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Discrete Type I Half-Logistic Weibull Distribution and its Properties

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Abstract

In this paper, we develop a discrete version of the type I half-logistic family of distributions. Several members of this family such as, discrete type I half-logistic version of uniform, Lomax, exponential, Fréchet and Weibull distributions are derived. Statistical properties of one of the members of this family, namely, the discrete type I half-logistic Weibull distribution, is studied in detail. The parameters of this distribution are estimated using the maximum likelihood method and a simulation study is conducted to evaluate the consistency of the method. Three data applications are illustrated to show the flexibility for fitting the proposed models to real-life data sets.

Key words: Data modeling; Discrete distributions; Hazard rate function; Order statistics; Weibull distribution.

AMS Subject Classifications: 60E05, 62E10.

1. Introduction

The logistic function is one of the oldest growth functions in the literature and is used to describe both population and organic growth. Various theoretical, methodological and applied issues relating to the logistic model are discussed in Balakrishnan (1991). Often in scientific enquiry we may come across observations which are discrete in nature. In a reliability study or life testing of equipment, it is difficult to quantify the length of life of the equipment on a continuous scale. In survival analysis, we may record the number of days of survival for lung cancer patients since therapy, or the times from remission to relapse are also usually recorded in number of days. But in some analysis, often the interest lies not only in counts but in changes in counts from a given origin, in such situation the variable of interest can take either zero, positive or negative value. The conventional discrete distributions such as, geometric, Poisson, binomial and negative binomial have wide but limited applicability in reliability, failure time modeling, etc. Thus, there is interest in developing new discrete

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family of distributions based on the well known continuous distributions. Among these, the discrete Weibull distribution of Nakagawa and Osaki (1975) is the most popular one.

Recently, discretization of the continuous distributions has attracted the researchers' attention and several forms of discrete lifetime distributions are being established in the literature. Some of the recent works on discretization of continuous distributions are the discrete Laplace distribution by Inusah and Kozubowski (2006), the discrete half-normal distribution by Kemp (2008), the discrete Burr and discrete Pareto distributions by Krishna and Pundir (2009), the discrete generalized exponential distribution by Gomez-Deniz (2010) and the discrete gamma distribution by Chakraborty and Chakravarty (2012).

Other notable works in this direction are the discrete additive Weibull distribution by Bebbington *et al.* (2012), the discrete inverse Weibull distribution by Jazi *et al.* (2010), the discrete generalized exponential distribution by Nekoukhou *et al.* (2012), discrete reduced modified Weibull distribution by Almalki and Nadarajah (2014), the discrete Lindley distribution by Bakouch *et al.* (2014), the discrete Logistic distribution by Chakraborty and Chakravarty (2016), the discrete log-logistic distribution by Para and Jan (2016), the discrete Weibull geometric distribution by Jayakumar and Babu (2018), the truncated discrete Mittag-Leffler distribution by Jayakumar and Sankaran (2018), discrete additive Weibull geometric distribution by Jayakumar and Babu (2019), the discrete Pareto type(IV) model by Ghosh (2020), among others. There are several methods available in the literature to discretize a continuous random variable, for more details, see Chakraborty (2015). This paper discusses the formation of distribution that are more appropriate to modeling discrete failure data in varying failure rate shape.

Let Y is discretized as $Y = \lfloor X \rfloor$, the largest integer less than or equal to X. Using the survival function $S_X(y)$, the discrete version of the random variable X can be derived by

$$P(Y = y) = P(X \geq y) - P(X \geq y+1) = S_X(y) - S_X(y+1); \quad y = 0, 1, 2, \dots . \tag{1}$$

The cumulative distribution function (cdf) of Y is given by,

$$\begin{aligned} F(y) = P(Y \leq y) &= P(Y < y) + P(Y = y) \\ &= 1 - S_X(y) + P(Y = y) \\ &= 1 - S_X(y+1) = P(X \leq y+1). \end{aligned}$$

Now, for evaluating the values of the cdf when the value of y can be integer or fractional the following general formula can be used

$$F(y) = P(X \leq \lfloor y \rfloor + 1),$$

where $\lfloor y \rfloor$ denotes the floor of y, i.e., the largest integer less or equal to y. In this paper we take y as $\lfloor y \rfloor$.

The discretization of a continuous distribution given in (1) retains the similar functional form of the survival function, so that many reliability characteristics remains unchanged. This motivated to use this technique of generating discretized version of continuous distribution. The half-logistic probability models are obtained as the models of the absolute value of the standard logistic models.

Recently Chipepa *et al.* (2022) introduced a generalized class of distributions named as exponentiated half-logistic generalized - G power series distribution by combining the exponentiated half-logistic generalized class of distributions and power series distribution. The hazard rate of this distribution exhibits increasing, decreasing, bathtub, bathtub followed by upside down bathtub, J and inverse-J shapes. One member of this family called type I half-logistic-G (TIHL-G) family of distributions was already studied by Cordeiro *et al.* (2016). This family is defined by

$$F(x;\lambda,\Theta)=\int_0^{-\log[1-G(x;\Theta)]}\frac{2\lambda e^{-\lambda t}}{(1+e^{-\lambda t})^2}dt=\frac{1-[1-G(x;\Theta)]^{\lambda}}{1+[1-G(x;\Theta)]^{\lambda}}, \tag{2}$$

where $G(x;\Theta)$ is the baseline *cdf* with parameter vector Θ and $\lambda>0$ is an additional shape parameter. When $\lambda=1$, this family becomes half-logistic-G (HL-G) family of distributions. The probability density function (pdf) of (2) is given by

$$f(x;\lambda,\Theta)=\frac{2\lambda g(x;\Theta)[1-G(x;\Theta)]^{\lambda-1}}{\Big[1+[1-G(x;\Theta)]^{\lambda}\Big]^2}, \tag{3}$$

where $g(x;\Theta)$ is the baseline pdf. Also, the survival function and hazard rate function (hrf) are respectively given by

$$S(x;\lambda,\Theta)=\frac{2[1-G(x;\Theta)]^{\lambda}}{1+[1-G(x;\Theta)]^{\lambda}}, \tag{4}$$

and

$$h(x;\lambda,\Theta)=\frac{\lambda g(x;\Theta)}{\Big[1-G(x;\Theta)\Big]\Big[1+[1-G(x;\Theta)]^{\lambda}\Big]}. \tag{5}$$

Different choices of $G(x;\Theta)$ in (2) leads to special models of this family of distributions. Some of the models are type I half-logistic normal, type I half-logistic gamma and type I half-logistic Fréchet discussed in Cordeiro *et al.* (2016). The objective of this paper is to introduce a discrete version of this family and study their mathematical properties.

The paper is organized as follows. In Section 2, we introduce a discrete type I half-logistic family of distributions. Some members of this family are introduced in Section 3. Section 4 discusses the construction of the type I half-logistic Weibull distribution and in Section 5, the maximum likelihood estimation of unknown parameters are discussed and a simulation study to asses the performance of the MLEs of the model parameters is also presented. Applications of this new discrete distribution for modeling real data sets are discussed in Section 6, and conclusions and future works are presented in Section 7.

2. Discretization of type I half-logistic family of distributions

Let X be a continuous random variable belonging to TIHL-G family of distributions with cdf given in (2). Let Y be the discrete analogue of X derived using the survival function (4) and by using the expression (1) as follows :

$$P(Y=y)=\frac{2\Big[\bar{G}^{\lambda}(y;\Theta)-\bar{G}^{\lambda}(y+1;\Theta)\Big]}{\Big[1+\bar{G}^{\lambda}(y;\Theta)\Big]\Big[1+\bar{G}^{\lambda}(y+1;\Theta)\Big]} \tag{6}$$

where $Y = \lfloor X \rfloor$, the largest integer less than or equal to X and $\bar{G}(y;\Theta) = 1 - G(y;\Theta)$. The corresponding cdf is given by

$$F(y) = \frac{1 - \bar{G}^{\lambda}(y+1;\Theta)}{1 + \bar{G}^{\lambda}(y+1;\Theta)}, \tag{7}$$

survival function is

$$S(y) = 1 - P(Y \le y) = \frac{2\bar{G}^{\lambda}(y+1;\Theta)}{1 + \bar{G}^{\lambda}(y+1;\Theta)}, \tag{8}$$

and hazard rate function is

$$h(y) = \frac{P(Y = y)}{P(Y \ge y)} = \frac{\bar{G}^{\lambda}(y;\Theta) - \bar{G}^{\lambda}(y+1;\Theta)}{\bar{G}^{\lambda}(y+1;\Theta)\left[1 + \bar{G}^{\lambda}(y;\Theta)\right]}. \tag{9}$$

The reverse hazard rate is

$$h^{*}(y) = \frac{P(Y = y)}{P(Y \le y)} = \frac{2\left[\bar{G}^{\lambda}(y;\Theta) - \bar{G}^{\lambda}(y+1;\Theta)\right]}{\left[1 + \bar{G}^{\lambda}(y;\Theta)\right]\left[1 + \bar{G}^{\lambda}(y+1;\Theta)\right]}. \tag{10}$$

The second rate of failure is given by

$$h^{**}(y) = log\left[\frac{S(y)}{S(y+1)}\right] = log\left[\frac{\bar{G}^{\lambda}(y+1;\Theta)}{\bar{G}^{\lambda}(y+2;\Theta)}\right] + log\left[\frac{1 + \bar{G}^{\lambda}(y+2;\Theta)}{1 + \bar{G}^{\lambda}(y+1;\Theta)}\right]. \tag{11}$$

2.1. Quantile function

The quantile function of the discrete type I half-logistic G family of distributions, say $Q(u)$, defined by $F(Q(u)) = u$, where $u \in (0,1)$ is given by

$$Q(u) = \left\lceil G^{-1}\left[1 - \left(\frac{1-u}{1+u}\right)^{\frac{1}{\lambda}}\right] - 1\right\rceil, \tag{12}$$

where $\lceil . \rceil$ denotes the ceiling value.
In particular, the median $= \left\lceil G^{-1}\left[1 - \left(\frac{1}{3}\right)^{\frac{1}{\lambda}}\right] - 1\right\rceil$.

