1 due to the last mentioned interaction viz.., $X_{11}X_{12}{}^2$.

<u>Case 1</u>: $\underline{\mathbf{D}}$ is design 7 x 3 in 3 plot blocks.

We have $7 = 3^2-3+1$. Confounding $X_{11}X_{12}{}^2$ we have the three groups of the combinations of $X_{11}X_{12}{}^2$ as follows: (**00**,11,22),(02,***10***,**21**) and (01,12,***20***). The combinations in **bold** type are the (a_rb_r)s and the ***bold and italics*** are the $(a'_rb'_r)$s. The type of the connectivity is one of linear type. We associate the combinations 00 and 10 with the same level, say 6, of F_1 and levels 21 and 20 with the level, say 5, of F_1. The AS-pseudo factors here are X_{11} and X_{12} respectively. We

associate the other combinations 01, 02, 11, 12 and 22 with the levels 0, 1, 2, 3 and 4 of F_1 respectively. The resulting 7 x 3 design **D** in 9 plots each of size 3 is as below (repeated combinations are shown in bold):
(**60**, 22, 41), (**61**, 20, 42), (**62,** 21, 40), (10, **62, 51**), (11, **60**, **52**), (12, **61**, **50**), (00, 32, **51**), (01, 30, **52**), (02, 31, **50**).

This design is disconnected for the estimation of the interaction F_1F_2 and the blocks form 3 groups each of 3 connected blocks through the treatment combinations (60 and 52), (61 and 50), and (62 and 51). It is not possible to get a connected 7 x 3 design in 9 blocks of 3 plots each since we have 21 treatments and the the rank of the **C**-matrix can utmost be 18. We also note that if **D** was a 7 x 2 design, it would be a connected design as X_{21} would also be AS-pseudo factor (Theorem 4).

Case 2(a): Disconnected **D** design 6 x 3 in 3 plot blocks.

Here we have $6 = 3^2-3$. Using the groups (**00**,11,***22***),(02,***10***,**21**) and (**01**,12,***20***) we have the circular type of connectivity by associating the combinations 00 and 10 with the same level, say 5, of F_1; levels 21 and 20 with the level, say 4, of F_1; and levels 01 and 22 with the level 3 of F_1.The remaining combinations 02, 11 and 12 will be associated with levels 0, 1 and 2 of F_1. The resulting 7 x 3 design **D** in 9 plots each of size 3 is as below.
(**50**, 12, **31**), (**51**, 10, **32**), (**52,** 11, **30**), (00, **52, 41**), (01, **50**, **42**), (02, **51**, **40**), (**30**, 22, **41**), (**31**, 20, **42**), (**32**, 21, **40**). Here again the design is disconnected and the blocks form 3 groups of 3 each and within each group the blocks will be circularly connected. We note that for the association scheme chosen we have $\sum_r (a'_r - a_r) =$ (2-0)+(1-2)+(2-0) = 0 (mod 3).
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Case 2(b): Connected **D** design 6 x 3 in 3 plot blocks.

Using the groups (**00**,11,***22***),(02,***10***,**21**) and (01,**12**,***20***) we have the circular type of connectivity by associating the combinations 00 and 10 with the same level, say 5, of F_1; levels 21 and 20 with the level, say 4, of F_1; and levels 12 and 22 with the level 3 of F_1.The remaining combinations 01, 02 and 11 will be associated with levels 0, 1 and 2 of F_1. The resulting 7 x 3 design **D** in 9 plots each of size 3 is as below.
(**50**, 22, **31**), (**51**, 20, **32**), (**52,** 21, **30**), (10, **52, 41**), (11, **50**, **42**), (12, **51**, **40**), (00, **32**, **41**), (01, **30**, **42**), (02, **31**, **40**). This design is disconnected. We note that for the association scheme chosen we have $\sum_r (a'_r - a_r) =$ (2-0)+(1-2)+(2-1) = 2 (mod 3) is non zero.

We now state the results above in the following theorem.

