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Yeh-Bradley conjecture on block designs with block size odd

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Abstract

Yeh and Bradley (1983) made a conjecture about the trend-free designs. Stufken (1988) gave some counterexamples for classes of designs with k is odd to show the conjecture is not true in general. This paper especially focuses on designs when the block size is odd. The conjecture is further investigated and several sufficient conditions for a design can be converted into a linear trend-free design by permuting the positions of treatments within blocks are obtained. Some designs in two-associate PBIB designs with $\lambda_1 = 0$ or λ_2=0 are proved can be converted into linear trend-free designs.

Key words: System of distinct representatives; Yeh-Bradley conjecture; Linear trend-free designs; Two-associate PBIB designs.

1 Introduction

In the classical block design setting we believe that the observations are affected by the treatment and the block effects only. Hence the order of the treatment applied to the experimental units in a block won't affect the observations. But in other situations, especially when the treatments are applied to the experimental units sequentially over time or spaces in a block, there is a probability that a systematic effect, or trend, influence the observations in addition to the treatment and the block effects. Facing this possible trend in the block, the usual analysis of block designs will not be proper any more.

Bradley and Yeh (1980) introduced the properties and theory of trend-free block designs. Trend-free block designs, can eliminate the trend effect by properly rearranging the treatment positions within blocks, will allow you to analyze the treatment effects as same as in the classical block design even if the trend effect is present.

Yeh and Bradley (1983) conjectured that every binary incomplete block design with parameters v, b, k and r can be converted into a linear trend-free block design by exchanges of plot positions for treatments within blocks if and only if $\frac{1}{2}r(k+1)$ is an integer. They also proved that the conjecture is true when $k = 2$. Stufken (1988) gave a family of counterexamples which are designs with k is odd and certain properties to show the conjecture is not correct in general. Chai and Majumdar (1993) proved that the conjecture is true for the following two kinds of designs (i) whenever k is even; (ii) BIBD (BBD) families. Majumdar (1996) showed that the Yeh-Bradley's conjecture is valid for Balanced Treatment Incomplete Block (BTIB) designs which are the high efficient designs for test-control treatment comparisons experiments. Lin and Stufken (1999) introduced and discusses a new algorithm to convert a given binary block design into a linear trend-free block design. Lin and Stufken (2002) elaborated the connection between the problem of finding strongly linear trend-free block design and a well-known problem in graph theory. Based on the connection, they found more classes of designs with some sufficient conditions can be converted into the linear trend-free block designs.

In this paper, we study the Yeh-Bradley conjecture further. Firstly, based on the ideas of the Stufken's counterexamples, we can construct more classes of designs that can not be converted into linear trend-free designs (see Theorem 3.1). Secondly, the connection of the truth of the Yeh-Bradley conjecture (i) between $D(v,b,k,r)$ and $D(\frac{v}{v_1},b,k,rv_1)$, with v is multiple of v_1; (ii) between $D(v,b,k,r)$ and $D(v,b,\alpha k,\alpha r)$ for positive integer α, are obtained (see Theorems 3.2 and 3.3). These two results can help us to focus on the smaller classes of designs when we seek for the truth of the conjecture. Thirdly, the conjecture is proved to be true for each binary design $d \in D(v = 3k, b, k, r)$ provided k is odd and r is even (see Theorem 3.6). Also, we derive several sufficient conditions for the designs can be rearranged into a linear trend-free designs (see Theorems 3.4, 3.6 and 3.7) and those results can be applied to two-associate-class PBIB designs with $\lambda_1 = 0$ or λ_2=0 (see Remark 3.1).

Some preliminary results and notations about linear trend-free block designs are given in Section 2. The main results and examples are given in Section 3. Section 4 has the concluding remarks.

2 Notation and preliminary results

We assume that the model for an observation in period l of block j, $1 \leq l \leq k$, $1 \leq j \leq b$ is

(2.1) $$y_{jl} = u + \sum_{i=1}^{v} \delta_{jl}^{i} \tau_i + \beta_j + \theta_1 \phi_1(l) + \epsilon_{jl}.$$

Here u is a general effect, $\tau_1, ..., \tau_v$ the treatment effects, $\beta_1, ..., \beta_b$ the block effects and θ_1 is the regression coefficient of $\phi_1(l)$. The common trend effect on period l of each block is $\theta_1 \phi_1(l)$. Moreover, $\phi_1(l)$ satisfies

$$\sum_{l=1}^{k} \phi_1(l) = 0, \qquad \sum_{l=1}^{k} \phi_1^2(l) = 1,$$

and

$$\delta_{jl}^{i} = \begin{cases} 1, & \text{if treatment } i \text{ is applied in period } l \text{ of block } j, \\ 0, & \text{otherwise,} \end{cases}$$

with $\sum_{i=1}^{v} \delta_{jl}^{i} = 1.$

A design d will be represented by a $k \times b$ array with elements from $S = \{1, 2, ..., v\}$. Thus, if the symbol i appears in cell (l, j) of d, it means that treatment i has to be applied in period l of block j under d. Let $D(v, b, k)$ be all connected designs in b blocks , k periods based on v treatments under model (2.1). To avoid trivialities we consider henceforth only classes $D(v, b, k)$ with $k \geq 2$. For $d \in D(v, b, k)$, let s_{dil} denote the number of times treatment i appears in row (period) l and $r_{di} = \sum_l s_{dil}$ denote the number of times treatment i occurs in the design. We shall use the notation:

$$D(v, b, k; r_1, ..., r_v) = \{d \in D(v, b, k) : r_{di} = r_i, 1 \leq i \leq v\},$$

$$D(v, b, k, r) = \{d \in D(v, b, k; r_1, ..., r_v) : r_i = r, 1 \leq i \leq v\}.$$

Bradley and Yeh (1980) showed that a design $d \in D(v, b, k, r)$ is linear trend-free if and only if

(2.2) $\sum_{l=1}^{k} s_{dil} \phi_1(l) = 0$, *i.e.*, $\sum_{j=1}^{b} \sum_{l=1}^{k} \delta_{jl}^{i} l = r(k+1)/2$, $i = 1, 2, ..., v$.