2.2. Probability generating function

The probability generating function (pgf) of discrete TIHL-G family of distributions is given by

$$P_Y(s) = E(s^Y) = 1 + 2(s-1)\sum_{y=1}^{\infty} \frac{s^{y-1}\bar{G}^{\lambda}(y+1;\Theta)}{1 + \bar{G}^{\lambda}(y+1;\Theta)}.$$

Then mean and variance are respectively,

$$E(Y) = \sum_{y=1}^{\infty} \frac{2\bar{G}^{\lambda}(y+1;\Theta)}{1 + \bar{G}^{\lambda}(y+1;\Theta)},$$

and

$$V(Y)=\sum_{y=1}^{\infty}\frac{(2y-1)\bar{G}^{\lambda}(y+1;\Theta)}{1+\bar{G}^{\lambda}(y+1;\Theta)}-\left[\sum_{y=1}^{\infty}\frac{2\bar{G}^{\lambda}(y+1;\Theta)}{1+\bar{G}^{\lambda}(y+1;\Theta)}\right]^2.$$

Also, the recurrence relation for generating probabilities, is

$$P_Y(y+1;\Theta,\lambda)=\frac{\Big[\bar{G}^{\lambda}(y+1;\Theta)-\bar{G}^{\lambda}(y+2;\Theta)\Big]\Big[1+\bar{G}^{\lambda}(y;\Theta)\Big]}{\Big[\bar{G}^{\lambda}(y;\Theta)-\bar{G}^{\lambda}(y+1;\Theta)\Big]\Big[1+\bar{G}^{\lambda}(y+2;\Theta)\Big]}P_Y(y;\Theta,\lambda).$$

Different choices of $G(y;\Theta)$ in (6) will give new family of discrete probability distributions. In the next section, we discuss some discrete probability models obtained from this family.

3. Some members of Discrete Type I Half Logistic- General (DTIHL-G) family

3.1. Discrete type I half-logistic uniform distribution

Let $X\sim U(0,\alpha)$ with cdf $G(x;\alpha)=\frac{x}{\alpha}, 0<x<\alpha$. Then the pmf, cdf and survival function of the discrete type I half-logistic uniform distribution are respectively,

$$P(Y=y)=\frac{2\alpha^{\lambda}\Big[(\alpha-y)^{\lambda}-(\alpha-(y+1))^{\lambda}\Big]}{\Big[\alpha^{\lambda}+(\alpha-y)^{\lambda}\Big]\Big[\alpha^{\lambda}+(\alpha-(y+1))^{\lambda}\Big]};\ y=0,1,...,\alpha-1,$$

$$F(y;\alpha,\lambda)=\frac{\alpha^{\lambda}-\Big[\alpha-(y+1)\Big]^{\lambda}}{\alpha^{\lambda}+\Big[\alpha-(y+1)\Big]^{\lambda}},$$

and

$$S(y;\alpha,\lambda)=\frac{2\Big[\alpha^{\lambda}-(y+1)\Big]^{\lambda}}{\alpha^{\lambda}+\Big[\alpha^{\lambda}-(y+1)\Big]^{\lambda}}.$$

3.2. Discrete type I half-logistic Lomax distribution

Let X follow the Lomax distribution with shape parameter $\alpha>0$ and scale parameter $\beta>0$ with cdf $G(x;\alpha,\beta)=1-(1+\beta x)^{-\alpha}$. Then the pmf, cdf and survival function of the discrete type I half-logistic Lomax distribution are given by

$$P(Y=y)=\frac{2\Big[(1+\beta y)^{-\alpha\lambda}-(1+\beta(y+1))^{-\alpha\lambda}\Big]}{\Big[1+(1+\beta y)^{-\alpha\lambda}\Big]\Big[1+(1+\beta(y+1))^{-\alpha\lambda}\Big]};\ y=0,1,\dots,$$

$$F(y;\alpha,\lambda)=\frac{1-(1+\beta y)^{-\alpha\lambda}}{1+(1+\beta y)^{-\alpha\lambda}},$$

and

$$S(y;\alpha,\lambda)=\frac{2(1+\beta(y+1))^{-\alpha\lambda}}{1+(1+\beta(y+1))^{-\alpha\lambda}}.$$

When $\lambda=1$, this distribution becomes discrete half-logistic Lomax distribution.

3.3. Discrete type I half-logistic exponential distribution

Let X follow the exponential distribution with parameter $\alpha > 0$ with cdf $G(x;\alpha) = 1 - e^{-\alpha x}$. By taking $q = e^{-\alpha}, 0 < q < 1$, we get the pmf, cdf and survival function of the discrete type I half-logistic exponential distribution are respectively,

$$P(Y = y) = \frac{2\Big[q^{\lambda y} - q^{\lambda(y+1)}\Big]}{\Big[1 + q^{\lambda y}\Big]\Big[q^{\lambda(y+1)}\Big]};\ y = 0, 1, \dots ,$$

$$F(y;\alpha,\lambda) = \frac{1 - q^{\lambda y}}{1 + q^{\lambda y}},$$

and

$$S(y;\alpha,\lambda) = \frac{2q^{\lambda(y+1)}}{1 + q^{\lambda(y+1)}}.$$

3.4. Discrete type I half-logistic Fréchet distribution

Let X follow the Fréchet distribution with scale parameter $\alpha > 0$ and shape parameter $\beta > 0$ with cdf $G(x;\alpha,\beta) = e^{-(\alpha/x)^{\beta}}$. By taking $q = e^{-\alpha^{\beta}}, 0 < q < 1$, we get the pmf, cdf and survival function of the discrete type I half-logistic Fréchet distribution as, respectively

$$P(Y = y) = \frac{2\Big[(1 - q^{y^{-\beta}})^{\lambda} - (1 - q^{(y+1)^{-\beta}})^{\lambda}\Big]}{\Big[1 + (1 - q^{y^{-\beta}})^{\lambda}\Big]\Big[1 + (1 - q^{(y+1)^{-\beta}})^{\lambda}\Big]};\ y = 0, 1, \dots ,$$

$$F(y;q,\beta,\lambda) = \frac{1 - (1 - q^{(y+1)^{-\beta}})^{\lambda}}{1 + (1 - q^{(y+1)^{-\beta}})^{\lambda}},$$

and

$$S(y;q,\beta,\lambda) = \frac{2(1 - q^{(y+1)^{-\beta}})^{\lambda}}{1 + (1 - q^{(y+1)^{-\beta}})^{\lambda}}.$$

In a similar way, by considering different choices of $G(y;\Theta)$ in (6), we can develop several discrete probability distributions. In the next section we study in detail the discrete type I half-logistic Weibull distribution.

4. Discrete type I half-logistic Weibull distribution

Let X follow the Weibull distribution with parameters $\alpha > 0$ and $\beta > 0$ with cdf and survival function are respectively $G(x;\alpha,\beta) = 1 - e^{-\alpha x^{\beta}}$ and $\bar{G}(x;\alpha,\beta) = e^{-\alpha x^{\beta}}$. By taking $q = e^{-\alpha}, 0 < q < 1$ and using (6), the pmf of the new distribution is given by

$$P(Y = y) = \frac{2(q^{\lambda y^{\beta}} - q^{\lambda(y+1)^{\beta}})}{(1 + q^{\lambda y^{\beta}})(1 + q^{\lambda(y+1)^{\beta}})}. \tag{13}$$

We call this distribution the discrete type I half-logistic Weibull (DTIHLW) distribution with parameters q, λ and β and is denoted by DTIHLW(q, λ, β).
When $\beta = 1$, the pmf becomes,

$$P(Y = y) = \frac{2q^{\lambda y}(1 - q^{\lambda})}{(1 + q^{\lambda y})(1 + q^{\lambda(y+1)})},$$

which is the discrete type I half-logistic exponential distribution.

5. Structural properties of DTIHLW(q, λ, β)

Figure 1 shows the shape of pmf of DTIHLW(q, λ, β) distribution for various selection of the parameter values.

