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Balanced and Partially Balanced Semi-Latin Rectangles with Block Size Two

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Abstract

Balanced Semi-Latin rectangles are the subclass of Semi-Latin rectangles which are generalizations of Latin squares and Semi-Latin squares. Such types of designs are more useful in plant disease experiments where we can consider plants as columns, half-leaves as experimental units, and the height of leaves as rows. In this paper, we study partially balanced semi-Latin rectangles which have all properties of balanced semi-Latin rectangles except that each pair of distinct treatments does not appear a constant number of times in the design. We also propose four construction methods of balanced and partially balanced semi-Latin rectangles. An R package *slr* has been developed to implement the proposed methods of constructions.

Key words: Semi-Latin Rectangles; Balanced Semi-Latin rectangles; Partially Balanced Semi-Latin rectangles; Canonical efficiency factor; Average efficiency factor.

1. Introduction

In agricultural experiments, plant leaves often serve as experimental units (plots), especially in plant disease studies, as detailed by Price (1946). Variability among plants and the impact of leaf height are significant sources of heterogeneity. To address this, row-column designs like Latin squares are recommended, where columns represent plants and rows correspond to leaf height. However, the number of available plants often exceeds the usable leaves per plant. In a notable experiment on tobacco mosaic virus, Youden (1937) introduced row-column designs with fewer rows than columns, now known as Youden squares.

Often row-column designs where more than one treatment can be allocated in each row-column intersection (henceforth called as block or cell) are useful. To this effect, Bailey and Monod (2001) introduced semi-Latin rectangles (SLR) which are a class of row-column

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designs where each block has the same size ($k > 1$) and each treatment appears more than or equal to once in each row and/or in each column. Uto and Bailey (2020) proposed balanced semi-Latin rectangles (BSLRs), a subclass that ensures balance among treatment pairs, and provided two algorithms for constructing such designs with block size two. Additionally, Uto and Bailey (2022) introduced regular-graph semi-Latin rectangles, which have the added property that treatment concurrences between any two distinct treatments differ by at most one and and proposed two different algorithms for constructing such designs.

Limited resource situations often make it challenging to implement a full BSLR design due to its large structure and resource requirements. In such cases, partially balanced semi-Latin rectangles (PBSLRs) offer greater practicality as they retain all the properties of BSLRs except for the condition that each pair of distinct treatments appears a constant number of times in the design. This distinction sets PBSLRs apart from both BSLRs and general SLRs, making them more suitable for constrained resource scenarios. In a PBSLR design, n_i number of pairs of a treatment appear in λ_i blocks, $i = 1, 2, ..., m$ (number of association classes). Also, each treatment appears in $hn_r(= pn_c)$ blocks (here, n_r and n_c denote the number of times a treatment appears in each row and in each column, respectively) and each block contains k distinct treatments. Hence, the sum of concurrences of a particular treatment with other treatments is $\sum_i^m n_i\lambda_i$, so

$$\begin{aligned} \sum_i^m n_i\lambda_i &= hn_r(k-1) = pn_c(k-1) \\ \sum_i^m n_i &= v - 1 \end{aligned} \tag{1}$$

A little work related to Semi-Latin rectangles with block size two for balanced situation are available in literature. Also, designs available in literature are obtainable by algorithmic methods as proposed by Uto and Bailey (2020). In this paper, we introduce the concept of partially balanced Semi-Latin rectangle designs, where the number of replications is drastically reduced compared to balanced designs and regular-graph SLRs. To avoid the complexity of algorithms, four different construction methods are developed for both balanced and partially balanced situations. These methods of construction are simple and easy to use for generation of these classes of semi-Latin rectangles. Further, an R package named *slr* (Yadav *et al.*, 2023) is developed in which the proposed methods are implemented and users can generate balanced and partially balanced semi-Latin rectangles by providing only the number of treatments.