Because of the symmetric properties of the orthogonal polynomial $\phi_1(l)$, (2.2) is true whenever

(2.3) $$s_{dil} = s_{di(k-l+1)}, \qquad l = 1, ..., [(k+1)/2], \qquad i = 1, ..., v.$$

Definition 2.1 (Chai and Majumdar (1993)). An array d_{tf}, derived from a design $d \in D(v, b, k, r)$ by permuting symbols within columns, will be called a linear trend-free version of d if it satisfies (2.2). The array d_{tf} will be called a strongly linear trend-free version of d if it satisfies (2.3).

Remark 2.1. Let $r(k+1)/2$ be an integer. Then the following three statements are equivalent. (i) Yeh and Bradley's conjecture is true in $D(v,b,k,r)$; (ii) each design $d \in D(v,b,k,r)$ has a linear trend-free version; (iii) each design $d \in D(v,b,k,r)$ can be converted into a linear trend-free block design by rearranging treatments within blocks.

Theorem 2.1 (Chai and Majumdar (1993)). Let $k, r_1, ..., r_v$ be even numbers. For each design in $D(v,b,k;r_1,r_2,...,r_v)$ there exists a strongly linear trend-free version.

Theorem 2.2. Let $d \in D(v,b,k,r)$ with k odd and r even. Suppose, in d, there exists a collection F, collects one symbol from each column of d, contains some symbols from the set $\{1,2,...,v\}$, repeats an even number of times each. Then d has a strongly linear trend-free version.

Remark 2.2. Theorem 2.2 is a special case of Theorem 3.2 of Chai and Majumdar (1993).

Theorem 2.3 (Stufken (1988)). Suppose k is odd and $(r,k) = 1$ ((r,k) denotes the greatest common divisor of r and k). If there exists positive integers α ($> k$), and β such that $\beta k = \alpha r - 1$, then there exists a design $d^S \in D(v^*, b^*, k, r)$, with $v^* = \alpha k$ and $b^* = \alpha r$, can't be converted into a linear trend-free block design. Hereafter, call d^S is Stufken-type counterexample design.

3 Main results

Yeh and Bradley's conjecture is still unsolved for some cases. The following theorem, based on the special construction of the Stufken's families, shows more designs can't be converted into the linear trend-free designs.

Theorem 3.1. Suppose k is odd and $(r,k) = 1$. If there exists positive integers α ($> k$), α_1 ($> k$), β and β_1 such that $\beta k = \alpha r - 1$ and $\beta_1 k = \alpha_1 r + 2$, then exists a design $d^* \in D(v^*, b^*, k, r)$, with $v^* = 2\alpha + \alpha_1$ and $b^* = 2\beta + \beta_1$, can't be converted into a linear trend-free design.

Proof. Construct the connected $d^* = [d_1|d_2|d_3]$, where d_i is a $k \times \beta$ array such that symbols $(i-1)\alpha+1, (i-1)\alpha+2, ..., (i-1)\alpha+\alpha-1$ appear r times and $(i-1)\alpha+\alpha$ appears $(r-1)$ times, for $i = 1, 2$ and d_3 is a $k \times \beta_1$ array such that symbols $2\alpha+1, 2\alpha+2, ..., 2\alpha+\alpha_1$ appear r times and symbols α and 2α appear only once and in the same column. Notice that d_1 and d_2 exist since $\beta k = \alpha r - 1$ and d_3 exists since $\beta_1 k = \alpha_1 r + 2$. Also, it is easy to see $d^* \in D(v^*, b^*, k, r)$ with $v^* = 2\alpha + \alpha_1$ and $b^* = 2\beta + \beta_1$. Now, suppose the constructed d^* can be converted into a linear trend-free block design. Call the resulting design $d^{**} = [d_1^*|d_2^*|d_3^*]$. In d_1^*, we have $\sum_{j=1}^{\beta} \sum_{l=1}^{k} \delta_{jl}^{i} l = (r)(k+1)/2$, $i = 1, 2, ..., \alpha-1$ and $\sum_{j=1}^{\beta} \sum_{l=1}^{k} \delta_{jl}^{\alpha} l = (r-1)(k+1)/2$, since $\sum_{j=1}^{\beta} \sum_{l=1}^{k} l = \beta k(k+1)/2$ and $\beta k = \alpha r - 1$. Similarly, in d_2^*, we have $\sum_{j=\beta+1}^{2\beta} \sum_{l=1}^{k} \delta_{jl}^{i} l = (r)(k+1)/2$, $i = \alpha+1, \alpha+2, ..., 2\alpha-1$ and $\sum_{j=\beta+1}^{2\beta} \sum_{l=1}^{k} \delta_{jl}^{2\alpha} l = (r-1)(k+1)/2$. d^{**} is a linear trend-free block design, hence, all symbols in d^{**} should satisfy equation (2.2). That implies $\sum_{j=2\beta+1}^{b^*} \sum_{l=1}^{k} \delta_{jl}^{\alpha} l$ and $\sum_{j=2\beta+1}^{b^*} \sum_{l=1}^{k} \delta_{jl}^{2\alpha} l$ both must equal to $(k+1)/2$. i.e., the symbols α and 2α must appear in the middle spot of the column in d_3^*. But, that is impossible, since symbols α and 2α are in the same column of d_3^*. Hence, the d^* can not be converted into a linear trend-free block design.

Corollary 3.1. (i) Choose $\beta_1 = (k-2)\beta+1$, hence $\alpha_1 = (k-2)\alpha$, then we get the Stufken-type counterexample design $d^S \in D(\alpha k, \alpha r, k, r)$. (ii) Let l be any positive integer. Choose $\beta_1 = (k-2)\beta + lr + 1$, hence $\alpha_1 = (k-2)\alpha + lk$, then we get the $d \in D((\alpha+l)k, (\alpha+l)r, k, r)$ can't be converted into a linear trend-free block design. This indicates if a Stufken-type counterexample design $d^S \in D(\alpha k, \alpha r, k, r)$ exists, for $v^* \geq v, b^* \geq b$ and $v^* r = b^* k$, there always exists a design, $d^* \in D(v^*, b^*, k, r)$ can not be converted into a linear trend-free block design.

