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Construction and classification of orthogonal arrays with small numbers of runs

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Abstract

We present a complete set of combinatorially non-isomorphic orthogonal arrays of types OA$(12, 2^s3^1)$, OA$(18, 3^s)$, OA$(18, 2^13^s)$, and OA$(20, 2^s5^1)$. To produce the complete catalog, we start from reduced sets of candidate orthogonal arrays and apply the isomorphic checking algorithm proposed in Clark and Dean (2001).

Key words: Combinatorial isomorphism; Design catalog; Design equivalence; Geometric isomorphism; Isomorphic designs.

1 Introduction

Orthogonal arrays have a great range of applications in many research areas (see, for example, Hedayat, Sloane and Stufken, 1999, Preface; Wu and Hamada, 2000, Chapter 7). Representative orthogonal arrays of various sizes are listed by many authors; for example, by Dey and Mukerjee (1999, Appendix A3) and Hedayat, Sloane and Stufken, (1999, Chapter 12).

An orthogonal array, OA$(n, m_1^{s_1}m_2^{s_2})$, has n rows and s_1+s_2 columns. There are m_1 distinct symbols in each of the first s_1 columns and m_2 distinct symbols in each of the last s_2 columns. An array of "strength t" has all combinations of symbols occurring the same number of times in every selection of t columns. For use in a factorial experiment, columns are assigned to factors at random and the distinct symbols within a column are assigned at random to the corresponding factor levels. The rows are randomly ordered to form the runs of the experiment. This means that several arrays could lead to the same design after randomization. Such arrays are called *equivalent* or *isomorphic*. Non-equivalent or non-isomorphic arrays can never result in the same design through randomization. Thus, access to a catalog of non-isomorphic arrays gives the widest possible scope for selecting a design for a given experiment.

Clark and Dean (2001) pointed out that there are two types of isomorphism of factorial designs depending upon whether factors are qualitative or quantitative. Called "combinatorial isomorphism" and "geometric isomorphism", respectively, by Cheng and Ye (2004), these are defined as follows.

Two designs d_1 and d_2 with quantitative factors are *geometrically isomorphic* if one can be obtained from the other by (i) reordering the factors (columns of the array), (ii) reordering the runs (rows of the array), and (iii) reversing the level label ordering for one or more factors. Geometrically isomorphic designs have identical statistical properties for the estimation of any given complete set of orthonormal factorial trend contrasts. For qualitative factors, designs are *combinatorially isomorphic* if one can be obtained from the other by (i) reordering the factors, (ii) reordering the runs, and (iii) relabeling the levels of one or more factors. This differs from geometric isomporphism in (iii) since a numerical ordering of the factor levels is no longer implied. Combinatorially isomorphic designs have identical statistical properties for estimation of *any* factorial contrasts.

Designs that are geometrically isomorphic are, by definition, also combinatorially isomorphic, but combinatorially isomorphic designs are not necessarily geometrically isomorphic. Note that some non-isomorphic designs may still be "model equivalent" in the sense of being equivalent for fitting a particular set of statistical models. For work on model equivalence, see, for example, Tsai, Gilmour and Mead (2000), Cheng and Wu (2001) but this is beyond the scope of this paper.

Recently, considerable effort has been expended in determining combinatorially isomorphic orthogonal arrays. Stufken and Tang (2007) gave a method of complete enumeration of non-isomorphic two-level orthogonal arrays of strength t with $t+2$ factors and run size $n = \lambda 2^t$, for integer λ. Sun, Li and Ye (2008) constructed a complete catalog of combinatorially non-isomorphic arrays $\mathrm{OA}(12, 2^s)$, $\mathrm{OA}(16, 2^s)$ and $\mathrm{OA}(20, 2^s)$ using the approach of building the array one column at a time. With the same approach, Tsai, Ye and Li (2006) obtained a complete catalog of geometrically isomorphic arrays $\mathrm{OA}(18, 2^1 3^s)$ and $\mathrm{OA}(18, 3^s)$. A purpose of the current paper is to classify these geometrically non-isomorphic designs into equivalence classes of combinatorially isomorphic designs, regarding the factors as qualitative instead of quantitative. In addition, we also present the complete set of combinatorially non-isomorphic arrays $\mathrm{OA}(12, 2^s 3^1)$ and $\mathrm{OA}(20, 2^s 5^1)$ using the arrays of Sun, Li and Ye (2008) as "base" designs. More details of our methods of construction and classification are given in Section 2. The results are summarized in Section 3 followed by some concluding remarks in Section 4.