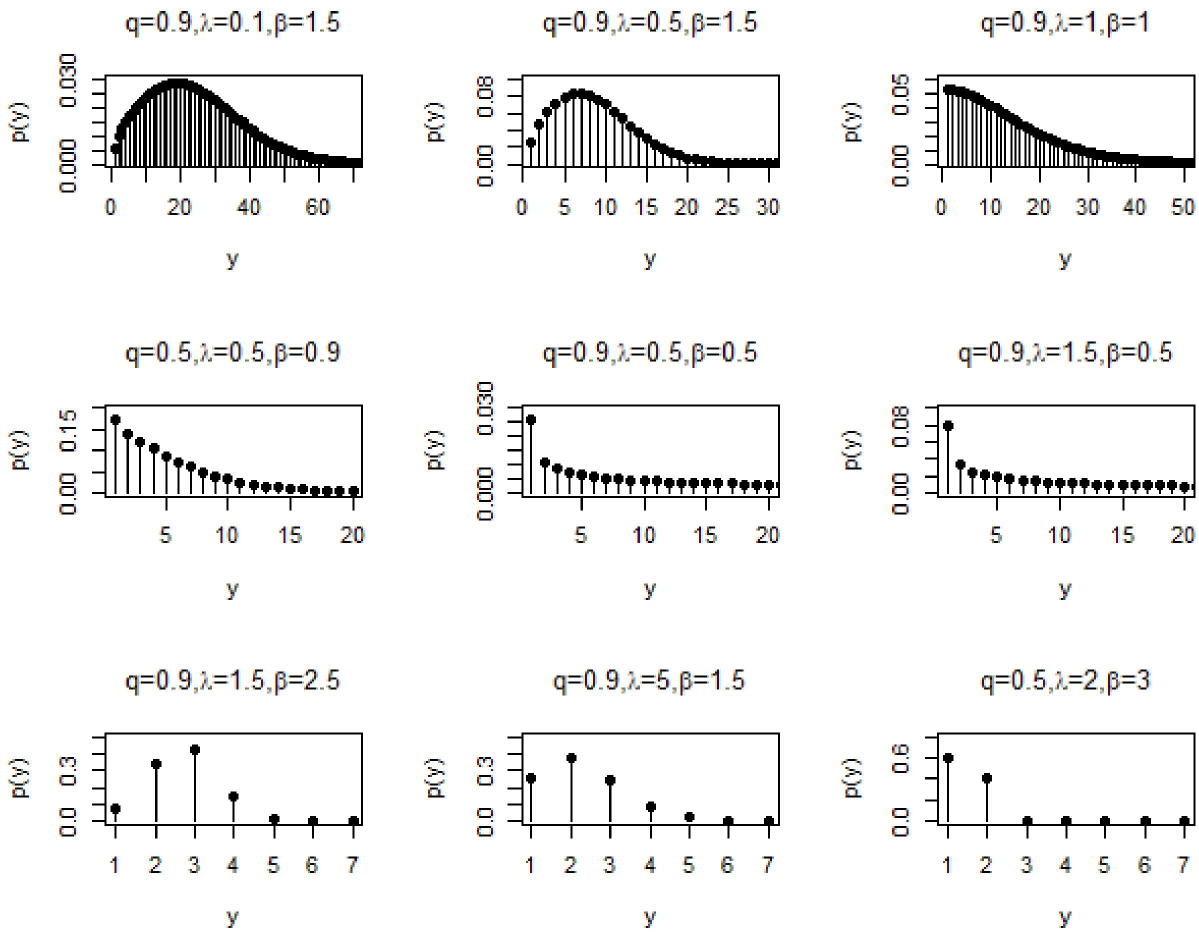

Figure 1: Shapes of the pmf of DTIHLW(q, λ, β) for various parameter values.

Theorem 1: The pmf of DTIHLW(q, λ, β) distribution is log-concave for $\beta \leq 1$.

Proof: From Kus *et al.* (2019), a distribution with pmf $p(y)$ is log-concave if

$$[p(y+1)]^2 > p(y)p(y+2), \tag{14}$$

for all $y \geq 0$.
Under $q \in (0,1)$, $\lambda > 0$ and $\beta \leq 1$ we have

$$\frac{[q^{\lambda(y+1)^\beta} - q^{\lambda(y+2)^\beta}]^2}{(1+q^{\lambda(y+1)^\beta})(1+q^{\lambda(y+2)^\beta})} - \frac{[q^{\lambda y^\beta} - q^{\lambda(y+1)^\beta}][q^{\lambda(y+2)^\beta} - q^{\lambda(y+3)^\beta}]}{(1+q^{\lambda y^\beta})(1+q^{\lambda(y+3)^\beta})} > 0,$$

for all $y \geq 0$. Thus (14) is satisfied by the pmf (13). □

5.1. Cumulative distribution function, survival and hazard rate functions of DTIHLW distribution

The cdf of DTIHLW(q, λ, β) distribution is obtained as

$$F(y; q, \lambda, \beta) = P(Y \leq y) = 1 - P(Y \geq y + 1) = \frac{1 - q^{\lambda(y+1)^\beta}}{1 + q^{\lambda(y+1)^\beta}}, \tag{15}$$

where $y = 0, 1, 2, ...; 0 < q < 1, \lambda > 0$ and $\beta > 0$.
In particular,

$$F(0) = \frac{1 - q^\lambda}{1 + q^\lambda},$$

and the proportion of positive values,

$$1 - F(0) = \frac{2q^\lambda}{1 + q^\lambda}.$$

Also,

$$P(a < Y \leq b) = \frac{1 - q^{\lambda(b+1)^\beta}}{1 + q^{\lambda(b+1)^\beta}} - \frac{1 - q^{\lambda(a+1)^\beta}}{1 + q^{\lambda(a+1)^\beta}}.$$

The survival function of DTIHLW(q, λ, β) is given by

$$S(y; q, \lambda, \beta) = P(Y > y) = 1 - P(Y \leq y) = \frac{2q^{\lambda(y+1)^\beta}}{1 + q^{\lambda(y+1)^\beta}}. \tag{16}$$

The hazard rate function is given by,

$$h(y; q, \lambda, \beta) = \frac{P(Y = y)}{P(Y \geq y)} = \frac{1 - q^{\lambda[(y+1)^\beta - y^\beta]}}{1 + q^{\lambda(y+1)^\beta}}, \tag{17}$$

provided $P(Y \geq y) > 0$. Here note that,

$$\lim_{y \to 0} h(y; q, \lambda, \beta) = \frac{1 - q^\lambda}{1 + q^\lambda}.$$

Also, when $\lambda > 0$ and $\beta > 1$,

$$\lim_{y \to \infty} h(y; q, \lambda, \beta) = 1,$$

when $\lambda > 0$ and $\beta < 1$,

$$\lim_{y \to \infty} h(y; q, \lambda, \beta) = 0,$$

and when $\lambda > 0$ and $\beta = 1$,

$$\lim_{y \to \infty} h(y; q, \lambda, \beta) = 1 - q^\lambda.$$

Figure 2 shows the shape of the hrf of DTIHLW(q, λ, β) for various choices of parameter values. The cumulative hazard function, $H(y; q, \lambda, \beta)$, is given by

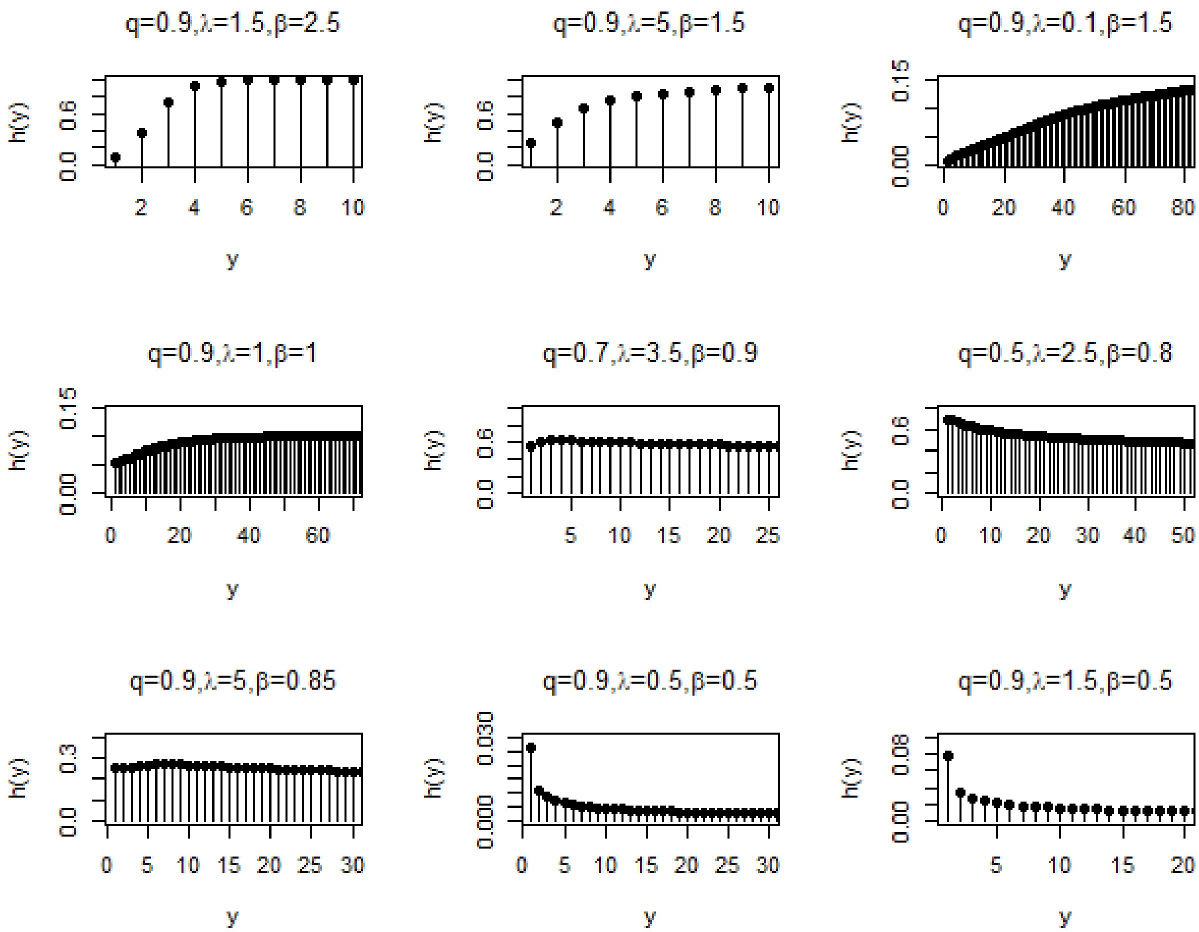

Figure 2: Shapes of the hrf of DTIHLW(q, λ, β) for various parameter values.

