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On Exploring Tails via Tail Equivalence

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Abstract

Tail equivalence between two distribution functions was introduced in Resnick, S.I. (1971). Tail equivalence and its applications, Journal of Applied Probability, 8(1), 136-156. After clarifying a few properties and giving examples of classes of tail equivalent distributions, this article looks briefly at some interesting applications of tail equivalence in establishing tail behaviours of mixtures and order statistics, in particular, of limit laws of normalised k-th upper order statistics from a random sample, for fixed integer k. The tail behaviours of such limit laws have been studied via tail equivalence. It turns out that tail equivalence simplifies much of the apparent difficulty in handling the tails of such limit laws. A consequence is a method of generating random observations from regularly varying tails having different exponents of regular variation.

Key words: Extreme value theory; Limit laws; Mixtures; Partial maximum; Tail equivalence; Upper order statistics.

AMS Subject Classifications: 60F05, 60G70, 62G30.

1. Introduction

Resnick (1971) introduces the concept of tail equivalence between two distribution functions (dfs) on the real line $\mathbb{R}$. Here tail refers to right tail and we confine to right tail in this article. Similar results for left tail can be derived from the results discussed here. Tail equivalence divides the class of all dfs on the real line into equivalence classes. In this article, after giving known definitions of heaviness of tail, illustrations of the use of tail equivalence to study the tail behaviour of limit laws of normalised mixtures and k-th upper order statistics from a random sample for fixed integer k are given, under fixed and random sample sizes. These results were derived by the author and co-workers in several articles.

1.1. Tail equivalence

Definition (Resnick, 1971): Two dfs F and G on $\mathbb{R}$ are said to be tail equivalent, denoted by $F \stackrel{T}{=} G$, if

$$\lim_{x\to\infty} \frac{1-F(x)}{1-G(x)} = A, 0 < A < \infty. \tag{1}$$

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We refer to Resnick (1971) for applications of tail equivalence in extreme value theory. The following are easy consequences of the definition:

- If $F \overset{T}{=} G$, and $r(F) = \sup\{x \in \mathbb{R} : F(x) < 1\}$ denotes the right extremity of F, then $r(F) = r(G)$, finite or infinite. This is because, otherwise, A in (1) will be 0 or ∞ according as $r(F) < r(G) \leq \infty$ or $r(G) < r(F) \leq \infty$, respectively.
- Since $F \overset{T}{=} F$ with $A = 1$ in (1), the relation $\overset{T}{=}$ is reflexive.
- If $F \overset{T}{=} G$ with the limit in (1) as A, then $G \overset{T}{=} F$ with the limit in (1) as $1/A$, so that the relation $\overset{T}{=}$ is symmetric.
- If $F \overset{T}{=} G$ with the limit in (1) as A, and $G \overset{T}{=} H$ with the limit in (1) as B, then $F \overset{T}{=} H$ with the limit in (1) as AB, so that the relation $\overset{T}{=}$ is transitive, proving that the relation is an equivalence relation.

Now we give some examples of tail equivalent families of dfs on $\mathbb{R}$.

Examples of classes of tail equivalent dfs:

- Family of exponential distributions with different location parameters:
  If $F(x;\mu) = 1 - e^{x-\mu}, x > \mu$, and 0 elsewhere, with $\mu \in \mathbb{R}$ as a location parameter, then $\lim_{x\to\infty} \frac{1-F(x;\mu_1)}{1-F(x;\mu_2)} = e^{-(\mu_1-\mu_2)}$. However, note that family of exponential distributions with different scale parameters, is not a tail equivalent class.
- Family of Pareto distributions with location and scale parameters:
  If $F(x;\mu,\sigma) = 1 - \frac{1}{\left(\frac{x-\mu}{\sigma}\right)}, x > \mu+\sigma$, and 0 elsewhere, with $\mu \in \mathbb{R}$ as a location parameter and $\sigma > 0$ as a scale parameter, then $\lim_{x\to\infty} \frac{1-F(x;\mu_1,\sigma_1)}{1-F(x;\mu_2,\sigma_2)} = \frac{\sigma_2}{\sigma_1}$.
- Family of log-Pareto distributions with scale and shape parameters:
  If $F(x;\mu,\sigma) = 1 - \frac{1}{\ln\left(\frac{x}{\mu}\right)^{\sigma}}, x > \mu e^{1/\sigma}$, and 0 elsewhere, with $\mu > 0$ as a scale parameter and $\sigma > 0$ as a shape parameter, then $\lim_{x\to\infty} \frac{1-F(x;\mu_1,\sigma_1)}{1-F(x;\mu_2,\sigma_2)} = \frac{\sigma_2}{\sigma_1}$.