2. Experimental set up

In plant disease experiments, generally, the leaves of plants are used as experimental units. The researcher is interested in comparing a set of treatments which may be different virus preparations or different doses of a virus, *etc.* A solution containing the treatment is rubbed onto the leaves, which develop infections, and some measurements of the disease such as the number of lesions on the leaves are taken. It has been found that there are two sources of heterogeneity in these types of experiments namely, differences between plants and influence of leaf height. So, row-column designs are applicable in this type of situation.

Table 1: Semi-Latin rectangle for 10 treatments in plant disease experiment

LH 5	1, 10	2, 1	3, 2	4, 3	5, 4	6, 5	7, 6	8, 7	9, 8	10, 9
LH 4	2, 9	3, 10	4, 1	5, 2	6, 3	7, 4	8, 5	9, 6	10, 7	1, 8
LH 3	3, 8	4, 9	5, 10	6, 1	7, 2	8, 3	9, 4	10, 5	1, 6	2, 7
LH 2	4, 7	5, 8	6, 9	7, 10	8, 1	9, 2	10, 3	1, 4	2, 5	3, 6
LH 1	5, 6	6, 7	7, 8	8, 9	9, 10	10, 1	1, 2	2, 3	3, 4	4, 5
	Plant1	**Plant2**	**Plant3**	**Plant4**	**Plant5**	**Plant6**	**Plant7**	**Plant8**	**Plant9**	**Plant10**

Here, rows refer to leaf height and columns to plants. Since, the number of plants available for the experiment is typically more than the number of usable leaves and their positions per plant, so a row-column design which has smaller number of rows than columns is useful in such situations. The researcher is interested to compare more than two treatments in p plants each with leaves at h heights, where typically $h < p$. Generally, the two half leaves of each of the hp leaves form the plots. Considering the leaf heights as rows and the plants as columns, we have two plots in the intersection of each row and each column. Clearly, the Semi-Latin rectangles are useful in such situations. In these designs, rows and columns can be randomized and treatments within blocks can also be randomly allocated.

For a plant disease experiment involving ten plants ($p = 10$) and pairs of half-leaves ($k = 2$) at five leaf heights ($h = 5$), with ten treatments ($v = 10$) can be designed as $(5 \times 10)/2$ SLR given in Table 1.

3. Preliminaries

In the context of assuming the additive effect, the model for the present study is expressed as:

$$\mathbf{y} = \mu\mathbf{1} + \boldsymbol{\Delta}'\boldsymbol{\tau} + \boldsymbol{\Phi}'\boldsymbol{\rho} + \boldsymbol{\Psi}'\boldsymbol{\gamma} + \mathbf{D}'\boldsymbol{\beta} + \boldsymbol{\epsilon} \tag{2}$$

where, $\mathbf{y}$ denotes $hpk \times 1$ vector of observations, $\mathbf{1}$ denotes $hpk \times 1$ vector of ones, $\boldsymbol{\Delta}'$ denotes $hpk \times v$ incidence matrix of observations versus treatments, $\boldsymbol{\tau}$ denotes $v \times 1$ vector of treatment effects, $\boldsymbol{\Phi}'$ denotes $hpk \times h$ incidence matrix of observations versus rows, $\boldsymbol{\rho}$ denotes $h \times 1$ vector of row effects, $\boldsymbol{\Psi}'$ denotes $hpk \times p$ incidence matrix of observations versus columns, $\boldsymbol{\gamma}$ denotes $p \times 1$ vector of column effects, $\mathbf{D}'$ denotes $hpk \times hp$ incidence matrix of observations versus blocks/cells, $\boldsymbol{\beta}$ denotes $hp \times 1$ vector of block effects, $\boldsymbol{\epsilon}$ denotes $hpk \times 1$ vector of random errors with Expectation$(\boldsymbol{\epsilon}) = \mathbf{0}$ and Dispersion$(\boldsymbol{\epsilon}) = \sigma^2\mathbf{I}_{hpk\times hpk}$ with $(.)'$ denoting transpose of a matrix and $\mathbf{I}_{t\times t}$ representing a $t \times t$ Identity matrix. The incidence matrices $\boldsymbol{\Delta}', \boldsymbol{\Phi}', \boldsymbol{\Psi}'$ and $\mathbf{D}'$ are defined in the usual manner. For instance, if $\boldsymbol{\Delta}' = (\delta_{ui})$, then

$$\delta_{ui} = \begin{cases} 1, & \text{if the } u^{th} \text{ observation corresponds to the } i^{th} \text{ treatment} \\ 0, & \text{otherwise} \end{cases}$$