Theorem 5: If an interaction involving only factors from i-th set of pseudo factors is confounded in **d*** involving a pseudo factor, say A and if $p_i \leq (s^{n_i} - s)$, it is possible to choose the association scheme such that the design **D** remains connected for the confounding of interaction A<u>Y</u> in **d***, where <u>Y</u> stand for some interaction involving factors from other than the i-th set..
These results will help in deciding the interactions to be confounded in **d*** and choosing the association schemes for different factors F_i. However, each of the confounded interactions along with their generalized interactions are to be considered for their effects on the connectivity of the design **D**.

We sum up with the presentation of the steps in obtaining **D** from **d*** along with live examples.

Step 1: Given the **D** choose the suitable s, keeping in view the resources available, including the experimental units and the block size. Choice of s, automatically determines the values of n_i's and thus n and s^n the size of experiment using one replication of **d***. As examples, we consider the two designs 2x3x5 and 3x5x7 for which the results are presented in the table below.

Design **D**	2x3x5 (30combinations)			3x5x7 (105 combinations)				5 x 6 (30 combinations)			
s	2	3	5	2	3	5	7	2	3	5	7
n_1	1	1	1	2	1	1	1	3	2	1	1
n_2	2	1	1	3	2	1	1	3	2	2	1
n_3	3	2	1	3	2	2	1	-	-	-	-
s^n	2^6 =64	3^4 =81	5^3 =125	2^8 =256	3^5 =243	5^4 =**625**	7^3 =343	2^6 =64	3^4 =81	5^3 =**125**	7^2 =49
AS-pseudo factors	X_{22}, & all X_{3j}	X_{11}, & all X_{3j}	X_{11} & X_{21}	X_{12}, all X_{2j} and X_{33}	All X_{2j} and all X_{3j}	X_{11}, all X_{3j}	X_{11} and X_{21}	All **but** X_{21}	All	X_{22}	All

Obviously, s=2 or 3 are the preferred in all the cases and may be s=5 in case of 2x3x5 design and s=7 in case of 3x5x7 and 5 x 6 in view of lesser number of the design points per replication of **d*.**

Step 2: Choose the block size. It is of the form s^r and, generally, the value of 'r' does not exceed 3, 2 and 1 for s = 2, 3, and 5 and above, respectively.

Step3: Choose the maximum number of AS-pseudo factors from each set of **d***. They can be as many as (s^{n_i} -p_i) or n_i whichever is smaller. Decide which of the q_i levels of F_i are to be associated with (r_i+1) combinations of the s^{n_i} combinations of the i-th set in the scheme of association. Using these choose the association scheme to include not necessarily the same but also other AS-pseudo factors for all the r_i or r_i+1 combinations to be associated with the level of F_i. In case any interaction involving factors from only one set are confounded in **d***, the association scheme has to be carefully chosen to have the circular connectivity and not linear one for this confounded interaction.

Step 4: Choose the interactions, including their generalized ones, to be confounded in **d***. They should each include factors from two or more sets and one of the AS-pseudo factors, unless the corresponding interaction in **D** is of little interest. If the interaction involves factors from only one set and if $p_i \leq$ (s^{n_i} - s), the scheme of association is to be chosen suitably for connectivity.

The interactions that can be confounded and the association schemes in the above designs for different values of s are illustrated for the small block sizes with connectivity..