Example 3.1. Let $r = 2$, $k = 5$ and $d^S \in D(40, 16, 5, 2)$ belong to Stufken's families. By Theorem 3.1, we can construct $d^* \in D(16 + \alpha_1, 6 + \beta_1, 5, 2)$ with $\alpha_1 > 5$ and $5\beta_1 = 2\alpha_1 + 2$ which do not have a linear trend-free version. Here, with minimum $\alpha_1 = 9$, hence $\beta_1 = 4$, a $d^* \in D(25, 10, 5, 2)$ is given.

$$d^* = \begin{bmatrix} 1 & 1 & 2 & 9 & 9 & 10 & 8 & 17 & 17 & 18 \\ 2 & 3 & 3 & 10 & 11 & 11 & 16 & 18 & 19 & 19 \\ 4 & 4 & 5 & 12 & 12 & 13 & 20 & 20 & 21 & 21 \\ 5 & 6 & 6 & 13 & 14 & 14 & 22 & 22 & 23 & 23 \\ 7 & 7 & 8 & 15 & 15 & 16 & 24 & 24 & 25 & 25 \end{bmatrix}.$$

Example 3.2. Let $d^S \in D(15, 10, 3, 2)$ belong to Stufken's families. For $v^* \geq 15$, $b^* \geq 10$ and $2v^* = 3b^*$, we can construct $d^* \in D(v^*, b^*, 3, 2)$ which does not have a linear trend-free version. Two designs $d_1^* \in D(18, 12, 3, 2)$ and $d_2^* \in D(21, 14, 3, 2)$ are given below.

Adopting the construction method in Theorem 3.1, we can write

$$d_1^* = \begin{bmatrix} 1 & 1 & 3 & 6 & 6 & 7 & 11 & 11 & 13 & 5 & 16 & 16 \\ 2 & 2 & 4 & 7 & 8 & 9 & 12 & 12 & 14 & 10 & 17 & 17 \\ 3 & 4 & 5 & 8 & 9 & 10 & 13 & 14 & 15 & 18 & 18 & 15 \end{bmatrix};$$

$$d_2^* = \begin{bmatrix} 1 & 1 & 3 & 6 & 6 & 7 & 11 & 11 & 13 & 5 & 16 & 16 & 17 & 17 \\ 2 & 2 & 4 & 7 & 8 & 9 & 12 & 12 & 14 & 10 & 18 & 18 & 19 & 19 \\ 3 & 4 & 5 & 8 & 9 & 10 & 13 & 14 & 15 & 21 & 20 & 21 & 20 & 15 \end{bmatrix}.$$

Check the 10th column of $d_1^*(d_2^*)$, we get the impossibilities of the linear trend-free version of $d_1^*(d_2^*)$.

Using the renaming techniques and the theory of the SDRs (system of distinct representatives; see Hall (1935)), Theorem 3.2 and Theorem 3.3 will show that the questions of the truth of the Yeh-Bradley conjecture in $D(v, b, \alpha k, \alpha r)$ and $D(\frac{v}{v_1}, b, k, rv_1)$ can be reduced to the class $D(v, b, k, r)$.

Theorem 3.2. Let k, α be odd integers and $(r, k) = 1$. If each design $d \in D(v, b, k, r)$ has a linear trend-free version d_{tf}, then each design $d^* \in D(v, b, \alpha k, \alpha r)$ has a linear trend-free version d_{tf}^*.

Proof. Suppose $d^* \in D(v, b, \alpha k, \alpha r)$. Let us derive an array d_1^* from d^* in the following fashion. Select any α cells of d^* that have the symbol i and replace i in these α cells by the ordered pair $(i, 1)$. Choose α other cells that have the symbol i from the $\alpha(r-1)$ remaining cells and replace by the order pair $(i, 2)$. Continue in this fashion until all $i's$ have been replaced by $(i, 1), (i, 2), ..., (i, r)$. Do this for each $i = 1, 2, ..., v$. Clearly, $d_1^* \in D(rv, b, \alpha k, \alpha)$. Let C_j denote the j_{th} column and S_j denotes the set of symbols in column j of d_1^*, $j = 1, ..., b$. It is easy to see that d_1^* satisfies $| S_{j_1} \cup S_{j_2} \cup ... \cup S_{j_t} | \geq kt$, for $1 \leq j_1 < ... < j_t \leq b$, $1 \leq t \leq b$. Thus, by Theorem 2.1 of Agrawal (1966), $S_1, S_2, ..., S_b$ possesses a $(k, k, ..., k)$ SDR, $(A_1, A_2, ..., A_b)$, say. Define $B_j = C_j \setminus A_j$, $j = 1, 2, ..., b$. Let $d_{11}^{**} = [A_1|A_2|\cdots|A_b]$ and $d_{12}^{**} = [B_1|B_2|\cdots|B_b]$. Permuting the symbol positions within columns of d_1^*, we can write d_1^* as

$$d_1^{**} = \left[\begin{array}{c|c|c|c} A_1 & A_2 & \cdots & A_b \\ \hline B_1 & B_2 & \cdots & B_b \end{array}\right] = \left[\begin{array}{c} d_{11}^{**} \\ \hline d_{12}^{**} \end{array}\right].$$

Now, replace each pair (i, g) by the symbol i, for $i = 1, ..., v$ and $g = 1, 2, ..., r$ in d_{11}^{**} and d_{12}^{**} to get d_{11} and d_{12}. Clearly $d_{11} \in D(v, b, k, r)$ and $d_{12} \in D(v, b, (\alpha - 1)k, (\alpha - 1)r)$. Applying Theorem 2.1, it is clearly that d_{12} has a strongly linear trend-free version $d_{12_{tf}}$. Write $d_{12_{tf}} = \begin{bmatrix} d^u_{12_{tf}} \\ d^l_{12_{tf}} \end{bmatrix}$, where $d^u_{12_{tf}}$ is a $((\alpha - 1)k/2) \times b$ array and so is $d^l_{12_{tf}}$. By the assumption of the theorem, d_{11} has a linear trend-free version $d_{11_{tf}}$. Then write $d^*_{tf} = \begin{bmatrix} d^u_{12_{tf}} \\ d_{11_{tf}} \\ d^l_{12_{tf}} \end{bmatrix}$ is a linear trend-free version of d^*.