Under model (2), the information matrix for estimating the linear functions of treatment effects is given by

$$\begin{aligned}\mathbf{C}_d =& \mathbf{R}_{\boldsymbol{\tau}} - (\mathbf{N}_1\mathbf{A}_{11}\mathbf{N}_1' + \mathbf{N}_2\mathbf{A}_{21}\mathbf{N}_1' + \mathbf{N}_3\mathbf{A}_{31}\mathbf{N}_1' + \mathbf{N}_1\mathbf{A}_{12}\mathbf{N}_2' + \mathbf{N}_2\mathbf{A}_{22}\mathbf{N}_2' + \mathbf{N}_3\mathbf{A}_{32}\mathbf{N}_2' + \\ & \mathbf{N}_1\mathbf{A}_{13}\mathbf{N}_3' + \mathbf{N}_2\mathbf{A}_{23}\mathbf{N}_3' + \mathbf{N}_3\mathbf{A}_{33}\mathbf{N}_3')\end{aligned}$$

where

$\mathbf{A}_{11} = \frac{1}{pk}\mathbf{I}_{h\times h} + \frac{1}{p^2}\left[\mathbf{J}_{h\times p}\mathbf{A}_{22}\mathbf{J}_{p\times h} + \mathbf{J}_{h\times hp}\mathbf{A}_{32}\mathbf{J}_{p\times h} + \mathbf{J}_{h\times p}\mathbf{A}_{23}\mathbf{J}_{hp\times h} + \mathbf{J}_{h\times hp}\mathbf{A}_{33}\mathbf{J}_{hp\times h}\right]$,

$\mathbf{A}_{12} = -\frac{1}{p}\,[\mathbf{J}_{h\times p}\mathbf{A}_{22} + \mathbf{J}_{h\times hp}\mathbf{A}_{32}]$,
$\mathbf{A}_{13} = (\frac{1}{p} - \frac{1}{h})\mathbf{J}_{h\times hp}\mathbf{A}_{33}, \mathbf{A}_{22} = \frac{1}{kh}\,[\mathbf{I}_{p\times p} + (pk^2 - p^3k^2)\mathbf{J}_{p\times p}]$,
$\mathbf{A}_{23} = -\frac{1}{h}\mathbf{J}_{p\times hp}\mathbf{A}_{33}, \mathbf{A}_{33} = (k\mathbf{I}_{hp\times hp} - \frac{pk}{h}\mathbf{J}_{hp\times hp})^{-}, \mathbf{R}_{\boldsymbol{\tau}} = hn_r\mathbf{I}_{v\times v}$,
$\mathbf{N}_1$ denotes incidence matrix of treatments versus rows, $\mathbf{N}_2$ denotes incidence matrix of treatments versus columns, $\mathbf{N}_3$ denotes incidence matrix of treatments versus blocks /cells, $\mathbf{J}_{r\times c}$ denotes an $r \times c$ matrix of ones and $(.)^{-}$ denotes generalized inverse of a matrix.

As we know that connected design is necessary to estimate all treatment contrast *i.e.* Rank $(\mathbf{C}_d) = v - 1$. For a connected design with v treatments a matrix $\mathbf{M}_d$ can be defined as

$$\mathbf{M}_d = \mathbf{R}_{\boldsymbol{\tau}}^{-1/2}\mathbf{C}_d\mathbf{R}_{\boldsymbol{\tau}}^{-1/2}$$

Then the positive eigenvalues of the matrix $\mathbf{M}_d$ say $\mu_i, 1 \leq i \leq v-1$ are known as the canonical efficiency factors of the design. Consider a equireplicate design with replication r, then, the canonical efficiency factors are $1/r$ times the positive eigenvalues of $\mathbf{C}_d$.