$$H(y; q, \lambda, \beta) = \sum_{t=0}^{y} h(t) = \sum_{t=0}^{y} \frac{1 - q^{\lambda[(t+1)^{\beta} - t^{\beta}]}}{1 + q^{\lambda(t+1)^{\beta}}}. \tag{18}$$

The mean residual life (MRL) function (see Jayakumar and Babu (2018)) is given by

$$L(y) = E[(Y - y)/Y \geq y] = \sum_{j \geq y} \prod_{i=y}^{j} \Big(1 - h(i)\Big) = \sum_{j \geq y} \prod_{i=y}^{j} \frac{1 + q^{-\lambda y^{\beta}}}{1 + q^{-\lambda(y+1)^{\beta}}}; \quad y = 0, 1, 2, \ldots . \tag{19}$$

Another expression for MRL by Roy and Gupta (1999) is given by

$$\mu(y) = E[(Y - y)/Y > y] = 1 + L(y+1) = 1 + \sum_{j \geq y+1} \prod_{i=y+1}^{j} \frac{1 + q^{-\lambda y^{\beta}}}{1 + q^{-\lambda(y+1)^{\beta}}}; \quad y = 0, 1, 2, \ldots . \tag{20}$$

When $y = 0$, then the MRL function is equal to the mean of the lifetime distribution, that is, $L(0) = \mu$. Thus, we have,

$$\mu(0) = \frac{\mu}{1 - p(0)} = \frac{\mu(1 + q^{\lambda})}{2q^{\lambda}}. \tag{21}$$

Also, the reverse hazard rate function is given by

$$h^*(y) = P(Y = y/Y \leq y) = \frac{2(q^{\lambda y^\beta} - q^{\lambda(y+1)^\beta})}{(1 + q^{\lambda y^\beta})(1 - q^{\lambda(y+1)^\beta})}. \tag{22}$$

The following Figure 3 shows the change of reverse hrf for given parameter values. The

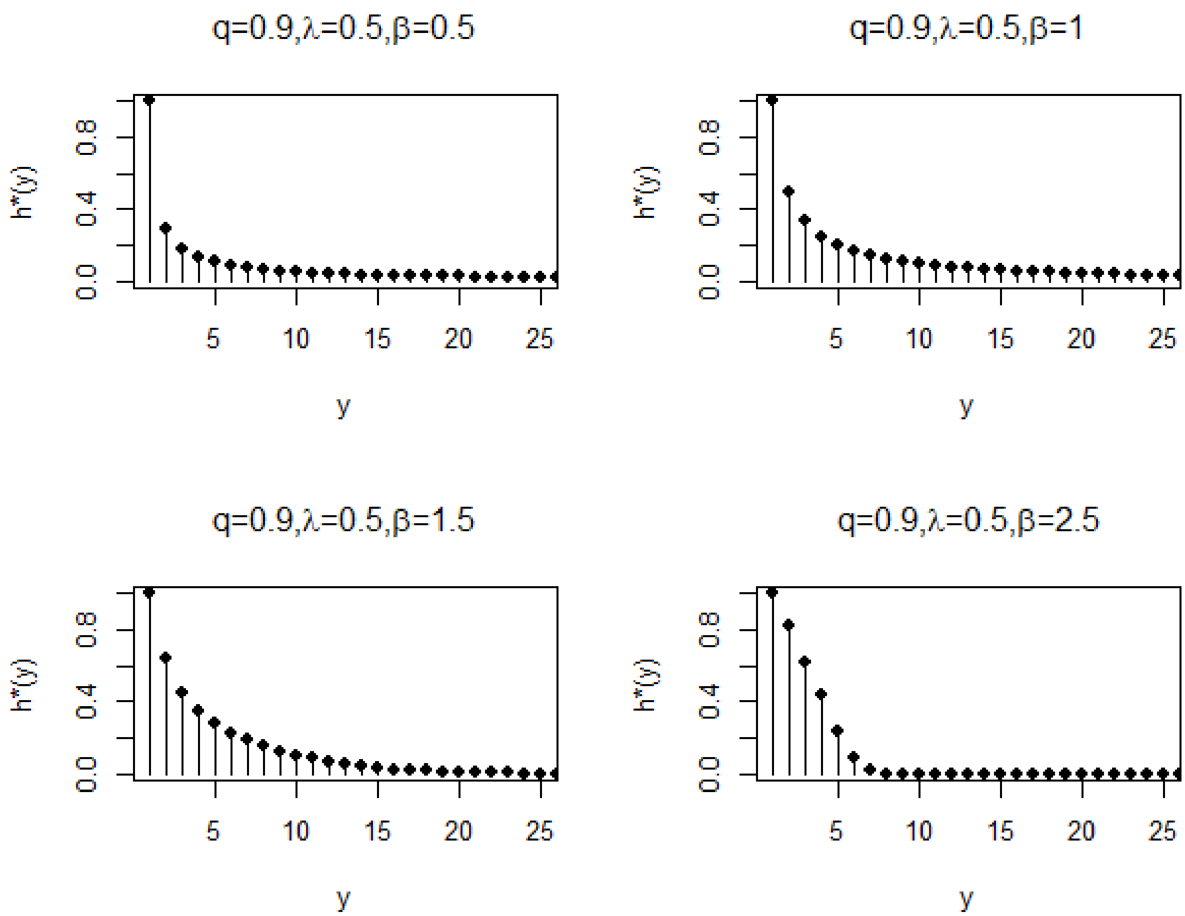

Figure 3: Reverse hrf of DTIHLW(q, λ, β) for various parameter values.

second rate of failure is given by

$$h^{**}(y) = \log\left\{\frac{S(y)}{S(y+1)}\right\} = \log\left\{\frac{q^{\lambda[(y+1)^\beta - (y+2)^\beta]}\left(1 + q^{\lambda(y+2)^\beta}\right)}{1 + q^{\lambda(y+2)^\beta}}\right\}. \tag{23}$$

5.2. Quantiles

The point y_u is known as the u^{th} quantile of a discrete random variable Y, if it satisfies $P(Y \leq y_u) \geq u$ and $P(Y \geq y_u) \geq 1 - u$, see Rohatgi and Saleh (2001). Then we have the following theorem.

Theorem 2: The u^{th} quantile $\phi(u)$ of DTIHLW(q, λ, β) is given by,

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left[ln\left(\frac{1-u}{1+u}\right) \Big/ \lambda\ ln(q) \right]^{\frac{1}{\beta}} - 1 \right\rceil, \tag{24}$$

where $\lceil y_u \rceil$ denotes the smallest integer greater than or equal to y_u.

Proof: Here first we assume that, $P(Y \leq y_u) \geq u$.
That is,

$$\begin{aligned} \frac{1 - q^{\lambda(y_u+1)^\beta}}{1 + q^{\lambda(y_u+1)^\beta}} &\geq u \\ &\Rightarrow 1 - q^{\lambda(y_u+1)^\beta} \geq u(1 + q^{\lambda(y_u+1)^\beta}) \\ &\Rightarrow \left[\frac{ln\left(\frac{1-u}{1+u}\right)}{\lambda\ ln(q)}\right]^{\frac{1}{\beta}} \leq y_u + 1 \\ &\Rightarrow y_u \geq \left[\frac{ln\left(\frac{1-u}{1+u}\right)}{\lambda\ ln(q)}\right]^{\frac{1}{\beta}} - 1, \end{aligned} \tag{25}$$

since $ln(q) < 0$.
Similarly, $P(Y \geq y_u) \geq 1 - u$ gives,

$$y_u \leq \left[\frac{ln\left(\frac{1-u}{1+u}\right)}{\lambda\ ln(q)}\right]^{\frac{1}{\beta}}. \tag{26}$$

From (25) and (26) we get,

$$\left[\frac{ln\left(\frac{1-u}{1+u}\right)}{\lambda\ ln(q)}\right]^{\frac{1}{\beta}} - 1 < y_u \leq \left[\frac{ln\left(\frac{1-u}{1+u}\right)}{\lambda\ ln(q)}\right]^{\frac{1}{\beta}}$$