2 Method of construction and classification

The necessary and sufficient conditions given by Clark and Dean (2001) for combinatorial isomorphism of designs d_1 and d_2 lead to an algorithm, *Deseq2*, which proceeds as follows. First, the Hamming distances between all pairs of runs are calculated, where the Hamming distance between runs i_1 and i_2 is defined to be the number of factors that are at different levels in these runs. A search is made for a column permutation $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_f\}$ of d_2 and an $n \times n$ row permutation matrix $\boldsymbol{R}$ such that, for *every* dimension $p = 1, 2, \ldots, f$,

$$\boldsymbol{H}_{d_1}^{\{1,2,\ldots,p\}} \;=\; \boldsymbol{R}(\boldsymbol{H}_{d_2}^{\{c_1,c_2,\ldots,c_p\}})\boldsymbol{R}', \tag{1}$$

where $\boldsymbol{H}_{d}^{\{c_1,c_2,\ldots,c_p\}}$ is the matrix of Hamming distances obtained from the subdesign consisting of factors (columns) $c_1, c_2, \ldots, c_p$. The algorithm was illustrated for 2-level designs by Clark and Dean (2001) and extended for 3-level designs by Katsaounis and Dean (2008). The extended algorithm was adapted for the current work.

In the remainder of this paper, we often refer to "combinatorial isomorhism" simply as "isomorphism". To reduce the computational burden of isomorphism checking, we started with a reduced set of candidate arrays that contained a manageable number of designs yet included all possible non-isomorphic cases.

The candidate set of orthogonal arrays $OA(12, 2^s3^1)$ was constructed by appending a three-level column to each non-isomorphic orthogonal array $OA(12, 2^s)$ in the catalog of Sun, Li, and Ye (2008) so that all symbols in the three-level column appear the same number of times. All possible arrangements of the three-level column were examined and those arrangements that gave orthogonality between the three-level factor and all two-level factors were retained. By this method, we obtained a set of arrays $OA(12, 2^s3^1)$ that includes all possible non-isomorphic cases, and in which it is clear that arrays $OA(12, 2^s3^1)$ constructed from non-isomorphic arrays $OA(12, 2^s)$ must be non-isomorphic. To reduce the number of orthogonal arrays in the candidate set, a preliminary screen for non-isomorphism was then performed. First the rows of each candidate design were sorted by the values of the first three columns (with the first column representing the three-level factor). All possible level permutations and column permutations were then applied to one of the arrays, and its rows resorted. If, in such a procedure, the two arrays become identical, then they are isomorphic and one of them can be removed from the list. Note again that this method does not identify all isomorphic pairs as the complete row permutations are not applied. The same approach was used to construct a set of candidate orthogonal arrays $OA(20, 2^s5^1)$ using the complete catalog of arrays $OA(20, 2^s)$ of Sun, Li, and Ye (2008).

In the case of arrays $OA(18, 2^p3^s)$, we took advantage of the complete catalog of geometrically non-isomorphic arrays $OA(18, 2^p3^s)$ of Tsai, Ye and Li (2006). This catalog was obtained using an efficient algorithm for checking geometric isomorphism based on a polynomial representation of factorial designs. To identify designs that are geometrically non-isomorphic but combinatorially isomorphic, we applied the algorithm *Deseq2* as described earlier in this section.

3 Results

3.1 OA(12,2^s3^1) and OA(20,2^s5^1)

Our construction method reveals that orthogonal arrays $OA(12, 2^s3^1)$ exist only for $s \leq 4$ and arrays $OA(20, 2^s5^1)$ exist only for $s \leq 8$. Using the screen for non-isomorphism as described above, we obtained a total of 15 orthogonal arrays $OA(12, 2^s)$ and 331 arrays $OA(20, 2^s)$

in the candidate list. We then applied the full isomorphism check (1) to those pairs of designs that were not already known to be non-isomorphic. The numbers of non-equivalent orthogonal arrays of each type are listed in the right hand side of Table 1 together with the number of orthogonal arrays in the candidate list that underwent the complete isomorphism check (1). We found three equivalence classes of arrays $\mathrm{OA}(12, 2^3 3^1)$ and representatives are shown (transposed) in the left hand side of Table 2. We found only one equivalence class of orthogonal arrays $\mathrm{OA}(12, 2^4 3^1)$ and a representative of this class is listed by Hedayat, Sloane, and Stufken (1999) and an alternative representative is shown in the right hand side of Table 2. We do not list all arrays $\mathrm{OA}(20, 2^s 5^1)$ because of limitation of space but they are available upon request or at http://www.umn.edu/∼wli. We found only one equivalence class of arrays $\mathrm{OA}(20, 2^8 5^1)$, and a representative can be constructed using the procedure described by Wang and Wu (1992).

Table 1: Number of Non-isomorphic Orthogonal Arrays together with the number of candidate OAs evaluated for complete isomorphism (in parentheses).