4. Methods of construction

In the present investigations, construction methods for generation of BSLR and PBSLR designs for block size two are developed. Four different construction methods are developed out of which two are for BSLR and another two are for PBSLR designs. The detailed construction methods are described in sequel.

Method 1: Construction of BSLR (v is odd)

The present approach for construction of BSLR design is based on the concept of generation of a design from an initial matrix. Once the initial matrix has been developed then the whole design can be generated from the initial matrix using specified procedure. Step 1 provides the method to get the initial matrix and Step 2 provides the procedure of generating the entire design from the initial matrix.

Step-1: Find a matrix of order $\frac{v-1}{2} \times 2$ such that elements of the first column of matrix is 1, where $v = 2n+3$, n is a non-negative integer. For obtaining the second column of the matrix, either arrange the elements from 2 to $\frac{v+1}{2}$ in increasing order starting from up to down or v to $\frac{v+3}{2}$ in decreasing order from down to up. For example, for $n = 2$, $v = 7$ matrix will be $\begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 5 \\ 1 & 6 \\ 1 & 7 \end{pmatrix}$.

Step-2: Take each row of the matrix one by one and add 1 to each of the element with mod v to obtain v blocks of size two. After doing cyclic rotation of these blocks, we get $v \times v$ row-column with intersection of size two from each row of the matrix. And after arranging these $v \times v$ row-column we get a $\left(v \times \frac{v(v-1)}{2}\right)/2$ balanced Semi-Latin rectangles of block size two. General parameters for the obtained design are $v = 2n+3, h = 2n+3, p = (n+1)(2n+3), k = 2, n_r = 2n+2, n_c = 2, \lambda = 2n+3$

For example, for $v = 7, h = 7, p = 21, k = 2, n_r = 6, n_c = 2, \lambda = 7$, a $(7 \times 21)/2$ balanced semi-Latin rectangle can be obtained as shown below.

Row/Col	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 1	1, 3	2, 4	3, 5
2	7, 1	1, 2	2, 3	3, 4	4, 5	5, 6	6, 7	7, 2	1, 3	2, 4
3	6, 7	7, 1	1, 2	2, 3	3, 4	4, 5	5, 6	6, 1	7, 2	1, 3
4	5, 6	6, 7	7, 1	1, 2	2, 3	3, 4	4, 5	5, 7	6, 1	7, 2
5	4, 5	5, 6	6, 7	7, 1	1, 2	2, 3	3, 4	4, 6	5, 7	6, 1
6	3, 4	4, 5	5, 6	6, 7	7, 1	1, 2	2, 3	3, 5	4, 6	5, 7
7	2, 3	3, 4	4, 5	5, 6	6, 7	7, 1	1, 2	2, 4	3, 5	4, 6

Row/Col	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**	**21**
1	4, 6	5, 7	6, 1	7, 2	1, 4	2, 5	3, 6	4, 7	5, 1	6, 2	7, 3
2	3, 5	4, 6	5, 7	6, 1	7, 3	1, 4	2, 5	3, 6	4, 7	5, 1	6, 2
3	2, 4	3, 5	4, 6	5, 7	6, 2	7, 3	1, 4	2, 5	3, 6	4, 7	5, 1
4	1, 3	2, 4	3, 5	4, 6	5, 1	6, 2	7, 3	1, 4	2, 5	3, 6	4, 7
5	7, 2	1, 3	2, 4	3, 5	4, 7	5, 1	6, 2	7, 3	1, 4	2, 5	3, 6
6	6, 1	7, 2	1, 3	2, 4	3, 6	4, 7	5, 1	6, 2	7, 3	1, 4	2, 5
7	5, 7	6, 1	7, 2	1, 3	2, 5	3, 6	4, 7	5, 1	6, 2	7, 3	1, 4