Theorem 3.3. Suppose v is multiple of v_1. If each design $d \in D(v, b, k, r)$ has a linear trend-free version d_{tf}, then each design $d^* \in D(\frac{v}{v_1}, b, k, rv_1)$ has a linear trend-free version d^*_{tf}.

Proof. Let $d^* \in D(\frac{v}{v_1}, b, k, rv_1)$. Using the same renaming process in Theorem 3.2, we can derive a d_1^* from d^* and $d_1^* \in D(v, b, k, r)$. By the assumption of the theorem, d_1^* has a linear trend-free version $d^*_{1_{tf}}$. Change all the new symbols back to the original symbols in $d^*_{1_{tf}}$ to get a $d^*_{tf} \in D(\frac{v}{v_1}, b, k, rv_1)$ which is a linear trend-free version of d^*.

Theorem 3.4. Suppose k is odd, r is even, and $S' \subseteq S$. Let $d \in D(v, b, k, r)$ be a binary design such that for one symbol (say symbol 1), the union of all columns that contain symbol 1 contains at least all the symbols in S'. Furthermore, suppose that the collection of columns not containing symbol 1 can be partitioned into two sets of columns X_1 and X_2, which satisfy (a) $| X_1 |$ is even and the columns in X_1 can be divided into $| X_1 | /2$ pairs of columns such that any two columns that form a pair have at least one symbol in common; and (b) the columns in X_2 are all disjoint and all the symbols in X_2 are contained in S'. Then d has a strongly linear trend-free version.

Proof. Suppose $C_1, C_2, ..., C_r, C_{r+1}, ..., C_{r+2t}, C_{r+2t+1}, ..., C_b$ are the columns of the array d of which $C_1, C_2, ..., C_r$ contain symbol 1. Let $Y = \{C_1, C_2, ..., C_r\}$, $X_1 = \{C_{r+1}, C_{r+2}, ..., C_{r+2t}\}$, $a_i \in C_{r+2i-1} \cap C_{r+2i}, 1 \leq i \leq t$ and $X_2 = \{C_{r+2t+1}, ..., C_b\}$. We have $| Y |= r$, $| X_1 |= 2t$ and $| X_2 |= b - r - 2t = q$.

If $S' = \phi$, then the result follows from Theorem 2.2; hence assume $S' \neq \phi$. Let $A_0 = \phi, A_i = (C_i \cap S') \setminus (A_0 \cup \cdots \cup A_{i-1})$, for $i = 1, 2, ..., r$. Suppose that $\{A_{i_1}, A_{i_2}, ..., A_{i_n}\}$ are all the non-empty set in the collection $\{A_0, A_1, ..., A_r\}$. Clearly, $n > q$ since $| A_i |\leq| C_j |$ for $1 \leq i \leq n$ and $b - q + 1 \leq j \leq b$. And $\{A_{i_1}, A_{i_2}, ..., A_{i_n}\}$ forms a partition of S'. It follows from Theorem 2 of Hall (1935) that there is a subset $\{j_1, j_2, ..., j_q\}$ of $\{i_1, i_2, ..., i_n\}$ with the property : $A_{j_l} \cap C_{b-l+1} \neq \phi$ for $l = 1, 2, ..., q$.

Hence $C_{j_l} \cap C_{b-l+1} \neq \phi$ for $l = 1, 2, ..., q$, where $\{C_{j_1}, ..., C_{j_q}\} \subset Y$. Note that $| Y | -$
$| C_{j_1}, ..., C_{j_q} |= r - q$ is even. Thus we can construct a $(1 \times b)$ row vector $\rho = \{\rho_1, ..., \rho_r, \rho_{r+1}, ...,$
$\rho_{r+2t}, \rho_{r+2t+1}, ..., \rho_b\}$ with the properties:

(i) $\rho_j \in C_j$, $j = 1, 2, ..., b$;

(ii) $\rho_{b-l+1} = \rho_{j_l}$, $l = 1, 2, ..., q$ (recall $\{j_1, j_2, ..., j_q\} \subset \{1, 2, ..., r\}$);

(iii) $\rho_j = 1$, $j \in \{1, 2, ..., r\} \setminus \{j_1, j_2, ..., j_q\}$;

(iv) $\{\rho_{r+1}, ..., \rho_{r+2t}\} = \{a_1, a_1, a_2, a_2 ..., a_t, a_t\}$.

Clearly ρ consists of some symbols from the set $\{1, 2, ..., v\}$, repeated an even number of times each. By Theorem 2.2, d has a strongly linear trend-free version.

Corollary 3.2. In the Theorem 3.4, the assumption "the union of all columns that contain symbol 1 contains at least all the symbols in S'" can be replaced by the union of even number of columns that contain symbol 1 contains at least all the symbols in S'" and the result remains valid.