1.2. Heavy tails

We refer to Praveena and Ravi (2023, 2025) and Nair *et al.* (2023) for definitions and results mentioned below and some recent work. We give some definitions now, followed by some examples.

Definitions:

- A df F on $\mathbb{R}$ is heavy tailed if $\limsup_{x\to\infty} \dfrac{1-F(x)}{e^{-x}} = \infty$.
- If not, F is said to be light tailed.
- A df F on $\mathbb{R}$ is super-heavy tailed to the right if $\limsup_{x\to\infty} \dfrac{1-F(x)}{x^{-\alpha}} = \infty$, for all $\alpha > 0$.

Examples:

- Pareto df $F(x) = 1 - \frac{1}{x^\alpha}, x > 0, \alpha > 0$, Weibull df with shape parameter greater than 1, are examples of heavy tailed dfs.
- Normal, exponential dfs are examples of light tailed dfs.
- Cauchy, Fréchet, Burr dfs are super-heavy tailed distributions.
- But there can be heavier tails like dfs log-Pareto, log-log-Pareto, *etc.*

1.3. Extremes and upper order statistics

The extreme value laws: If $X_1, X_2, \ldots,$ are independent and identically distributed random variables with common df F, $M_n = \max\{X_1, \ldots, X_n\}$, and $\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x), \quad x \in \mathcal{C}(G)$, the set of all continuity points of the limit df G, then we denote this as $F \in \mathcal{D}_l(G)$. It is known that G is a type of the extreme value laws, given by:

- Fréchet law: $\Phi_\alpha(x) = \exp(-x^{-\alpha}),\ 0 \leq x,$
- Weibull law: $\Psi_\alpha(x) = \exp(-|x|^\alpha),\ x < 0,$
- Gumbel law, $\Lambda(x) = \exp(-\exp(-x)),\ x \in \mathbb{R}$; where $\alpha > 0$ a parameter.

Max stability: The extreme value laws satisfy the following stability property:

$$\Phi_\alpha^n\left(n^{1/\alpha}x\right) = \Phi_\alpha(x), \quad \Psi_\alpha^n\left(n^{-1/\alpha}x\right) = \Psi_\alpha(x), \quad \Lambda^n(x + \log n) = \Lambda(x), x \in \mathbb{R}.$$

1.4. Order statistics and k-th extremes

We denote the order statistics of $\{X_1, \ldots, X_n\}$ by $X_{1:n} \leq \ldots \leq X_{n:n}$ and assume that $F \in \mathcal{D}_l(G)$ for some G. The df of the k-th upper order statistic $X_{n-k+1:n}$, for a fixed positive integer k is given by

$$F_{k:n}(x) = P\left(X_{n-k+1:n} \leq x\right) = \sum_{i=0}^{k-1} \binom{n}{i} F^{n-i}(x)(1 - F(x))^i, x \in \mathbb{R}.$$

The limit $G_k(x) = \lim_{n\to\infty} F_{k:n}(a_n x + b_n)$ is given by

$$G_k(x) = G(x) \sum_{i=0}^{k-1} \frac{(-\log G(x))^i}{i!}, \quad x \in \{y : G(y) > 0\}.$$

2. Applications of tail equivalence to tail behaviour

The following questions on tails of dfs were answered by using tail equivalence.

2.1. Questions and motivation

- If $F(.) = \alpha F_1(.) + (1-\alpha)F_2(.)$ is a mixture df with component dfs F_1, F_2, how are the tails of F related to those of F_1, F_2? Or, how is $F \in \mathcal{D}_l(.)$ related to $F_i \in \mathcal{D}_l(.), i = 1, 2$?
- What is the tail of G_k like? Or, $G_k \in \mathcal{D}_l(.)$?