Design	Block size	Interactions to be confounded (AS-pseudo factors in **bold** font)	Factor	Association scheme + (Groups of combinations to associate with a level of F_i)
2 x 3 x 5	4	X_{11} $\mathbf{X_{22}}$, $\mathbf{X_{22}}$ $\mathbf{X_{31}}$, X_{21} $\mathbf{X_{32}}$ and X_{11} $\mathbf{X_{33}}$	F_2 F_3	(00,01) or (10,11) (000,001), (101,111), (010,110)
2 x 3 x 5	3	$\mathbf{X_{11}}$ X_{21}, X_{21} $\mathbf{X_{31}}$ and $\mathbf{X_{11}}$ X^2_{11} $\mathbf{X_{32}}$	F_1 F_3	(0,1) or (0,2) or (1,2) (00,01), (12,02), (10,11), (20,21)
3 x 5 x 7	8*	X_{11} $\mathbf{X_{21}}$, $\mathbf{X_{22}}$ X_{31}, $\mathbf{X_{23}}X_{32}$, $\mathbf{X_{12}}$ X_{31} and X_{11} $\mathbf{X_{22}}$ $\mathbf{X_{33}}$	F_1 F_2 F_3	(00,01) or (10,11) (000,001), (101,111), (010,110) (000,001)
3 x 5 x 7	9*	X_{11} $\mathbf{X_{21}}$, $\mathbf{X_{22}}$ X_{31} and X_{11} $\mathbf{X_{32}}$	F_2 F_3	(00,01), (12,02), (10,11), (20,21) (00,01), (12,02)
5 x 6	4	$\mathbf{X_{11}}$ X_{21}, $\mathbf{X_{12}}$ $\mathbf{X_{22}}$, $\mathbf{X_{13}}$ $\mathbf{X_{23}}$ and $\mathbf{X_{11}}$ $\mathbf{X_{13}}$ $\mathbf{X_{22}}$	F_1 F_2	(000,001), (100,110), (011,111) (000,001), (100,110)
5 x 6	3	$\mathbf{X_{11}}$ $\mathbf{X_{21}}$, $\mathbf{X_{12}}$ $\mathbf{X_{22}}$ and $\mathbf{X_{11}}$ $\mathbf{X_{22}}$	F_1 F_2	(00,01), (12,02), (21,11), (20,22) (00,01), (12,02), (21,11)

+ Remaining combinations to be associated individually with a level of F_i.

*since $p_i \leq (s^{n_i} - s)$ is not satisfied for F_3, smaller block size is not possible without confounding its main effect.

Step5: Choose the number of replications of **d*** to be used for obtaining **D**. Many a times for values of p_i very near to s^{n_i}, the designs **D** (for example, 4 x 5 in 5 plot blocks) obtained from a single replication of **d*** provide a small number of degrees of freedom for error and necessitate repetition of the process using a second replication of **d***, not necessarily confounding the same set of interactions and using the same association scheme. In case of **D** such as 2 x 3^2 in 3 plot blocks one of the components of the interaction F_2F_3 (with 2 of the 4 degrees of freedom) gets completely confounded, thus necessitating use of another replication of **d*** so that the other component of F_2F_3 gets confounded and the design **D** from the two replications of **d*** will be balanced for the interaction F_2F_3. While the association scheme in different replications need not be the same the, the q_i levels of F_i to be associated with (r_i+1) combinations of the i-th set of pseudo factors of **d*** need to remain unchanged over the replications for orthogonality.

An indicative list of designs obtained through this technique, along with the interactions confounded and the AS-pseudo factors of **d*** is presented in Table 3 in Appendix 1.

4 Analysis of variance of data

The analysis of variance of data, with the availability of computers, does not prove problematic. Obviously, in the design **D** the factors are orthogonal to one other since the combinations of different sets of pseudo factors of **d*** are orthogonal to one another, and these from different sets were used to be replaced by the levels of different factors F_i after merger within the sets. Thus, the interactions which are not confounded can be estimated orthogonally. The normal equations for the estimation of any main effect or interaction effects can be freed from one other but for the block effects, in the case of interactions corresponding to those involving at least one of the AS-pseudo factors. However, these after adjustment for the block effects also can be shown to remain free of other confounded effects since the block effects in **d*** are free of the effects of interactions not confounded in it.