Example 3.3. Let $d \in D(15, 12, 5, 4)$ and we can write

$$d = \begin{bmatrix} 1 & 1 & 1 & 1 & 13 & 13 & 11 & 11 & 12 & 12 & 2 & 3 \\ 2 & 3 & 4 & 5 & 15 & 7 & 13 & 12 & 13 & 10 & 4 & 5 \\ 6 & 7 & 8 & 9 & 14 & 15 & 12 & 14 & 15 & 14 & 6 & 11 \\ 9 & 2 & 10 & 3 & 10 & 8 & 14 & 7 & 9 & 15 & 7 & 8 \\ 5 & 4 & 11 & 8 & 2 & 3 & 4 & 5 & 6 & 6 & 9 & 10 \end{bmatrix}.$$

Carefully observing this design d, we find (i) $S' = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$; (ii) $X_1 = \{C_5, C_6, C_7, C_8, C_9, C_{10}\}$; (iii) $X_2 = \{C_{11}, C_{12}\}$. By Theorem 3.4, we have a $\rho = (2, 3, 1, 1, 13, 13, 11, 11, 12, 12, 2, 3)$ collected from each column and each symbol in $\{2, 3, 1, 13, 11, 12\}$ repeats twice. Hence a strongly linear trend-free version of d is obtained and can be written as

$$d_{tf} = \begin{bmatrix} 6 & 2 & 8 & 5 & 14 & 8 & 14 & 12 & 9 & 15 & 4 & 11 \\ 1 & 7 & 10 & 9 & 15 & 3 & 4 & 7 & 13 & 10 & 6 & 5 \\ 2 & 3 & 1 & 1 & 13 & 13 & 11 & 11 & 12 & 12 & 2 & 3 \\ 9 & 1 & 4 & 3 & 10 & 7 & 13 & 5 & 15 & 6 & 7 & 10 \\ 5 & 4 & 11 & 8 & 2 & 15 & 12 & 14 & 6 & 14 & 9 & 8 \end{bmatrix}.$$

Example 3.4. Let $d \in D(15, 18, 5, 6)$ and we can write

$$d = \begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 4 & 15 \\ 6 & 5 & 2 & 3 & 3 & 5 & 1 & 1 & 8 & 2 & 3 & 4 & 10 & 7 & 2 & 5 & 2 & 3 \\ 7 & 8 & 7 & 4 & 11 & 8 & 9 & 6 & 10 & 6 & 9 & 9 & 11 & 10 & 9 & 12 & 5 & 6 \\ 11 & 13 & 9 & 10 & 13 & 9 & 12 & 8 & 11 & 8 & 11 & 10 & 12 & 12 & 10 & 13 & 7 & 8 \\ 15 & 14 & 12 & 13 & 14 & 14 & 15 & 12 & 15 & 14 & 14 & 13 & 15 & 13 & 14 & 15 & 13 & 11 \end{bmatrix}.$$

The first four columns, compare to $r = 6$, already contain all the symbols. Hence, by Corollary 3.2, we can get a strongly linear trend-free design d_{tf}. Let us write

$$d_{tf} = \begin{bmatrix} 6 & 5 & 1 & 4 & 3 & 8 & 1 & 8 & 11 & 2 & 11 & 13 & 15 & 7 & 10 & 5 & 4 & 15 \\ 7 & 8 & 9 & 10 & 13 & 14 & 12 & 6 & 15 & 14 & 9 & 9 & 10 & 12 & 14 & 12 & 13 & 11 \\ 1 & 1 & 2 & 3 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 2 & 3 \\ 11 & 13 & 12 & 13 & 14 & 9 & 9 & 12 & 10 & 6 & 14 & 10 & 12 & 13 & 9 & 15 & 7 & 8 \\ 15 & 14 & 7 & 1 & 11 & 5 & 15 & 1 & 8 & 8 & 3 & 4 & 11 & 10 & 2 & 13 & 5 & 6 \end{bmatrix}.$$

Theorem 3.5. Suppose k is odd, r is even and $v \geq 2k$. Let $d \in D(v, b, k, r)$ be a binary design such that for one symbol (say symbol 1), the union of all columns that contain symbol 1 contain at least $v-k-1$ symbols. Then d has a strongly linear trend-free version.

Proof. Suppose $C_1, C_2, ..., C_r$, $C_{r+1}, .., C_{2r}$, $C_{2r+1}, ..., C_{2r+2t}$, $C_{2r+2t+1}, ..., c_b$ are the columns of the array d of which $C_1, C_2, ..., C_r$ all contain symbol 1. Let $Y = \{C_1, C_2, ..., C_r\}$. If Y contains all symbols $1, 2, ..., v$, then by Theorem 3.3 of Chai and Majumdar (1993), d has a strongly linear trend-free version. Let $S_1 = \{2, 3, ..., k+2\}$. Suppose Y contains $v-k-1$ symbols, namely contains $S \setminus S_1$. Without loss of generality, let $C_{r+1}, .., C_{2r}$ contain symbol 2, $C_{2r+2j-1}$ and C_{2r+2j} has at least one symbol, say a_j, in common, $1 \leq j \leq t$ and $C_{2r+2t+1}, ..., C_b$ are all disjoint columns. Notice that we always can make no $C_l = (3, ..., k+2)$, $2r+2t+1 \leq l \leq b$, since if that happens, for the sake of the connectedness of d, we can either find a pair of columns C_{r+i_1} and C_{r+i_2}, $1 \leq i_1, i_2 \leq r$, such that $C_l \cap C_{i_1} \neq \phi$ and $C_{i_2} \neq (3, ..., k+2)$ to replace C_l or find a pair of columns $C_{2r+2j-1}$ and C_{2r+2j}, $1 \leq j \leq t$, such that $C_l \cap C_{2r+2j-1} \neq \phi$ and $C_{2r+2j} \neq (3, ..., k+2)$ to replace C_l. Let $C_j^* = C_j \setminus \{3, 4, ..., k+2\}$, $2r+2t+1 \leq j \leq b$ and S' denotes a collection of all the symbols in $C^*_{2r+2t+1} \cup ... \cup C^*_b$. Let $q = b-(2r+2t)$, $A_0 = \phi, A_i = (C_i \cap S') \setminus (A_0 \cup \cdots \cup A_{i-1})$, for $i = 1, 2, ..., r$. Suppose that $\{A_{i_1}, A_{i_2}, ..., A_{i_n}\}$ are all the non-empty set in the collection $\{A_0, A_1, ..., A_r\}$. Clearly, $n \geq q$, otherwise $q-1 \geq n$ implies $(q-1)(k-1) \geq n(k-1) \geq (q-1)k$ (minimum number of symbols in $C^*_{2r+2t+1} \cup ... \cup C^*_b$) which is impossible. And $\{A_{i_1}, A_{i_2}, ..., A_{i_n}\}$ forms a partition of S'. If follows from Theorem 2 of Hall (1935), that there is a subset $\{j_1, j_2, ..., j_q\}$ of $\{i_1, i_2, ..., i_n\}$ with the property : $A_{j_l} \cap C_{b-l+1} \neq \phi$ for $l = 1, 2, ..., q$.