Hence, $\phi(u)$ is an integer given by,

$$\phi(u) = \lceil y_u \rceil = \left\lceil \left[ln\left(\frac{1-u}{1+u}\right) \Big/ \lambda\ ln(q) \right]^{\frac{1}{\beta}} - 1 \right\rceil,$$

This completes the proof. □

Let U be a random number drawn from a uniform distribution on $(0, 1)$, then a random number Y following DTIHLW(q, λ, β) distribution is obtained by using the expression (24). In particular, the median is given by,

$$\phi\left(\frac{1}{2}\right) = \lceil y_{\frac{1}{2}} \rceil = \left\lceil \left[\frac{-1.099}{\lambda\ ln(q)}\right]^{\frac{1}{\beta}} - 1 \right\rceil.$$

5.3. Probability generating function of DTIHLW(q, λ, β)

The pgf of DTIHLW(q, λ, β) distribution is

$$P_Y(s) = 1 + 2(s-1) \sum_{y=1}^{\infty} \frac{s^{y-1} q^{\lambda(y+1)^\beta}}{1 + q^{\lambda(y+1)^\beta}}. \tag{27}$$

Then the mean is

$$E(Y) = \sum_{y=1}^{\infty} \frac{2q^{\lambda(y+1)^\beta}}{1 + q^{\lambda(y+1)^\beta}}, \tag{28}$$

and the variance is

$$V(Y) = \sum_{y=1}^{\infty} \frac{2(2y-1) q^{\lambda(y+1)^\beta}}{1 + q^{\lambda(y+1)^\beta}} - \left[\sum_{y=1}^{\infty} \frac{2q^{\lambda(y+1)^\beta}}{1 + q^{\lambda(y+1)^\beta}} \right]^2. \tag{29}$$

5.4. Moments

The r^{th} moment about origin of DTIHLW(q, λ, β) is given by

$$\mu_r\prime = E(Y^r) = 2 \sum_{y=0}^{\infty} \frac{y^r (q^{\lambda y^\beta} - q^{\lambda(y+1)^\beta})}{(1 + q^{\lambda y^\beta})(1 + q^{\lambda(y+1)^\beta})}. \tag{30}$$

For given values of the parameters, (30) can be numerically computed using R-programming. Table 1 shows the raw and central moments, skewness, and kurtosis for the given values of q, λ and β.

5.5. Order statistics

Let $Y_1, Y_2, ..., Y_n$ be n random samples taken from DTIHLW(q, λ, β) and let $Y_{(1)}$,$Y_{(2)}$,..., $Y_{(n)}$ denote the corresponding order statistics. Then the cdf for the k^{th} order statistic, say $Z = Y_{(k)}$, is given by

$$F_Z(z) \quad = \quad \sum_{j=k}^{n} \binom{n}{j} F^j(z)[1 - F(z)]^{n-j}. \tag{31}$$

Using the binomial expansion for $[1 - F(z)]^{n-j}$, we get

$$\begin{aligned} F_Z(z) &= \sum_{j=k}^{n} \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} (-1)^i [F(z)]^{i+j} \\ &= \sum_{j=k}^{n} \sum_{i=0}^{n-j} \binom{n}{j} \binom{n-j}{i} (-1)^i \left[\frac{1 - q^{\lambda(z+1)^\beta}}{1 + q^{\lambda(z+1)^\beta}} \right]^{i+j}. \end{aligned} \tag{32}$$

Table 1: Moments, skewness and kurtosis for $q = 0.5, \lambda = 1.0$ and various choices of β.

Parameter	Raw moments	Central moments	Skewness	Kurtosis
$\beta = 0.5$	$\mu_1' = 6.42$ $\mu_2' = 190.36$ $\mu_3' = 12507.7$ $\mu_4' = 1482469$	$\mu_2 = 149.2$ $\mu_3 = 9370.9$ $\mu_4 = 1203285$	5.14	54.09
$\beta = 1.0$	$\mu_1' = 1.53$ $\mu_2' = 5.15$ $\mu_3' = 23.98$ $\mu_4' = 143.84$	$\mu_2 = 2.81$ $\mu_3 = 7.50$ $\mu_4 = 53.02$	1.59	6.69
$\beta = 1.5$	$\mu_1' = 0.98$ $\mu_2' = 1.74$ $\mu_3' = 3.75$ $\mu_4' = 9.56$	$\mu_2 = 0.78$ $\mu_3 = 0.53$ $\mu_4 = 2.11$	0.76	3.44
$\beta = 2$	$\mu_1' = 0.79$ $\mu_2' = 1.04$ $\mu_3' = 1.57$ $\mu_4' = 2.69$	$\mu_2 = 0.42$ $\mu_3 = 0.09$ $\mu_4 = 0.47$	0.32	2.69
$\beta = 2.5$	$\mu_1' = 0.71$ $\mu_2' = 0.78$ $\mu_3' = 0.94$ $\mu_4' = 1.25$	$\mu_2 = 0.29$ $\mu_3 = -0.02$ $\mu_4 = 0.19$	-0.107	2.42
$\beta = 3$	$\mu_1' = 0.67$ $\mu_2' = 0.69$ $\mu_3' = 0.72$ $\mu_4' = 0.78$	$\mu_2 = 0.24$ $\mu_3 = -0.06$ $\mu_4 = 0.10$	-0.54	1.82

The pmf of the k^{th} order statistics is obtained as

$$\begin{aligned} f_Z(z) &= F_Z(z) - F_Z(z-1) \\ &= \sum_{j=k}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i \\ &\quad \left(\left[\frac{1-q^{\lambda(z+1)^\beta}}{1+q^{\lambda(z+1)^\beta}}\right]^{i+j} - \left[\frac{1-q^{\lambda z^\beta}}{1+q^{\lambda z^\beta}}\right]^{i+j}\right) \\ &= \sum_{j=k}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\binom{n-j}{i}(-1)^i \\ &\quad \frac{\left[(1-q^{\lambda(z+1)^\beta})(1+q^{\lambda z^\beta})\right]^{i+j} - \left[(1+q^{\lambda(z+1)^\beta})(1-q^{\lambda z^\beta})\right]^{i+j}}{\left[(1+q^{\lambda(z+1)^\beta})(1+q^{\lambda z^\beta})\right]^{i+j}}. \end{aligned} \tag{33}$$

5.6. Infinite divisibility

From Steutel and van Harn (2004), we have the following result.

Lemma 1: If $p_y = P(Y = y), y \in Z_+$, is infinitely divisible, then we have $p_y \leq e^{-1} = 0.3679$, for all $y \in N$.

From the above Lemma, using (13), we have arrived the condition that DTIHLW distribution is infinitely divisible for a given q, λ and β if it satisfies

$$q^{\lambda y^\beta} \leq q^{\lambda(y+1)^\beta} + \frac{1}{2e}(1 + q^{\lambda y^\beta})(1 + q^{\lambda(y+1)^\beta}).$$

But we can show that $p_y > 0.3679$ for some values of $y \in N, \lambda, \beta$ and q. We take $\lambda = 1.5, \beta = 2$ and $q = 0.5$, then we have $p_1 = 0.4920 > 0.3679$. This shows that the DTIHLW distribution is not infinitely divisible.

5.7. Stress-strength parameter

The stress-strength parameter R is a measure of component reliability. Let the random variable Y be the strength of a component which is subjected to a random stress Z. The estimation of R when Y and Z are independently and identically distributed (iid) has been discussed in the literature by many authors. For a detailed study, one can see Kotz *et al.* (2003). In discrete case, the stress-strength model is defined as,

$$R = P(Y > Z) = \sum_{y=0}^{\infty} p_Y(y)\ F_Z(y), \tag{34}$$

where, $p_Y(y)$ and $F_Z(y)$ are the pmf and cdf of the independent discrete random variables Y and Z, respectively. The stress-strength models are useful in various fields such as medicine, engineering, and psychology. Let $Y \sim DTIHLW(\theta_1)$ and $Z \sim DTIHLW(\theta_2)$, where $\theta_1 = (q_1, \lambda_1, \beta_1)^T$ and $\theta_2 = (q_2, \lambda_2, \beta_2)^T$. Then, using (13) and (15), we have,

$$R = \sum_{y=0}^{\infty} \frac{2\Big[q_1^{\lambda_1 y^{\beta_1}} - q_1^{\lambda_1 (y+1) y^{\beta_1}}\Big]\Big[1 - q_2^{\lambda_2 (y+1)^{\beta_2}}\Big]}{\Big[1 + q_1^{\lambda_1 y^{\beta_1}}\Big]\Big[1 + q_1^{\lambda_1 (y+1)^{\beta_1}}\Big]\Big[1 + q_2^{\lambda_2 (y+1)^{\beta_2}}\Big]}. \tag{35}$$

The stress strength reliability parameter for different parameter values are numerically computed and presented in Table 2. We see that the value of stress-strength parameter is decreasing when β_1 increases and increasing when β_2 increases.