2.2. On mixtures

The following discussion is from Praveena *et al.* (2019).

- If F is the mixture df, $r(F) = \max\{r(F_1), r(F_2)\}$.
- If $F \in \mathcal{D}_l(.)$ with some norming constants and $r(F_1) < r(F_2)$, then $F_2 \in \mathcal{D}_l(.)$ with the same norming constants. This is because $F \stackrel{T}{=} F_2$.
- If $F \in \mathcal{D}_l(.)$ with some norming constants and $r(F_1) = r(F_2)$, then nothing can be said about the max domains to which F_1, F_2 may belong to. Examples have been given.
- If $F_1 \stackrel{T}{=} F_2$ and one of them belong to $\mathcal{D}_l(.)$ with some norming constants, then $F \in \mathcal{D}_l(.)$ with the same norming constants.

2.3. On k-th extremes via tail equivalence

The discussion here is from Ravi and Manohar (2018).

A recurrence relation: For any df F, fixed integer $k \geq 1$, define $F_k(x) = F(x) \sum_{i=0}^{k-1} \dfrac{(-\ln F(x))^i}{i!}$, $x \in \{y : F(y) > 0\}$. The df F_k satisfies the recurrence relation

$$F_{k+1}(x) = F_k(x) + \frac{F(x)}{k!}(-\ln F(x))^k, \quad k \geq 1, \ \ x \in \{y : F(y) > 0\}.$$

The pdf of F_{k+1} is

$$f_{k+1}(x) = \frac{f(x)}{k!}(-\ln F(x))^k, \quad k \geq 1, \ \ x \in \{y : F(y) > 0\}.$$

A result for fixed sample size: If F is a df with pdf f, then for every positive integer k, $(1 - F(x))^k$ is tail of the df $H_k(x) = 1 - (1 - F(x))^k$, $x \in \mathbb{R}$, and H_k is also absolutely continuous with pdf $H_k'(x) = k\{1 - F(x)\}^{k-1} f(x), x \in \mathbb{R}$. Further, the following are true:

- If $F \in D_l(\Phi_\alpha)$, then $r(H_k) = r(F) = \infty$, and $H_k \in D_l(\Phi_{k\alpha})$ with $a_n = F^{-}(1 - (1/n)^{1/k}), b_n = 0$.
- If $F \in D_l(\Psi_\alpha)$ then $r(H_k) = r(F) < \infty$, and $H_k \in D_l(\Psi_{k\alpha})$ with $a_n = r(F) - F^{-}(1 - (1/n)^{1/k}), b_n = r(F)$.

- If $F \in D_l(\Lambda)$, $a_n = v(b_n)$ and $b_n = F^-\left(1 - \frac{1}{n}\right)$ then $r(H_k) = r(F)$, and $H_k \in D_l(\Lambda)$ with $a_n = \dfrac{v(b_n)}{k}, b_n = H_k^-(1 - 1/n)$.

Another result for fixed sample size: Let rv X have absolutely continuous df F with pdf f and k be a positive integer. Then for $F_k(x) = F(x)\sum_{i=0}^{k-1} \dfrac{(-\ln F(x))^i}{i!}$, $x \in \{y : F(y) > 0\}$, the following results are true:

- F_k is a df with $r(F_k) = r(F)$, pdf $f_k(x) = \dfrac{f(x)}{(k-1)!}(-\ln F(x))^{k-1}$, $x \in \{y \in \mathbb{R} : F(y) > 0\}$; and $\lim_{x \to r(F)} \dfrac{1 - F_k(x)}{(1 - F(x))^k} = \dfrac{1}{k!}$, so that $F_k \overset{TE}{=} H_k$.
- If $F \in D_l(\Phi_\alpha)$, then $r(F_k) = r(F) = \infty$, and $F_k \in D_l(\Phi_{k\alpha})$ with $a_n = F^-(1 - (k!/n)^{1/k}), b_n = 0$.
- If $F \in D_l(\Psi_\alpha)$ then $r(F_k) = r(F) < \infty$, and $F_k \in D_l(\Psi_{k\alpha})$ with $a_n = r(F) - F^-(1 - (k!/n)^{1/k}), b_n = r(F)$.
- If $F \in D_l(\Lambda)$, $a_n = v(b_n)$ and $b_n = F^-\left(1 - \frac{1}{n}\right)$ then $r(F_k) = r(F)$, and $F_k \in D_l(\Lambda)$ with $a_n = \dfrac{v(b_n)}{k}, b_n = F_k^-(1 - 1/n)$.