Hence $C_{j_l} \cap C_{b-l+1} \neq \phi$ for $l = 1, 2, ..., q$, where $\{C_{j_1}, ..., C_{j_q}\} \subset Y$. Note that $| Y | -$
$| C_{j_1}, ..., C_{j_q} | = r-q$ is even. Thus we can construct a $(1 \times b)$ row vector $\rho = \{\rho_1, ..., \rho_r, \rho_{r+1}, ...,$
$\rho_{b-q}, \rho_{b-q+1}, ..., \rho_b\}$ with the properties:

(i) $\rho_j \in C_j$, $j = 1, 2, ..., b$;

(ii) $\rho_{b-l+1} = \rho_{j_l}$, $l = 1, 2, ..., q$ (recall $\{j_1, j_2, ..., j_q\} \subset \{1, 2, ..., r\}$);

(iii) $\rho_j = 1$, $j \in \{1, 2, ..., r\} \setminus \{j_1, j_2, ..., j_q\}$;

(iv) $\rho_i = 2$, $r+1 \leq i \leq 2r$;

(v) $\{\rho_{2r+1}, ..., \rho_{2r+2t}\} = \{a_1, a_1, a_2, a_2..., a_t, a_t\}$.

Clearly ρ consists of some symbols from the set $\{1, 2, ..., v\}$, repeated an even number of times each. By Theorem 2.2, d has a strongly linear trend-free version. If Y contains more than $v-k-1$ symbols, then the proof is followed similarly as the above. Hence, the proof is completed.

Theorem 3.6. Suppose k is odd, r is even and $v = 3k$. Then each binary design $d \in D(v, b, k, r)$ has a strongly linear trend-free version.
Proof. We have two cases. **Case 1.** r=2. Hence, b=6. Without loss of generality, we can write

$$d = \left[\begin{array}{c|c|c|c|c|c} 1 & 1 & 2 & 2 & & \\ & & & & & \\ C_{11} & C_{12} & C_{21} & C_{22} & C_3 & C_4 \\ & & & & & \\ & & & & & \end{array}\right],$$

where $(1, C_{11})$, $(1, C_{12})$ are those two columns contain symbol 1, $(2, C_{21})$, $(2, C_{22})$ are those two columns contain symbol 2 and C_5, C_6 are the remaining two columns. If $C_{11} \cup C_{12} \cup C_{21} \cup C_{22}$ does not contain symbols 3,4,..., v, say symbol 3 is missing, then C_5, C_6 must contain symbol 3. Therefore, symbols 1, 1, 2, 2, 3, 3 are chosen from the columns of the d, by Theorem 2.2, d has a linear trend-free version. Hence, $C_{11} \cup C_{12} \cup C_{21} \cup C_{22}$ must contain symbols 3, 4,..., v and C_5 and C_6 are disjoint. Without loss of generality, let $C_5 = (3, ..., k+2)$ and $C_6 = (k+3, ..., 2k+2)$. Also, let $S' = S \setminus (\{1, 2\} \cup C_5 \cup C_6) = \{2k+3, ..., 3k\}$. Suppose symbol $x \in C_5$ and symbol $y \in C_6$. We claim that x and y can 't be in the same column of $C_{11} \cup C_{12} \cup C_{21} \cup C_{22}$. If that happens, say x and y in C_{11}, then (i) C_{12} can not contain any symbol from C_5 and C_6, otherwise, $C_{11} \cap C_5 \neq \phi$ (or $C_{11} \cap C_6 \neq \phi$) and $C_{12} \cap C_6 \neq \phi$ (or $C_{12} \cap C_5 \neq \phi$), the job is done; (ii) C_{12} contains only symbols from S'. But, S' has only $k-2$ symbols and C_{12} has $k-1$ spaces. (i) and (ii) proves that claim. If $x \in C_{11}$ and $y \in C_{12}$, then $C_{11} \cap C_5 = x$ and $C_{12} \cap C_6 = y$, the proof is done. Hence, suppose $x \in C_{11}$ and $y \in C_{21}$. Then, $C_{11} \cup C_{12}$ must contain all symbols from C_5 and some symbols from S' and $C_{21} \cup C_{22}$ must contain all symbols from C_6 and some symbols from S'. k symbols from $C_5(C_6)$ have to distribute to both columns C_{11} (C_{21}) and C_{12} (C_{22}), since either column has only $k-1$ spaces. The number of the remaining open spaces is odd in $C_{11} \cup C_{12}$ ($C_{21} \cup C_{22}$)