5.8. Likelihood function of DTIHLW distribution

Consider a random sample $(y_1, y_2, ..., y_n)$ of size n, from the DTIHLW(q, λ, β). Then, the likelihood function is given by,

$$L = \frac{2^n \prod_{i=1}^n (q^{\lambda y_i^\beta} - q^{\lambda(y_i+1)^\beta})}{\prod_{i=1}^n (1 + q^{\lambda y_i^\beta}) \prod_{i=1}^n (1 + q^{\lambda(y_i+1)^\beta})}. \tag{36}$$

The log-likelihood function is,

$$\begin{aligned} ln(L) &= n\ ln(2) + \sum_{i=1}^n ln(q^{\lambda y_i^\beta} - q^{\lambda(y_i+1)^\beta}) \\ &\quad - \sum_{i=1}^n ln(1 + q^{\lambda y_i^\beta}) - \sum_{i=1}^n ln(1 + q^{\lambda(y_i+1)^\beta}). \end{aligned} \tag{37}$$

Table 2: Value of stress-strength parameter (R) for various choices of parameters.

$q_1 = 0.5, q_2 = 0.5$				
$\lambda_1 = 0.5, \lambda_2 = 0.5$				
$\beta_1 \rightarrow$ $\beta_2 \downarrow$	0.5	1.0	1.5	2.0
0.5	0.5236	0.3268	0.2742	0.2528
1.0	0.7374	0.5576	0.4348	0.3744
1.5	0.7989	0.7091	0.6062	0.5253
2.0	0.8257	0.7799	0.7213	0.6592
$\lambda_1 = 0.5, \lambda_2 = 1.5$				
$\beta_1 \rightarrow$ $\beta_2 \downarrow$	0.5	1.0	1.5	2.0
0.5	0.8333	0.7337	0.6743	0.6424
1.0	0.8889	0.8569	0.8193	0.7860
1.5	0.9029	0.8928	0.8796	0.8645
2.0	0.9082	0.9054	0.9016	0.8971
$\lambda_1 = 1, \lambda_2 = 0.5$				
$\beta_1 \rightarrow$ $\beta_2 \downarrow$	0.5	1.0	1.5	2.0
0.5	0.3477	0.2549	0.2325	0.2236
1.0	0.5363	0.3837	0.3224	0.2969
1.5	0.6242	0.5129	0.4342	0.3916
2.0	0.6679	0.6013	0.5392	0.4946

The likelihood equations are the following

$$\begin{aligned}\frac{\partial ln(L)}{\partial q} &= \sum_{i=1}^{n} \frac{y_i^{\beta} q^{\lambda y_i^{\beta}-1} - (y_i+1)^{\beta} q^{\lambda (y_i+1)^{\beta}-1}}{q^{\lambda y_i^{\beta}} - q^{\lambda (y_i+1)^{\beta}}} \\ &\quad - \sum_{i=1}^{n} \frac{y_i^{\beta} q^{\lambda y_i^{\beta}-1}}{1+q^{\lambda y_i^{\beta}}} - \sum_{i=1}^{n} \frac{(y_i+1)^{\beta} q^{\lambda (y_i+1)^{\beta}-1}}{1+q^{\lambda (y_1+1)^{\beta}}} = 0, \end{aligned} \tag{38}$$

$$\begin{aligned}\frac{\partial ln(L)}{\partial \lambda} &= \sum_{i=1}^{n} \frac{y_i^{\beta} q^{\lambda y_i^{\beta}} - (y_i+1)^{\beta} q^{\lambda (y_i+1)^{\beta}}}{q^{\lambda y_i^{\beta}} - q^{\lambda (y_i+1)^{\beta}}} \\ &\quad - \sum_{i=1}^{n} \frac{y_i^{\beta} q^{\lambda y_i^{\beta}}}{1+q^{\lambda y_i^{\beta}}} - \sum_{i=1}^{n} \frac{(y_i+1)^{\beta} q^{\lambda (y_i+1)^{\beta}}}{1+q^{\lambda (y_i+1)^{\beta}}} = 0, \end{aligned} \tag{39}$$

and

$$\begin{aligned}\frac{\partial ln(L)}{\partial \beta} &= \sum_{i=1}^{n} \frac{ln(y_i) y_i^{\beta} q^{\lambda y_i^{\beta}} - ln(y_i+1)(y_i+1)^{\beta} q^{\lambda (y_i+1)^{\beta}}}{q^{\lambda y_i^{\beta}} - q^{\lambda (y_i+1)^{\beta}}} \\ &\quad - \sum_{i=1}^{n} \frac{ln(y_i) y_i^{\beta} q^{\lambda y_i^{\beta}}}{1+q^{\lambda y_i^{\beta}}} - \sum_{i=1}^{n} \frac{ln(y_i+1)(y_i+1)^{\beta} q^{\lambda (y_i+1)^{\beta}}}{1+q^{\lambda (y_i+1)^{\beta}}} = 0. \end{aligned} \tag{40}$$

These equations do not have explicit solutions and their solutions must be obtained numerically by using statistical software like *nlm* or *optim* package in R programming.

We compute the maximized unrestricted and restricted log-likelihood ratio (LR) test statistic for testing on some DTIHLW submodels. The LR test statistic can be used to check whether DTIHLW distribution for a given data set is statistically superior to the submodels. For example, $H_0 : \beta = 1$ versus $H_1 : \beta \neq 1$ is equivalent to compare the DTIHLW distribution and DTIHLE distribution. Here the LR test statistic reduces to $\omega = 2[l(\hat{q}, \hat{\lambda}, \hat{\beta}) - l(\hat{q}', \hat{\lambda}', 1)]$, where $(\hat{q}, \hat{\lambda}, \hat{\beta})$ and $(\hat{q}', \hat{\lambda}')$ are the MLEs under H_1 and H_0, respectively. The test statistic ω is asymptotically (as $n \to \infty$) distributed as $\chi^2_{(k)}$, where k is the length of the parameter vector of interest. The LR test rejects H_0 if $\omega > \chi^2_{(k,\alpha)}$ where $\chi^2_{(k,\alpha)}$ denotes the upper $(1-\alpha)100\%$ quantile of the $\chi^2_{(k)}$ distribution.

5.9. Simulation study

This section demonstrates the performance of the MLEs of the model parameters of DTIHLW distribution using Monte Carlo simulation for various sample sizes and for selected parameter values. The algorithm for the simulation study are as follows:
Step 1. Input the value of replication (N);
Step 2. Specify the sample size n and the values of the parameters q, λ and β;
Step 3. Generate u_i from $U(0,1),\ i = 1, 2, ..., n$;
Step 4. Obtain the random observations from the DTIHLW distribution using (24);
Step 5. Compute the MLEs of the three parameters;
Step 6. Repeat steps 3 to 5, N times;
Step 7. Compute the parameter estimate, standard error of estimate, average bias, mean square error (MSE) and coverage probability (CP) for each parameter.
Here the expected value of the estimator is
$E(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}\hat{\theta}_i$, $E(SE(\hat{\theta})) = \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(-\frac{\partial^2 \log(L)}{\partial\theta_i^2}\right)}$,
Average Bias $= \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)$, $MSE(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2$ and
CP = Probability of $\theta_i \in \left(\hat{\theta}_i \pm 1.96\sqrt{-\frac{\partial^2 \log(L)}{\partial\theta_i^2}}\right)$.
We take random samples of size n=50, 100, 200 and 500 respectively. The MLEs of the parameter vector $\theta = (q, \lambda, \beta)^T$ are determined by maximizing the log-likelihood function given in (37) by using the *optim* package in **R** programming based on each generated samples. This simulation is repeated 1000 times and the average estimate and its standard error, average bias, MSE and CP are computed and presented in Table 3. From Table 3, it can be seen that, as sample size increases the estimates of bias and MSE are decreases. Also note that the CP values are quite closer to the 95% nominal level.

6. Applications

In order to check the use of DTIHLW distribution for real life data modeling, we consider three data sets. The first data set is continuous measurement of flood peaks (in m^3/s) of the Wheaton River near Carcross in Yukon Territory, Canada for the years 1958-1984. This data was analyzed by Choulakian and Stephens (2001). The data are as follows: 1.7 2.2 14.4 1.1 0.4 20.6 5.3 0.7 13.0 12.0 9.3 1.4 18.7 8.5 25.5 11.6 14.1 22.1 1.1 2.5 14.4 1.7 37.6 0.6 2.2 39.0 0.3 15.0 11.0 7.3 22.9 1.7 0.1 1.1 0.6 9.0 1.7 7.0 20.1 0.4 14.1 9.9 10.4 10.7

Table 3: The MLE, standard error, average bias, MSE and CP for given parameters.