2.3.1. Results for random sample size

Uniform k-th extremes: Suppose that n in the previous section is replaced by a discrete uniform rv N_n with $P(N_n = r) = \frac{1}{n}, r = m+1, m+2, \ldots, m+n$, N_n independent of the iid rvs $X_1, X_2, \ldots,$ $m \geq 1$ a fixed integer. We look at the tail behaviour of the limit of linearly normalized $X_{N_n-k+1:N_n}$. Observe that $X_{N_n-k+1:N_n}$ is well defined for $1 \leq k \leq m$. We have $F_{k:N_n}(x) = P(X_{N_n-k+1:N_n} \leq x) = \sum_{r=m}^{\infty} P(X_{N_n-k+1:N_n} \leq x, N_n = r)$
$= \sum_{r=m}^{\infty} \sum_{i=0}^{k-1} \binom{r}{i} F^{r-i}(x)(1 - F(x))^i P(N_n = r)$, $x \in \mathbb{R}$. The following results are true:

- If $F \in D_l(G)$ for some max stable df G then $\lim_{n\to\infty} F_{k:N_n}(a_n x + b_n)$ is equal to $U_{k,G}(x) = k\{\frac{1-G(x)}{-\ln G(x)}\} - G(x)\sum_{l=1}^{k-1}(k-l)\frac{(-\ln G(x))^{l-1}}{l!}$, $x \in \{y \in \mathbb{R} : G(y) > 0\}$, $G = \Phi_\alpha$ or Ψ_α or Λ.
- For any df F, and fixed integer $k \geq 1$, let $U_{k,F}(x) = k\{\frac{1-F(x)}{-\ln F(x)}\} - F(x)\sum_{l=1}^{k-1}(k-l)\frac{(-\ln F(x))^{l-1}}{l!}$, $x \in \{y : F(y) > 0\}$. If X has df F, pdf f, k is a fixed positive integer, and $U_{1,F}(x) = \dfrac{1 - F(x)}{-\ln F(x)}$, $x \in \{y : F(y) > 0\}$, then $U_{1,F}$ is a df with $r(U_{1,F}) = r(F)$, pdf $u_{1,F}(x) = \dfrac{f(x)}{F(x)} \dfrac{U_{1,F}(x) - F(x)}{-\ln F(x)} = \dfrac{f(x)}{F(x)} \left\{ \dfrac{1 - F(x) + F(x)\ln F(x)}{(-\ln F(x))^2} \right\}$, $x \in \{y \in \mathbb{R} : F(y) > 0\}$; and $\lim_{x\to r(F)} \dfrac{1 - U_{1,F}(x)}{1 - F(x)} = \dfrac{1}{2}$ so that $U_{1,F} \overset{T}{=} F$.

For the family of dfs $U_{k,F}$ and $H_k(x) = 1-(1-F(x))^k, x\in\mathbb{R}$, the following results are true.

- $U_{k,F}$ is a df with $r(U_{k,F}) = r(F)$, pdf $u_{k,F}(x) = \dfrac{kf(x)}{(-\ln F(x))^2}\left\{\dfrac{1}{F(x)} - \sum_{l=0}^{k}\dfrac{(-\ln F(x))^l}{l!}\right\}$, $x\in\{y\in\mathbb{R}: F(y)>0\}$, $\lim_{x\to r(F)}\dfrac{1-U_{k,F}(x)}{(1-F(x))^k} = \dfrac{1}{(k+1)!}$, and $U_{k,F} \stackrel{T}{=} H_k$.
- If $F\in D_l(\Phi_\alpha)$, then $r(U_{k,F}) = r(F) = \infty$, $U_{k,F}\in D_l(\Phi_{k\alpha})$ with $a_n = F^{-}(1-((k+1)!/n)^{1/k}), b_n = 0$.
- If $F\in D_l(\Psi_\alpha)$ then $r(U_{k,F}) = r(F) < \infty$, $U_{k,F}\in D_l(\Psi_{k\alpha})$ with $a_n = r(F) - F^{-}(1-((k+1)!/n)^{1/k}), b_n = r(F)$.
- If $F\in D_l(\Lambda)$, $a_n = v(b_n)$ with $b_n = F^{-}\left(1-\frac{1}{n}\right)$ then $r(U_{k,F}) = r(F)$, $U_{k,F}\in D_l(\Lambda)$ with $F = U_{k,F}, G=\Lambda$, $a_n = \dfrac{v(b_n)}{k}, b_n = U^{-}_{k,F}(1-1/n)$.
- The df $U_{k,F}$ satisfies the recurrence relation