, after all k symbols of C_5 (C_6) are filled into them. That means at least one symbol from S' should appear in $C_{11} \cup C_{12}$, say in C_{11} and $C_{21} \cup C_{22}$, say in C_{21}. Then we can have $C_{11} \cap C_{21} \neq \phi$, $C_{12} \cap C_5 \neq \phi$ and $C_{22} \cap C_6 \neq \phi$. By Theorem 2.2, the proof is completed. **Case 2.** $r > 2$. Let $C_1, C_2, ..., C_b$ be the columns of d. Without loss of generality, we can write

$$d = \left[\begin{array}{c|c|c|c|c} 1 \cdots 1 & 2 \cdots 2 & & & \\ d_1 & d_2 & d_3 & C_{b-1} & C_b \\ & & & & \end{array}\right],$$

where (i) $d_1((k-1) \times r)$ represents $C_1, ..., C_r$, but without symbol 1, (ii) $d_2((k-1) \times r)$ represents $C_{r+1}, ..., C_{2r}$, but without symbol 2, (iii) $d_1 \cup d_2$ covers symbols 3,...,v, (iv) $d_3(k \times 2t)$ has $2t$ columns, namely $C_{2r+1}, ..., C_{2r+2t}$, such that $C_{2r+2j-1} \cap C_{2r+2j} \neq \phi$, $1 \leq j \leq t$, (v) $C_{b-1} = (3, ..., k+2)$ and $C_b = (k+3, ..., 2k+2)$ are disjoint. Let symbol $x \in C_{b-1}$ and symbol $y \in C_b$. Based on the same arguments in Case 1, x and y can not appear simultaneously in (same column or different columns of) d_1, d_2 and d_3. Suppose symbol $x \in d_1$ and $y \in d_2$. Recall that $S' = \{2k+3, ..., 3k\}$. If d_1 (d_2) contains no symbol from S', then $C_{r+1}, ..., C_{2r}$ ($C_1, ..., C_r$) contain $2k-1$ symbols, by Theorem 3.5, the proof is done. If a symbol from S' appears in d_1 and d_2, then the finding of the pair of columns which have the common symbol between C_{b-1}, C_b and the columns of d_1 and d_2 is solved. Hence, we can let $S' = S'_1 \cup S'_2$, where S'_1 and S'_2 both not empty and disjoint and S'_1 (S'_2) denotes the collection of symbols from S' appear in d_1 (d_2). Suppose symbol $p \in S'_1$ and symbol $q \in S'_2$. Then p and q can not appear simultaneously in both columns $C_{2r+2j-1}$, C_{2r+2j}, $1 \leq j \leq t$, since (i) if $p \in C_{2r+2j-1}$ and $q \in C_{2r+2j}$, then two columns, say $C_{1*} \ni x$ ($C_{2*} \ni y$) and $C_{1**} \ni p$ ($C_{2**} \ni q$), from d_1 (d_2) are chosen, such that $C_{2r+2j-1} \cap C_{1**} = \{p\}$, $C_{2r+2j} \cap C_{2**} = \{q\}$ $C_{1*} \cap C_{b-1} = \{x\}$ and $C_{2*} \cap C_b = \{y\}$, the job is done, (ii) (a) if $\{p, q\} \subset C_{2r+2j-1}$ and $x \in C_{2r+2j}$, then two columns, $C_{2**} \ni q$ and $C_{2*} \ni y$, from d_2 are chosen, such that $C_{2r+2j-1} \cap C_{2**} = \{q\}$, $C_{2r+2j} \cap C_{b-1} = \{x\}$ and $C_{2*} \cap C_b = \{y\}$ the job is done; (b) if $\{p, q\} \subset C_{2r+2j-1}$ and $y \in C_{2r+2j}$, the finding for the common symbol between columns can be done similarly as in (a). Now, we claim that t pairs columns

$C_{2r+2j-1} \cap C_{2r+2j} \neq \phi$, $1 \leq j \leq t$, in d_3, must have at least one of the two types. Type I. $\{x, q\} \subset C_{2r+2j-1}$ and $x \in C_{2r+2j}$ for some j. Type II. $\{y, p\} \subset C_{2r+2j-1}$ and $y \in C_{2r+2j}$ for some j. If there is no such pairs of columns in d_3, then the design d is a disconnected design. If we have a type I pair in d_3, then pick up two columns, $C_{2*} \ni y$ and $C_{2**} \ni q$, from d_2, such that $C_{2r+2j-1} \cap C_{2**} = \{q\}$, $C_{2r+2j} \cap C_{b-1} = \{x\}$ and $C_{2*} \cap C_b = \{y\}$; if we have a type II pair in d_3, then pick up two columns, $C_{1**} \ni p$ and $C_{1*} \ni x$, from d_1, such that $C_{2r+2j-1} \cap C_{1**} = \{p\}$, $C_{2r+2j} \cap C_b = \{y\}$ and $C_{1*} \cap C_{b-1} = \{x\}$. Hence, by Theorem 2.2, d has a linear trend-free version.

Theorem 3.7. Suppose k is odd, r is even. Let $d \in D(v, b, k, r)$ be a binary design which satisfies the following assumptions:

(a) There exists even numbers of columns $A_1, A_2, ..., A_{2s}$ such that each A_i contains one common symbol (say symbol 1) and another even numbers of columns $B_1, B_2, ..., B_{2t}$ such that each B_j contains another common symbol (say symbol 2),

(b) $\{\cup_{i=1}^{2s} A_i\} \cup \{\cup_{j=1}^{2t} B_j\} \supseteq \{1, 2, ..., v-1, v\}$,

(c) There exists one column P_o(other than those A_i's and B_j's) that contains symbol 1 and symbol 2.

Then d has a strongly linear trend-free version.

Proof. Without loss of generality, we can write

$$d = \left[\begin{array}{c|c|c|c|c} 1 & 1\cdots 1 & 2\cdots 2 & & \\ 2 & & & & \\ & d_1 & d_2 & d_3 & d_4 \\ & & & & \\ & & & & \end{array}\right],$$

where $d_1((k-1) \times 2s)$ represents those A_i's without symbol 1 and $d_2((k-1) \times 2t)$ represents those B_j's without symbol 2. Furthermore, $d_3(k \times 2l)$ is an array in which $(2j-1)_{th}$ column and $(2j)_{th}$ column has at least one symbol in common, say symbol b_j, for $1 \leq j \leq l$ and all columns in $d_4(k \times (b-1-2s-2t-2l))$ are disjoint. Let $b-1-2s-2t-2l = m$ and $P_1, P_2, ..., P_n, P_{n+1}, ..., P_{m-1}, P_m$ be the columns in d_4.