Parameter(θ)	Samples(n)	$E(\hat{\theta})(E(SE(\hat{\theta})))$	Average bias	MSE	CP
$q = 0.5$	50	0.368(0.223)	-0.121	0.115	87.5
	100	0.412(0.192)	-0.108	0.101	88.1
	200	0.432(0.115)	-0.097	0.093	90.2
	500	0.486(0.102)	-0.066	0.088	92.8
$\lambda = 1.5$	50	1.733(0.196)	0.228	0.114	91.6
	100	1.692(0.188)	0.119	0.091	91.9
	200	1.544(0.094)	0.105	0.077	92.7
	500	1.539(0.023)	0.099	0.061	94.1
$\beta = 0.9$	50	0.633(0.251)	-0.129	0.025	92.8
	100	0.701(0.190)	-0.116	0.021	93.5
	200	0.876(0.022)	-0.064	0.009	94.2
	500	0.899(0.013)	-0.027	0.003	94.9
$q = 0.8$	50	0.759(0.142)	-0.183	0.226	91.2
	100	0.773(0.136)	-0.086	0.149	93.4
	200	0.792(0.084)	-0.043	0.063	93.8
	500	0.803(0.048)	0.008	0.041	94.7
$\lambda = 1.0$	50	0.836(0.362)	-0.161	0.447	90.7
	100	0.881(0.143)	-0.133	0.219	91.3
	200	0.934(0.081)	-0.096	0.124	93.3
	500	0.962(0.016)	-0.077	0.019	93.9
$\beta = 1.5$	50	1.543(0.116)	0.039	0.118	94.1
	100	1.514(0.108)	0.026	0.103	94.4
	200	1.503(0.099)	0.021	0.081	95.2
	500	1.501(0.086)	0.018	0.011	95.8

30.0 3.6 5.6 30.8 13.3 4.2 25.5 3.4 11.9 21.5 27.6 36.4 2.7 64.0 1.5 2.5 27.4 1.0 27.1 20.2 16.8 5.3 9.7 27.5 2.5 27.0 1.9 2.8. Since the data set is continuous, here first we discretize the data by considering the floor value (y) and fitted the new distribution for the y values.

The second data set is the daily ozone level measurements (in ppm x 1000) taken from Nadarajah (2008) and are as follows: 7 115 79 31 9 8 45 61 23 28 19 23 35 59 21 23 32 48 22 44 28 4 7 65 24 13 18 11 27 44 21 73 12 1 10 110 23 28 36 30 85 89 20 80 41 6 97 122 32 135 34 21 82 73 16 14 23 52 168 24 18 39 20 45 13 14 71 108 9 18 11 29 16 21 46 16 37 63 44 13 12 59 84 7 20 64 118 36 37 50 76 23 13 39 85 14 49 9 96 30 32 16 78 14 64 78 91 18 40 35 47 20 77 66 97 11.

The third data set is from Eliwa *et al.* (2021) which represents the daily new deaths due to COVID-19 in China from 23 January to 28 March, 2019. The data are: 8 16 15 24 26 26 38 43 46 45 57 64 65 73 73 86 89 97 108 97 146 121 143 142 105 98 136 114 118 109 97 150 71 52 29 44 47 35 42 31 38 31 30 28 27 22 17 22 11 7 13 10 14 13 11 8 3 7 6 9 7 4 6 5 3 5.

We fit DTIHLW(q, λ, β) distribution for the three data sets. The fit of the data sets are compared with six competitive models, respectively, type I half-logistic exponential (DTIHLE) distribution, a sub model of the proposed distribution, discrete Weibull geometric

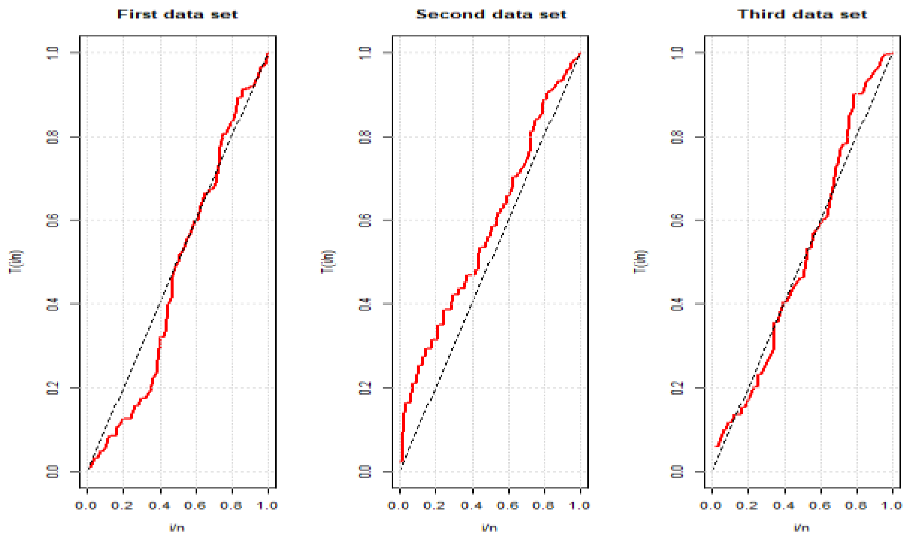

Figure 4: The TTT plots of the three data sets.

(DWG) distribution of Jayakumar and Babu (2018), exponentiated discrete Weibull (EDW) distribution of Nekoukhou and Bidram (2015), discrete modified Weibull (DMW) distribution of Nooghabi *et al.* (2011), discrete logistic (DLOG) distribution of Chakraborty and Chakravarty (2016), discrete Weibull (DW) distribution of Nakagawa and Osaki (1975).

Descriptive statistics of the three data sets are shown in Table 4. The Total Time on Test (TTT) plot of the three data sets are shown in Figure 4.

Table 4: Descriptive statistics for the three data sets

Data	Samples(n)	Mean	SD	Min.	Max.	Skewness	Kurtosis
First set	72	12.204	12.297	0.1	64.0	1.304	3.189
Second set	116	42.129	32.988	1	168	1.242	1.290
Third set	66	49.742	43.873	3	150	0.837	2.450

The values of the log-likelihood function ($-\log L$), the statistics Kolmogorov-Smirnov (K-S), Akaike Information Criterion (AIC), Akaike Information Criterion with correction(CAIC) and Bayesian Information Criterion(BIC) are calculated for the seven distributions in order to verify which distribution fits better to these data. The better distribution corresponds to smaller $K-S$, $-\log L$, AIC, $CAIC$, BIC values and high p value. Here, AIC$=-2\log L+2k$, CAIC$=-2\log L + (\frac{2kn}{n-k-1})$ and BIC$=-2\log L + k\log n$ where, L is the likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters and n is the sample size.

The values in Table 5 shows that the $DTIHLW$ distribution leads to a better fit compared to the other six models. Figure 5, shows the fitted pdf and cdf with the empirical distribution of the first data set. The LR test statistic to test the hypothesis $H_0 : \beta = 1$ versus $H_0 : \beta \neq 1$ for the first data set is $\omega = 7.264 > 3.841$ with p value 0.0070. So we reject the null hypothesis.

Table 5: Parameter estimates and goodness of fit for the first data set

Model	ML estimates	-log L	AIC	CAIC	BIC	K-S	p-value
DTIHLW	$\hat{q} = 0.758$ $\hat{\lambda} = 0.794$ $\hat{\beta} = 0.765$	251.99	509.97	510.32	516.80	0.109	0.351
EDW	$\hat{q} = 0.866$ $\hat{\lambda} = 0.831$ $\hat{\beta} = 1.089$	252.24	510.48	510.83	517.31	0.125	0.208
DWG	$\hat{q} = 0.123$ $\hat{\lambda} = 0.900$ $\hat{\beta} = 0.912$	252.25	510.49	510.85	517.33	0.136	0.140
DMW	$\hat{q} = 0.917$ $\hat{\lambda} = 0.870$ $\hat{\beta} = 1.016$	253.53	513.06	513.41	519.89	0.137	0.133
DTIHLE	$\hat{q} = 0.121$ $\hat{\lambda} = 0.051$	255.62	515.24	515.42	519.79	0.187	0.013
DW	$\hat{q} = 0.779$ $\hat{\beta} = 0.630$	257.52	519.04	519.21	523.59	0.159	0.051
DLOG	$\hat{q} = 0.664$ $\hat{\lambda} = 9.382$	280.19	564.37	564.55	568.93	0.279	$2.7\text{x}10^{-5}$

Table 6: Parameter estimates and goodness of fit for the second data set

Model	ML estimates	-log L	AIC	CAIC	BIC	K-S	p-value
DTIHLW	$\hat{q} = 0.954$ $\hat{\lambda} = 0.428$ $\hat{\beta} = 1.133$	545.24	1096.49	1096.70	1104.75	0.079	0.464
EDW	$\hat{q} = 0.939$ $\hat{\lambda} = 0.878$ $\hat{\beta} = 1.666$	548.47	1102.93	1103.15	1111.19	0.148	0.013
DWG	$\hat{q} = 0.162$ $\hat{\lambda} = 0.966$ $\hat{\beta} = 0.879$	560.95	1127.91	1128.12	1136.17	0.207	$9.3\text{x}10^{-5}$
DMW	$\hat{q} = 0.978$ $\hat{\lambda} = 0.828$ $\hat{\beta} = 1.010$	551.36	1108.71	1108.93	1116.98	0.104	0.159
DTIHLE	$\hat{q} = 0.101$ $\hat{\lambda} = 0.014$	548.59	1101.18	1101.28	1106.68	0.105	0.153
DW	$\hat{q} = 0.989$ $\hat{\beta} = 1.158$	547.65	1099.31	1099.41	1104.81	0.100	0.193
DLOG	$\hat{q} = 0.946$ $\hat{\lambda} = 38.115$	567.74	1139.48	1139.59	1144.99	0.134	0.032