Method 2: Construction of BSLR (v is even)

In this construction method, the process involves two steps as:

Step-1: Take $(v-1)$ pairs of treatments such that the second treatment of each pair is v namely $(1, v), (2, v), (3, v), ..., (v-1, v)$, where, $v = 2n+2, n$ is any positive integer. Now, develop a group of $\frac{v}{2}$ pairs where first pair will be from already generated $(v-1)$ pairs and rest of the $(\frac{v}{2}-1)$ pairs by adding 1 mod $(v-1)$ in the first element of previous pair and second element of each pair by adding 1 mod $(v-1)$ from the last pair to first pair in the group. For example, for $n = 2, v = 6$ pairs are (1, 6), (2, 6), (3, 6), (4, 6), and (5, 6) and groups of pairs from each pair are

Group 1: (1, 6), (2, 5), (3, 4)
Group 2: (2, 6), (3, 1), (4, 5)
Group 3: (3, 6), (4, 2), (5, 1)
Group 4: (4, 6), (5, 3), (1, 2)
Group 5: (5, 6), (1, 4), (2, 3)

Step-2: Find $\frac{v}{2} \times \frac{v}{2}$ row-column from each group by arranging in cyclic form and after arranging these $\frac{v}{2} \times \frac{v}{2}$ row-column one by one, we get a $\left(\frac{v}{2} \times \frac{v}{2}(v-1)\right)/2$ BSLR of block size two.

General parameters for the obtained design are $v = 2n+2, h = n+1, p = (n+1)(2n+1), k = 2, n_r = 2n+1, n_c = 1, \lambda = n+1$

For example, for $v = 8, h = 4, p = 28, k = 2, n_r = 7, n_c = 1, \lambda = 4$, by following the above steps, we get a $(4 \times 28)/2$ BSLR as depicted below.

Row/Col	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**
1	1, 8	2, 7	3, 6	4, 5	2, 8	3, 1	4, 7	5, 6	3, 8	4, 2	5, 1	6, 7	4, 8	5, 3
2	4, 5	1, 8	2, 7	3, 6	5, 6	2, 8	3, 1	4, 7	6, 7	3, 8	4, 2	5, 1	7, 1	4, 8
3	3, 6	4, 5	1, 8	2, 7	4, 7	5, 6	2, 8	3, 1	5, 1	6, 7	3, 8	4, 2	6, 2	7, 1
4	2, 7	3, 6	4, 5	1, 8	3, 1	4, 7	5, 6	2, 8	4, 2	5, 1	6, 7	3, 8	5, 3	6, 2
Row/Col	**15**	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**
1	6, 2	7, 1	5, 8	6, 4	7, 3	1, 2	6, 8	7, 5	1, 4	2, 3	7, 8	1, 6	2, 5	3, 4
2	5, 3	6, 2	1, 2	5, 8	6, 4	7, 3	2, 3	6, 8	7, 5	1, 4	3, 4	7, 8	1, 6	2, 5
3	4, 8	5, 3	7, 3	1, 2	5, 8	6, 4	1, 4	2, 3	6, 8	7, 5	2, 5	3, 4	7, 8	1, 6
4	7, 1	4, 8	6, 4	6, 4	1, 2	5, 8	7, 5	1, 4	2, 3	6, 8	1, 6	2, 5	3, 4	7, 8

Method 3: Construction of PBSLR (v is even)

In this methodology for constructing PBSLR designs, the approach revolves around generating the design from an initial matrix. After the initial matrix is established, the entire design can be derived from this matrix using a specific procedure. Step 1 outlines the method to obtain the initial matrix, and Step 2 details the procedure for generating the entire design from the initial matrix.