The procedure for the analysis of data from **D** proceeds in the same manner as in the case of block designs. First, we obtain the Sums of Squares (SS) due to the blocks unadjusted. Then we proceed to get the SS due to the different main effects, then the SS due the interactions from lower to the higher order interactions after adjusting for the block effects, wherever necessary. The reduction in the degrees of freedom need be made due to the confounded interaction components, if any. The SS due to any v-factor interaction along with the lower order interactions of these factors is obtained as for the treatment SS (adjusted for blocks) in block designs by considering the v-factor combinations as the treatments and ignoring all other factors for the purpose. We notice that in case the design is connected for these combinations, their partitioned incidence matrix will be made up of that of a disconnected design and/or cyclic design(s). For example in the above discussed design 6x3 in 3 plot blocks, under case 2(b), the S.S. due to the main effect F_2 with 2 d.f. can be obtained directly as if it were a randomized block design with 9 replications. The S.S. due to F_2 with 5 d.f. is obtained as in the case of a connected block design with the lay out as (1,3,5), (0,4,5) and (2,3,4). The S.S. due to the treatments with 17 d.f. is obtained treating **D** as a connected block design with the treatment combinations as the treatments and the S.S. due to interaction F_1F_2 with 10 d.f. is obtained by subtraction. The error S.S. carries only 1 d.f.

5 Use of fractional replication

The technique above can be extended not only to the case of multiple replications of **d*** but also when a fractional replication of it used. We explain with the help of an example of design **D**, for a $7x3^3$ experiment with factors A B and C at levels 7, 3, 3 and 3 respectively, in 9 plots each of size 9, using **d*** a (1/3) 3^5 design with pseudo factors A_1 and A_2 (corresponding to factor A),; and B and C. We use the defining contrast $I = A_1A_2BC$, and confound A_1B^2 and A_2C^2 for blocking. The combinations 00 and 20 are used to denote one of the levels of A, the combinations 21 and 22 to denote another level, and the remaining combinations to denote individually a level of A. Thus both A_1 and A_2 will be the AS-pseudo factors. The resulting design will be a disconnected one as below:

Block 1: (0000, 0202, 1011, 2022, 3101, 4112, 5120, 6210, 6221)
Block 2: (0102, 0001, 1110, 2121, 3299, 4211, 5222, 6012, 6020)
Block 3: (0201, 0100, 1212, 2220, 3002, 4010, 5021, 6111, 6122)
Block 4: (0021, 0220, 1002, 2010, 3122, 4100, 5111, 6201, 6212)
Block 5: (0120, 0022, 1101, 2112, 3221, 4202, 5210, 6000, 6011)
Block 6: (0222, 0121, 1200, 2211, 3020, 4001, 5012, 6102, 6110)
Block 7: (0012, 0211, 1020, 2001, 3110, 4121, 5102, 6222, 6200)
Block 8: (0111, 0010, 1122, 2100, 3212, 4220, 5201, 6021, 6002)
Block 9: (0210, 0112, 1221, 2202, 3011, 4022, 5000, 6120, 6101)

The main effects A, B, C and D as also the two-factor interactions BC, BD, CD are orthogonal to blocks. If the two factor interactions between the symmetric factors B, C and D and the 3rd and higher order interactions are considered negligible this design provides the estimation of the main effects and the two factor interactions involving the factor A since the AS-pseudo factors connects the design for such estimation.

References

Banerjee, A.K. (1977). On construction and analysis of p x q confounded asymmetrical factorial designs. *J. Ind. Soc. Agril. Statist.,* **29**, 42-52.

Das, M.N. (1960). Fractional replications as asymmetrical factorial designs. *J.Ind. Soc. Agril. Statist.*, **12**, 159-174.

Gupta, V.K., Nigam, A.K., Parsad, R., Bhar, L.M. and Behera, S.K. (2011). Resolvable block designs for factorial experiments with full main effects efficiency. *J. Ind. Soc. Agril. Statist*, **65(3)**, 303-315.

Handa, D.P. (1990). Investigation on design and analysis of factorial experiments. *Unpublished Ph.D. Thesis, IASRI (ICAR), New Delhi*.

Malhotra R. (1989). Some studies of design and analysis of factorial experiments. U*npublished Ph.D. Thesis, IASRI (ICAR), New Delhi.*

Parsad, R., Gupta, V.K. and Srivastava, R. (2007). Designs for cropping systems research. *J. Statist. Plann. Inf.*, **137**, 1687-1703.

Yates, F. (1937). The Design and Analysis of Factorial Experiments. *Imp. Bur. Soil Sc., Tech. Comm*. No. 35.

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