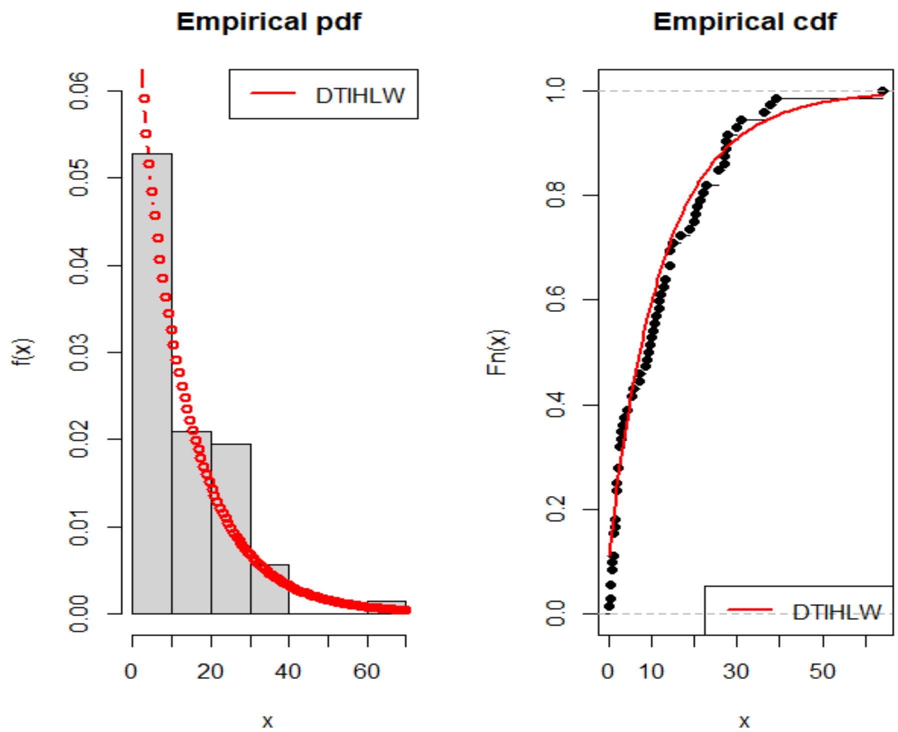

Figure 5: Fitted pdf and cdf plots for the first data set

The values in Table 6 indicates that the DTIHLW distribution leads to a better fit compared to the other six models. Figure 6, shows the fitted pdf and cdf with the empirical distribution of the second data set. The LR test statistic to test the hypothesis $H_0 : \beta = 1$ versus $H_0 : \beta \neq 1$ for the second data set is $\omega = 6.7 > 3.841$ with p value 0.0096. So we reject the null hypothesis. The values in Table 7 indicates that the DTIHLW distribution leads to a better fit compared to the other six models. Figure 7, shows the fitted pdf and cdf with the empirical distribution of the second data set. The LR test statistic to test the hypothesis $H_0 : \beta = 1$ versus $H_0 : \beta \neq 1$ for the third data set is $\omega = 42.3 > 3.841$ with p value $8.02x10^{-11}$. So we reject the null hypothesis.

7. Conclusion and future works

The discrete version of the Type I half logistic distributions was introduced. Several members of this family such as discrete type I half-logistic uniform, discrete type I half-logistic Lomax, discrete type I half-logistic exponential, discrete type I half-logistic Fréchet and discrete type I half-logistic Weibull distributions were specified. Some properties of the discrete type I half-logistic Weibull distribution were studied. The three parameters of the new distribution were estimated using maximum likelihood method and a simulation study was conducted to check the performance of the method. Three real data applications shows that this model is suitable for modeling discrete data. Since the likelihood equations of the present distribution are highly non-linear equations and it is difficult to study the existence and uniqueness of the MLE's of parameters, so we propose further studies in this direction as future work.

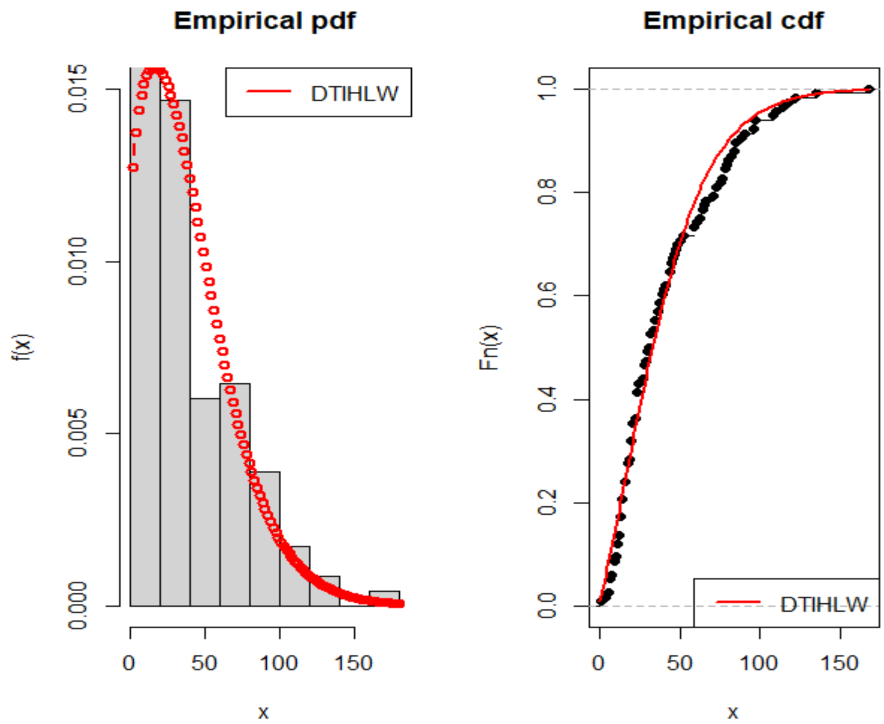

Figure 6: Fitted pdf and cdf plots for the second data set.

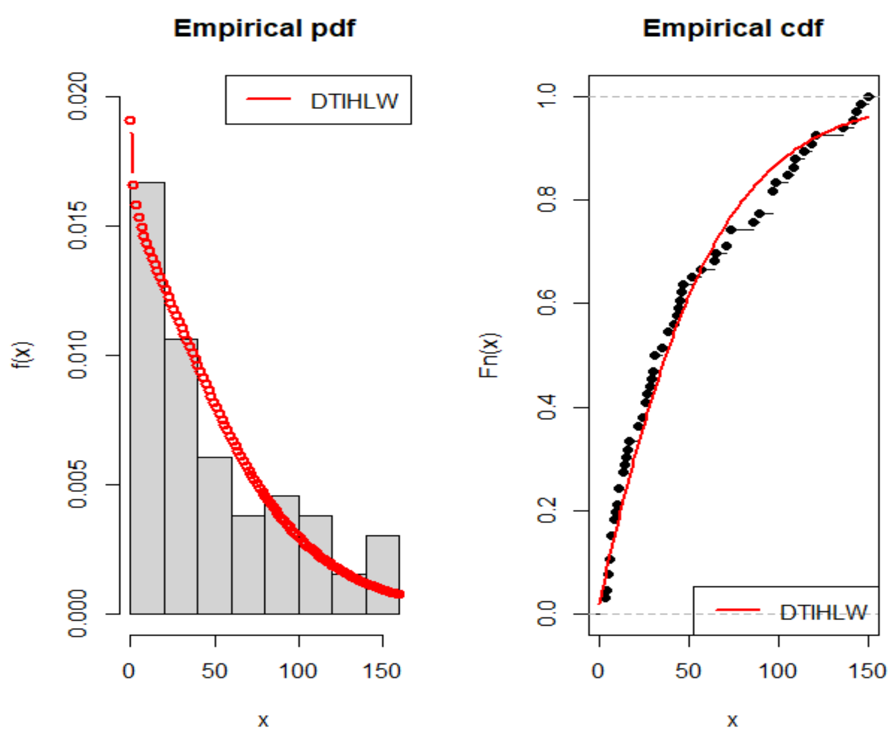

Figure 7: Fitted pdf and cdf plots for the third data set.

Table 7: Parameter estimates and goodness of fit for the third data set.

Model	ML estimates	-log L	AIC	CAIC	BIC	K-S	p-value
DTIHLW	$\hat{q} = 0.798$ $\hat{\lambda} = 0.169$ $\hat{\beta} = 0.922$	324.82	655.64	656.02	662.21	0.091	0.644
EDW	$\hat{q} = 0.899$ $\hat{\lambda} = 0.702$ $\hat{\beta} = 1.716$	325.96	657.92	658.31	664.49	0.114	0.357
DWG	$\hat{q} = 0.645$ $\hat{\lambda} = 0.412$ $\hat{\beta} = 0.739$	1137.19	2280.37	2280.76	2286.94	0.568	2.2x10^{-16}
DMW	$\hat{q} = 0.877$ $\hat{\lambda} = 0.437$ $\hat{\beta} = 1.001$	352.59	711.18	711.57	717.74	0.256	3.5x10^{-4}
DTIHLE	$\hat{q} = 0.305$ $\hat{\lambda} = 0.042$	345.95	695.89	696.08	700.27	0.255	3.6x10^{-4}
DW	$\hat{q} = 0.784$ $\hat{\beta} = 0.531$	357.11	718.22	718.41	722.59	0.391	$3.5x10^{-9}$
DLOG	$\hat{q} = 0.971$ $\hat{\lambda} = 4.885$	366.55	737.10	737.29	741.41	0.494	$2.2x10^{-14}$

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Conflict of interest

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