$\mathrm{OA}(18, 3^3)$	4(13)	$\mathrm{OA}(12, 3^1 2^3)$	3(10)
$\mathrm{OA}(18, 3^4)$	12(133)	$\mathrm{OA}(12, 3^1 2^4)$	1(6)
$\mathrm{OA}(18, 3^5)$	10(332)		
$\mathrm{OA}(18, 3^6)$	8(478)	$\mathrm{OA}(20, 5^1 2^3)$	10(22)
$\mathrm{OA}(18, 3^7)$	3(284)	$\mathrm{OA}(20, 5^1 2^4)$	15(82)
$\mathrm{OA}(18, 2^1 3^3)$	15(121)	$\mathrm{OA}(20, 5^1 2^5)$	38(154)
$\mathrm{OA}(18, 2^1 3^4)$	48(1836)	$\mathrm{OA}(20, 5^1 2^6)$	30(65)
$\mathrm{OA}(18, 2^1 3^5)$	19(1332)	$\mathrm{OA}(20, 5^1 2^7)$	4(6)
$\mathrm{OA}(18, 2^1 3^6)$	12(1617)	$\mathrm{OA}(20, 5^1 2^8)$	1(2)
$\mathrm{OA}(18, 2^1 3^7)$	3(726)		

3.2 OA$(18, 3^s)$ and OA$(18, 2^1 3^s)$

The numbers of non-isomorphic orthogonal arrays $\mathrm{OA}(18, 3^s)$ and $\mathrm{OA}(18, 2^1 3^s)$ are shown on the left hand side of Table 1 together with the number of geometric non-isomorphic orthogonal arrays in parentheses. One can see that the number of combinatorially non-

Table 2: Non-isomorphic arrays $OA(12, 2^s 3^1)$, s=3,4; rows correspond to factors and columns to runs

$OA(12, 2^3 3^1)$	$OA(12, 2^4 3^1)$
1 000011112222	1 000011112222
001100110011	001100110011
010101010101	010100111100
011010010110	011001010101
	011010101001
2 000011112222	
001100110011	
010101010101	
011000111100	
3 000011112222	
001100110011	
010101010101	
100110011001	

isomorphic OAs is much smaller than the number of geometrically non-isomorphic arrays.

Our results show there exist only three non-isomorphic orthogonal arrays $OA(18, 3^7)$. The first one is well known, and the other two were reported by Xu, Cheng and Wu (2004). We also show that there exist only three non-isomorphic arrays $OA(18, 2^1 3^7)$. These correspond to each of the three non-isomorphic arrays $OA(18, 3^7)$ augmented with an additional two-level factor. The (transposed) arrays are listed in the leftmost column of Table 3. The first one is well known and can be found in Taguchi (1987). The other two, as far as we know, have not been reported before.

All non-isomorphic orthogonal arrays $OA(18, 3^s)$ are listed in Table 3 in Appendix. Table **??** (in Appendix) lists the number of geometric-isomorphic designs corresponding to each of these arrays. One can observe that these numbers vary greatly among orthogonal arrays of the same size. For example, the three non-isomorphic $OA(18, 3^7)$s are equivalent to 204, 30, 50 geometrically non-isomorphic designs respectively. The number of non-isomorphic arrays $OA(18, 2^1 3^s)$ for $s < 7$ is much larger than that of arrays $OA(18, 3^s)$, so we are unable to list all non-isomorphic $OA(18, 2^1 3^s)$s. However, they are available from the authors upon request or at http://www.umn.edu/∼wli.

3.3 OA$(9, 3^s)$

It can be seen that the first listed OA$(18, 3^3)$ and the first listed OA$(18, 3^4)$ in Table 3 in Appendix consist of a duplicated OA$(9, 3^3)$ and a duplicated OA$(9, 3^4)$, respectively. No other 18-run orthogonal array consists of duplicated 9-run orthogonal arrays for $s \geq 3$. Thus, each of the OA$(9, 3^3)$ and the OA$(9, 3^4)$ must be the unique case of its size. This follows since if there were to exist two arrays OA$(9, 3^3)$ that are not isomorphic, their duplicates must be two non-isomorphic arrays OA$(18, 3^3)$. But we find only one OA$(18, 3^3)$ equivalence class consisting of duplicated arrays OA$(9, 3^3)$. Therefore, there is only one OA$(9, 3^3)$ equivlance class. By the same argument, there is only one equivlance class of arrays OA$(9, 3^4)$.