Step-1: First construct a matrix of order $\frac{v}{2} \times 2$ such that row sums of the matrix are constant which is $(v+1)$, where $v = 2n+2$, n is any positive integer. First column of the matrix is obtained by arranging the numbers from up to down as $1, 2, ... \frac{v}{2}$ and second column is obtained by arranging the numbers as $(\frac{v}{2}+1), (\frac{v}{2}+2), ..., v$ starting from down to up. For example, for $n = 3, v = 8$, an initial matrix of order 4×2 is $\begin{pmatrix} 1 & 8 \\ 2 & 7 \\ 3 & 6 \\ 4 & 5 \end{pmatrix}$.

Step-2: Considering the matrix of order $\frac{v}{2} \times 2$ which is obtained in step-1 as first column of size two and adding one to each element with mod v, we get total v columns of size two. Then the resulting design is $(\frac{v}{2} \times v)/2$ PBSLR of block size two.

General parameters for the obtained design are $v = 2n+2, h = n+1, p = 2n+2, k = 2, n_r = 2, n_c = 1, \lambda_1 = 0, \lambda_2 = 2, n_1 = n, n_2 = n+1$.

Association scheme for the designs constructed by this method is well known two-associate rectangular association scheme in which if we arrange treatments in two rows such that first row contain odd numbered treatments and second row contains even numbered treatments then the two treatments are (i) first associates if they lie in same row, (ii) second associates otherwise.

The parameters of the association scheme are:
$v = 2n+2, n_1 = n, n_2 = n+1, P_1 = \begin{pmatrix} n-1 & 0 \\ 0 & n+1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & n \\ n & 0 \end{pmatrix}$.

For example, for $v = 8, h = 4, p = 8, k = 2, n_r = 2, n_c = 1, \lambda_1 = 0, \lambda_2 = 2, n_1 = 3, n_2 = 4$, a $(4 \times 8)/2$ PBSLR is obtained following above steps and is given below.

Row/Col	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1, 8	2, 1	3, 2	4, 3	5, 4	6, 5	7, 6	8, 7
2	2, 7	3, 8	4, 1	5, 2	6, 3	7, 4	8, 5	1, 6
3	3, 6	4, 7	5, 8	6, 1	7, 2	8, 3	1, 4	2, 5
4	4, 5	5, 6	6, 7	7, 8	8, 1	1, 2	2, 3	3, 4

Now, for association scheme, arrange v treatments as

$$\begin{array}{c} 1, 3, 5, 7 \\ 2, 4, 6, 8 \end{array}$$

Then, association scheme for above designs is

Treatment	1st associates	2nd associates
1	3, 5, 7	2, 4, 6, 8
2	4, 6, 8	1, 3, 5, 7
3	1, 5, 7	2, 4, 6, 8
4	2, 6, 8	1, 3, 5, 7
5	1, 3, 7	2, 4, 6, 8
6	2, 4, 8	1, 3, 5, 7
7	1, 3, 5	2, 4, 6, 8
8	2, 4, 6	1, 3, 5, 7

with parameters of the association scheme as $v = 8, n_1 = 3, n_2 = 4, P_1 = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$.

Method 4: Construction of PBSLR (v is odd)

In this construction method, the process involves two steps as given below.

Step-1: To get the initial row of p columns, first two columns are always $(v, 1)$ and $(v, 2)$ and last column is $((v+1)/2, (v-1)/2)$. Remaining $p - 3$ columns are obtained as $(v - k, k), (v - k, k + 2)$, where $k = 1, 2, ..., (v - 3)/2$ and $v = 2n + 3, n$ is any non-negative integer. For example, if $v = 7$ then first two columns will be (7, 1), (7, 2), last column will be (4, 3) and remaining columns will be (6, 1), (6, 3), (5, 2), (5, 4).

Step-2: By considering the initial row obtained in step-1, adding 1 mod v to each element of all pairs, one can get a $(v \times v)/2$ PBSLR of block size two.