Applying the same technique in finding the common symbols between C_i's and X_2 in Theorem 3.4 and without loss of generality we get

(i) Let $\{P_1, P_2, ..., P_n, P_{n+1}, ..., P_{m-1}, P_m\} = P \cup Q$, where P contains P_1, P_2,..., P_n and Q contains the remaining columns;

(ii) For each P_h in P, one can find $a_h \in \{2, 3, 4, ..., v\}$ and A_{i_h} such that $a_h \in P_h \cap A_{i_h}$, i_h's are all distinct, $1 \leq h \leq n$ and $\{i_1, i_2, ..., i_n\} \subseteq \{1, 2, ..., 2s\}$;

(iii) For each P_l in Q, one can find $a_l \in \{1, 3, 4, ..., v\}$ and B_{i_l} such that $a_l \in P_l \cap B_{i_l}$, i_l's are all distinct, $1 \leq l \leq m$ and $\{i_1, i_2, ..., i_m\} \subseteq \{1, 2, ..., 2t\}$.

Now, we have two cases:

Case 1. n is odd. Hence $m-n$ is even. We can get a $\rho = (1, a_1, a_2, ..., a_n, 1, ..., 1, a_{n+1}, ..., a_m, 2, ..., 2, b_1, b_1, ..., b_l, b_l, a_1, ..., a_n, a_{n+1}, ..., a_m)$. Notice that all the symbols appear in ρ repeated an even number of times.

Case 2. n is even. Hence $m-n$ is odd. We get a $\rho = (2, a_1, a_2, ..., a_n, 1, ..., 1, a_{n+1}, ..., a_m, 2, ..., 2, b_1, b_1, ..., b_l, b_l, a_1, ..., a_n, a_{n+1}, ..., a_m)$. Similar to Case 1, all the symbols appear in ρ repeated an even number of times.

Therefore, by Theorem 2.2, d has a strongly linear trend-free version.

Corollary 3.3. In Theorem 3.7, if those $2s + 2t$ columns, which contain $A_1, A_2, ..., A_{2s}$ and $B_1, B_2, ..., B_{2t}$, contain at least $v - k + 2$ symbols, by the similar proof techniques in Theorem 3.5, then the result is still valid.

Example 3.5. Let $d \in D(12, 24, 3, 6)$ and we can write d as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 9 & 9 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 2 & 7 & 1 \\ 2 & 3 & 4 & 5 & 6 & 11 & 12 & 5 & 6 & 10 & 12 & 8 & 6 & 5 & 6 & 4 & 7 & 5 & 8 & 8 & 9 & 9 & 11 & 4 \\ 3 & 7 & 8 & 9 & 10 & 3 & 4 & 7 & 8 & 11 & 3 & 9 & 11 & 10 & 11 & 10 & 12 & 10 & 12 & 11 & 12 & 10 & 12 & 8 \end{bmatrix}.$$

Let $C_1, C_2, ..., C_{24}$ be columns of d. We find (i) Those $A_i's$ are C_2, C_3, C_4 and C_5; (ii) Those $B_i's$ are C_6, C_7; (iii) P_0 is C_1; (iv) $n = 2$, $m = 3$ and P_1, P_2 are C_{22}, C_{23}; P_3 is C_{24}. Hence, by Theorem 3.7, a d_{tf} is obtained and we can write d_{tf} as

$$\begin{bmatrix} 1 & 3 & 4 & 1 & 10 & 11 & 12 & 5 & 6 & 11 & 3 & 8 & 11 & 5 & 6 & 10 & 7 & 10 & 12 & 8 & 9 & 2 & 12 & 8 \\ 2 & 7 & 1 & 9 & 1 & 2 & 4 & 2 & 2 & 9 & 9 & 3 & 3 & 4 & 4 & 5 & 5 & 6 & 6 & 7 & 7 & 9 & 7 & 4 \\ 3 & 1 & 8 & 5 & 6 & 3 & 2 & 7 & 8 & 10 & 12 & 9 & 6 & 10 & 11 & 4 & 12 & 5 & 8 & 11 & 12 & 10 & 11 & 1 \end{bmatrix}.$$

Remark 3.1. Theorems 3.4, 3.5, 3.7 and Corollaries 3.2 and 3.3 all state the designs with various sufficient conditions can be converted into linear trend-free block designs. Those results can be applied to some optimal designs classes, especially to two-associate-class PBIB designs with $\lambda_1 = 0$ or $\lambda_2 = 0$. For examples, (i) Group-divisible designs with k odd, r even, λ_1=0, $v = mn$ and $n \leq k + 2$; (ii) Triangular design with k odd, r even, λ_2=0, $v = n(n-1)/2$ and $2n \leq k + 6$, both classes of designs can be proved to have the linear trend-free version by Theorem 3.6. I believe that a lot of two-associate-class PBIB designs classes can have the linear trend-free version. The interested readers can go and check designs from the tables of two-associate-class PBIB designs by Clatworthy (1973).

4 Concluding remark

Yeh-Bradley conjecture is answered, true or false, on more classes. But, still a few classes have no answer on it. For examples, two classes, $D(v, b, k, r)$ with k odd, $(r, k) \neq 1$ and $D(v, b, k, r)$ with k, r both odd, do not have the solution in general. For the class $D(v, b, k, r)$ with k odd, $(r, k) \neq 1$, my feeling is Yeh-Bradley conjecture might be true in the class. Since, so far as I know, all the counterexamples for the Yeh-Bradley conjecture are either the Stufken-type or its related counterexamples and Stufken-type counterexamples must have (r, k)=1. For the class $D(v, b, k, r)$ with k, r both odd, I think it is not easy to find a strongly linear trend-free version in the class. Other type of linear trend-free designs, other than the strongly linear trend-free designs (ie., $s_{dil} = s_{di(k-l+1)}, l = 1, ..., [(k + 1)/2], i = 1, ..., v$) should be characterized and hope to succeed in this class.

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