4 Concluding remarks

The results in this paper, together with the results of Sun, Li and Ye (2008) on orthogonal arrays OA$(12, 2^s)$ and OA$(20, 2^s)$, complete the classification of 9-run, 12-run, 18-run and 20-run orthogonal arrays under combinatorial isomorphism. Hedayat, Sloane, and Stufken (1999, Table 12.7) list OA$(12, 2^s 6^1)$ and OA$(20, 2^s 10^1)$ only $s \leq 2$. For $s = 2$, there exists only one equivalence class in each case, and the case $s = 1$ is trivial, as only full factorials are possible. The non-existence of OA$(12, 2^s 6^1)$ and OA$(20, 2^s 10^1)$ for $s \geq 3$ can be verified numerically by attempting to augment a third two-level column to the arrays OA$(20, 2^2 10^1)$ and OA$(12, 2^2 6^1)$. The cases that still remain to be classified among the orthogonal arrays with fewer than 20 runs are mixed-level orthogonal arrays with 8 runs and 16 runs, in particular, OA$(8, 2^s 4^t)$ and OA$(16, 2^s 4^t)$. This work is underway using an approach similar to that presented in this paper.

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Appendix

Table 3: Non-isomorphic arrays OA$(18, 3^s)$ and OA$(18, 2^1 3^7)$; rows correspond to factors and columns to runs

	OA$(18, 3^3)$
1	000000111111222222 001122001122001122 001122112200220011
2	000000111111222222 001122001122001122 001122120201120201
3	000000111111222222 001122001122001122 001212120102120201
4	000000111111222222 001122001122001122 010212021201120102

	OA$(18, 2^t 3^7)$
1	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010212211020202110 012120020112212010 012120201021120201 100110101001011010
2	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010212211020202110 012120021012212001 012120200121120210 100110101001100101
3	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010221121020202110 012102200121122010 012120201012210201 100110101001011010

	OA$(18, 3^4)$
1	000000111111222222 001122001122001122 001122112200220011 001122220011112200
2	000000111111222222 001122001122001122 001122112200220011 001212120012121200
3	000000111111222222 001122001122001122 001122112200220011 010212120102021201
4	000000111111222222 001122001122001122 001122120201120201 001212112020221010
5	000000111111222222 001122001122001122 001122120201120201 001212121020212010
6	000000111111222222 001122001122001122 001122120201120201 010212102021221100
7	000000111111222222 001122001122001122 001122120201120201 010212102120221001
8	000000111111222222 001122001122001122 001122120201120201 010212121020202101
9	000000111111222222 001122001122001122 001212120102120201 001221121020212010
10	000000111111222222 001122001122001122 001212120102120201 010122022110212001
11	000000111111222222 001122001122001122 001212120102120201 010221021120212001
12	000000111111222222 001122001122001122 010212021201120102 011220212010020121

	OA$(18, 3^5)$
1	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010212211020202110
2	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010221121020202110
3	000000111111222222 001122001122001122 001122120201120201 010212102021221100 010212221100102021
4	000000111111222222 001122001122001122 001122120201120201 010212102021221100 012120020121212010
5	000000111111222222 001122001122001122 001122120201120201 010212102120221001 010212221001102120
6	000000111111222222 001122001122001122 001122120201120201 010212102120221001 012120020121212010
7	000000111111222222 001122001122001122 001212120102120201 010122022110212001 011220212001021021
8	000000111111222222 001122001122001122 001212120102120201 010122022110212001 012201121020200121
9	000000111111222222 001122001122001122 001212120102120201 010221021120212001 011220212001021021
10	000000111111222222 001122001122001122 010212021201120102 011220212010020121 012021201210211002

	OA$(18, 3^6)$
1	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010212211020202110 012120020112212010
2	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010212211020202110 012120021012212001
3	000000111111222222 001122001122001122 001122112200220011 010212120102021201 010221121020202110 012102200121122010
4	000000111111222222 001122001122001122 001122120201120201 010212102021221100 010212221100102021 012120020121212010
5	000000111111222222 001122001122001122 001122120201120201 010212102120221001 010212221001102120 012120020121212010
6	000000111111222222 001122001122001122 001212120102120201 010122022110212001 011220212001021021 012201121020200121
7	000000111111222222 001122001122001122 001212120102120201 010221021120212001 011220212001021021 012102122010200121
8	000000111111222222 001122001122001122 010212021201120102 011220212010020121 012021201210211002 012102122001201021

Table 4: Number of geometrically non-isomorphic $\mathrm{OA}(18, 3^s)$ designs that are combinatorially isomorphic

$\mathrm{OA}(18, 3^3)$									1	2	3	4
geo. distinct									2	4	5	2
$\mathrm{OA}(18, 3^4)$	1	2	3	4	5	6	7	8	9	10	11	12
geo. distinct	2	6	3	7	16	4	24	10	13	14	30	4
$\mathrm{OA}(18, 3^5)$			1	2	3	4	5	6	7	8	9	10
geo. distinct			20	36	6	15	30	123	40	17	42	3
$\mathrm{OA}(18, 3^6)$					1	2	3	4	5	6	7	8
geo. distinct					186	24	44	15	123	43	38	5
$\mathrm{OA}(18, 3^7)$										1	2	3
geo. distinct										204	30	50