General parameters for the design so obtained are $v = 2n + 3, h = 2n + 3, p = 2n + 3, k = 2, n_r = 2, n_c = 2, \lambda_1 = 3, \lambda_2 = 2, n_1 = 2, n_2 = 2n$.

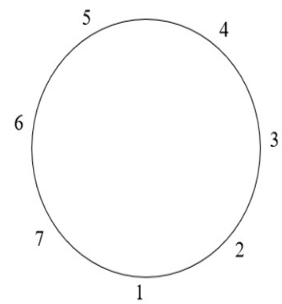
Association scheme for the designs constructed by this method is well known two-associate circular association scheme in which if we arrange the treatments (v) on the circle serially, then two treatments are first associates if they lie adjacently, second associates otherwise.

The parameters of the association scheme are:
$v = 2n + 3, n_1 = 2, n_2 = 2n, P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 2n-1 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2n-2 \end{pmatrix}$.

For example, for $v = 7, h = 7, p = 7, k = 2, n_r = 2, n_c = 2, \lambda_1 = 3, \lambda_2 = 2, n_1 = 2, n_2 = 4$, we obtained a $(7 \times 7)/2$ PBSLR as shown below.

Row/Col	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	7, 1	7, 2	6, 1	6, 3	5, 2	5, 4	4, 3
2	1, 2	1, 3	7, 2	7, 4	6, 3	6, 5	5, 4
3	2, 3	2, 4	1, 3	1, 5	7, 4	7, 6	6, 5
4	3, 4	3, 5	2, 4	2, 6	1, 5	1, 7	7, 6
5	4, 5	4, 6	3, 5	3, 7	2, 6	2, 1	1, 7
6	5, 6	5, 7	4, 6	4, 1	3, 7	3, 2	2, 1
7	6, 7	6, 1	5, 7	5, 2	4, 1	4, 3	3, 2

Now, for association scheme, arrange v treatments as



Then, association scheme for above design will be

Treatment	1st associates	2nd associates
1	2, 7	3, 4, 5, 6
2	1, 3	4, 5, 6, 7
3	2, 4	1, 5, 6, 7
4	3, 5	1, 2, 6, 7
5	4, 6	1, 2, 3, 7
6	5, 7	1, 2, 3, 4
7	1, 6	2, 3, 4, 5

with parameters of the association scheme as $v = 7, n_1 = 2, n_2 = 4, P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

Remark 1: According to the definition of BSLR, each treatment appears an equal number of times in each row, an equal number of times in each column, and at most once in a block. Therefore, BSLR with block size two exists only for $v > 2$ because if $v = 2$, then each block, row, and column would contain the same content, which has no meaningful interpretation in the context of BSLR.

Remark 2: An R package *slr* has been developed to implement the proposed methods of constructions of balanced and partially balanced semi-Latin rectangles. The package is available on CRAN in the webpage `https://cran.r-project.org/web/packages/slr/index.html`. It comes with two main functions namely *bslr(v,k)* and *pbslr(v,k)* which can be used to construct a balanced SLR and a partially balanced SLR, respectively, for given number of treatments v and $k = 2$.

Remark 3: Using the proposed methods of constructions (Method 1, 2, 3 and 4), several designs were constructed and catalogued for $3 \leq v \leq 30$. Though designs can also be constructed for $v > 30$ using proposed methods, however, a general proof of the results could not be provided by us. Thus, it may be considered as a limitation of the proposed methods.

5. Efficiency of semi-Latin rectangles

In this Section, we provide the average efficiency factor of SLR designs generated using the developed methods of constructions. The average efficiency factor (John, 1992) of a design is determined by taking the harmonic mean of the canonical efficiency factors, which are obtained by multiplying the nonzero eigenvalues of its information matrix by $1/r$. Here, the lists of BSLR and PBSLR designs with their average efficiency factor $(\bar{E})$ for $v \leq 30$ are given in Table 2 and Table 3, respectively.