$$U_{k+1}(x) = U_{k,F}(x) + U_{1,F}(x) - F(x)\sum_{l=1}^{k}\frac{(-\ln F(x))^{l-1}}{l!}.$$

Geometric k-th extremes: Let N_n be a shifted geometric rv with pmf $P(N_n = r) = p_n q_n^{r-m}, r = m, m+1, m+2,\ldots,$ $0<p_n<1, q_n = 1-p_n$ and $\lim_{n\to\infty} np_n = 1$.

- If $F\in D_l(G)$ for some max stable law G, then for fixed integer $k, 1\le k\le m$, $\lim_{n\to\infty} F_{k:N_n}(a_nx+b_n)$ is equal to

$$R_{k,G}(x) = 1-\left(\frac{-\ln G(x)}{1-\ln G(x)}\right)^k, \; x\in\{y\in\mathbb{R}: G(y)>0\}, \quad \text{with}$$

$$R_{k,G}(x) = \begin{cases} 1-\left(\frac{1}{1+x^\alpha}\right)^k & \text{if } G(x) = \Phi_\alpha(x),\\ 1-\left(\dfrac{(-x)^\alpha}{1+(-x)^\alpha}\right)^k & \text{if } G(x) = \Psi_\alpha(x),\\ 1-\left(\frac{e^{-x}}{1+e^{-x}}\right)^k & \text{if } G(x) = \Lambda(x).\end{cases}$$

  The first two are Burr distributions of XII kind (Burr, 1942) and the last is the logistic distribution.

- If X has df F, pdf f, k a positive integer and $R_{k,F}$ is as defined above, then the following are true:
- $R_{k,F}$ is a df with pdf $r_{k,F}(x) = \dfrac{kf(x)(-\ln F(x))^{k-1}}{F(x)(1-\ln F(x))^{k+1}}$, $x\in\{y\in\mathbb{R}: F(y)>0\}$, $r(R_{k,F}) = r(F)$, and $\lim_{x\to r(F)}\dfrac{1-R_{k,F}(x)}{(1-F(x))^k} = 1$, and $R_{k,F}\stackrel{T}{=} H_k$.

- If $F \in D_l(\Phi_\alpha)$, then $r(R_{k,F}) = r(F) = \infty$, and $R_{k,F} \in D_l(\Phi_{k\alpha})$ with $a_n = F^{-}(1-(1/n)^{1/k}), b_n = 0$.

- If $F \in D_l(\Psi_\alpha)$ then $r(R_{k,F}) = r(F) < \infty$, and $R_{k,F} \in D_l(\Psi_{k\alpha})$ with $F = R_{k,F}, G = \Psi_{k\alpha},\ a_n = r(F) - F^{-}(1-(1/n)^{1/k}), b_n = r(F)$.

- If $F \in D_l(\Lambda)$, $a_n = v(b_n)$ and $b_n = F^{-}\left(1 - \dfrac{1}{n}\right)$ then $r(R_{k,F}) = r(F)$, and $R_{k,F} \in D_l(\Lambda)$ with $F = R_{k,F}, G = \Lambda,\ a_n = \dfrac{v(b_n)}{k}, b_n = R^{-}_{k,F}(1-1/n)$.

The Burr connection: Burr (1942) proposed twelve explicit forms of dfs which have since come to be known as the Burr system of distributions. A number of well-known distributions such as the uniform, Rayleigh, logistic, and log-logistic are special cases of Burr dfs.

A df W is said to belong to the Burr family if it satisfies the differential equation $\dfrac{dW(x)}{dx} = W(x)(1-W(x))h(x, W(x))$, where $h(x, W(x))$ is a non-negative function for x for which the function is increasing, $h(x, W(x))$ could be $h(x, W(x)) = \dfrac{h_1(x)}{W(x)}$ where $h_1(x) \geq 0$. Then $\dfrac{dW(x)}{dx} = (1-W(x))h_1(x)$.

The dfs $R_{k,F}$ belong to the Burr family.

Negative Binomial k-th extremes: Let N_n be a shifted negative binomial rv with $P(N_n = l) = \binom{l-m+r-1}{l-m} p_n^r q_n^{l-m}, r = m, m+1, m+2, \ldots,$ where $0 < p_n < 1, q_n = 1 - p_n$ and $\lim_{n\to\infty} np_n = 1$.

If $F \in D_l(G)$ for some G, then for fixed integer $k, 1 \leq k \leq m$, $\lim_{n\to\infty} F_{k:N_n}(a_n x + b_n)$ is equal to

$$T_{k,G}(x) = \sum_{l=0}^{k-1} \binom{l+r-1}{l} \frac{(-\ln G(x))^l}{(1-\ln G(x))^{r+l}},\ x \in \{y \in \mathbb{R} : G(y) > 0\}.$$

The df $T_{k,F}$ satisfies the recurrence relation

$$T_{k+1,F}(x) = T_{k,F}(x) + \binom{k+r-1}{k} \frac{(-\ln F(x))^k}{(1-\ln F(x))^{k+r}},\quad k \geq 1, x \in \mathbb{R}.$$

Its pdf is $t_{k+1,F}(x) = \dfrac{1}{B(r,k+1)} \dfrac{f(x)}{F(x)} \dfrac{(-\ln F(x))^k}{(1-\ln F(x))^{r+k+1}},\quad k \geq 1, x \in \mathbb{R}$.

Let rv X have df F with pdf f and k be a fixed positive integer. Then for $T_{k,F}$, the following results are true.

- $T_{k,F}$ is a df with pdf $t_{k,F}(x) = \dfrac{1}{B(r,k)} \dfrac{f(x)}{F(x)} \dfrac{(-\ln F(x))^{k-1}}{(1-\ln F(x))^{r+k}},\ x \in \{y \in \mathbb{R} : F(y) > 0\}$, right extremity $r(T_{k,F}) = r(F)$, and $\lim_{x\to r(F)} \dfrac{1-T_{k,F}(x)}{(1-F(x))^k} = \dfrac{k}{B(r,k)}$.

- If $F \in D_l(\Phi_\alpha)$, then $r(T_{k,F}) = r(F) = \infty$, and $T_{k,F} \in D_l(\Phi_{k\alpha})$ with $a_n = F^{-}(1-(1/n)^{1/k}), b_n = 0$.
- If $F \in D_l(\Psi_\alpha)$ then $r(T_{k,F}) = r(F) < \infty$, and $T_{k,F} \in D_l(\Psi_{k\alpha})$ with $a_n = r(F) - F^{-}(1-(1/n)^{1/k}), b_n = r(F)$.
- If $F \in D_l(\Lambda)$, $a_n = v(b_n)$ and $b_n = F^{-}\left(1 - \frac{1}{n}\right)$ then $r(T_{k,F}) = r(F)$, and $T_{k,F} \in D_l(\Lambda)$ with $a_n = \frac{v(b_n)}{k}, b_n = T^{-}_{k,F}(1-1/n)$.

3. Conclusion

In this article, tail behaviour of several interesting tails are explored through the concept of tail equivalence which simplifies several proofs. After recalling the definition of tail equivalence and clarifying some simple properties of tail equivalence, the article explores tail behaviour of mixtures of dfs and the limit laws of linearly normalised k upper order statistics of a random sample of size n, when n is fixed and n is replaced by Uniform, Geometric and Negative Binomial random sample sizes. Several results stated here can be used to simulate random observations from a variety of tails.

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Conflict of interest

The author does not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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