6. Concluding remarks

In this paper we proposed construction methods for obtaining balanced semi-Latin rectangles for any number of treatments with block size two. Another major contribution of this paper is introduction of the concept of partially balanced semi-Latin rectangles. Four methods of construction of semi-Latin rectangles are developed out of which two are for balanced and two are for partially balanced designs. The methods enable construction of balanced and partially balanced semi-Latin rectangles for all permissible number of treatments with block size two. While the proposed methods (Methods 1 and 2) are not superior to those suggested by Uto and Bailey (2020), the methods offer a simpler and easier alternative for constructing designs compared to the methods proposed by Uto and Bailey (2020). We also have provided the average efficiency factors of designs generated using the proposed methods of constructions for the number of treatments up to 30. Further, we have developed an R package called *slr* which implements the proposed methods of constructions.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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Table 2: Efficiency of balanced semi-Latin rectangles with $v \leq 30$

Method 1							
n	v	h	p	n_r	n_c	λ	$\bar{E}$
0	3	3	3	2	2	3	0.75
1	5	5	10	4	2	5	0.625
2	7	7	21	6	2	7	0.5833
3	9	9	36	8	2	9	0.5625
4	11	11	55	10	2	11	0.55
5	13	13	78	12	2	13	0.5417
6	15	15	105	14	2	15	0.5357
7	17	17	136	16	2	17	0.5313
8	19	19	171	18	2	19	0.5278
9	21	21	210	20	2	21	0.525
10	23	23	253	22	2	23	0.5227
11	25	25	300	24	2	25	0.5208
12	27	27	351	26	2	27	0.5192
13	29	29	406	28	2	29	0.5179
Method 2							
1	4	2	6	3	1	2	0.6667
2	6	3	15	5	1	3	0.6
3	8	4	28	7	1	4	0.5714
4	10	5	45	9	1	5	0.5556
5	12	6	66	11	1	6	0.5455
6	14	7	91	13	1	7	0.5385
7	16	8	120	15	1	8	0.5333
8	18	9	153	17	1	9	0.5294
9	20	10	190	19	1	10	0.5263
10	22	11	231	21	1	11	0.5238
11	24	12	276	23	1	12	0.5217
12	26	13	325	25	1	13	0.52
13	28	14	378	27	1	14	0.5185
14	30	15	435	29	1	15	0.5172

Table 3: Efficiency of partially balanced semi-Latin rectangles with $v \leq 30$

Method 3								
n	v	h	p	n_r	n_c	λ_1	λ_2	$\bar{E}$
1	4	2	4	2	1	0	2	0.6
2	6	3	6	2	1	0	2	0.5556
3	8	4	8	2	1	0	2	0.5385
4	10	5	10	2	1	0	2	0.5294
5	12	6	12	2	1	0	2	0.5238
6	14	7	14	2	1	0	2	0.52
7	16	8	16	2	1	0	2	0.5172
8	18	9	18	2	1	0	2	0.5152
9	20	10	20	2	1	0	2	0.5135
10	22	11	22	2	1	0	2	0.5122
11	24	12	24	2	1	0	2	0.5111
12	26	13	26	2	1	0	2	0.5102
13	28	14	28	2	1	0	2	0.5094
14	30	15	30	2	1	0	2	0.5088
Method 4								
0	3	3	3	2	2	3	-	0.75
1	5	5	5	2	2	3	2	0.62
2	7	7	7	2	2	3	2	0.5799
3	9	9	9	2	2	3	2	0.5602
4	11	11	11	2	2	3	2	0.5483
5	13	13	13	2	2	3	2	0.5404
6	15	15	15	2	2	3	2	0.5348
7	17	17	17	2	2	3	2	0.5305
8	19	19	19	2	2	3	2	0.5272
9	21	21	21	2	2	3	2	0.5245
10	23	23	23	2	2	3	2	0.5223
11	25	25	25	2	2	3	2	0.5205
12	27	27	27	2	2	3	2	0.5189
13	29	29	29	2	2	3	2	0.5176