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An Adroit Strategy Randomized Response Model Using Fuzzy Numbers

Tanveer Ahmad Tarray, Muzafar Rasool Bhat and Peer Bilal Ahmad
Islamic University of Science and Technology, Pulwama, Kashmir

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Abstract

The nub of this paper is to consider an unrelated question randomized response model using allocation problem in two-stage stratified random sampling based on Singh and Tarray (2014) model and minimize the variance subject to cost constraint. The costs (measurement costs and total budget of the survey) in the cost constraint are assumed as fuzzy numbers, in particular triangular and trapezoidal fuzzy numbers due to the ease of use. The problem formulated is solved by using Lagrange multipliers technique and the optimum allocation obtained in the form of fuzzy numbers is converted into crisp form using α-cut method at a prescribed value of α. Numerical illustrations are also given in support of the present study and the results are formulated through LINGO.

Key words: Unrelated randomized response technique, Optimum allocation, stratified random sampling, Sensitive attribute, Fuzzy Logic.

1. Introduction

Randomized response technique (RRT) was introduced by Warner (1965) mainly to cut down the possibility of (i) reduced response rate and (ii) inflated response bias experienced in direct or open survey relating to sensitive issues. Warner himself pointed out how one may get a biased estimate in an open survey when a population consists of individuals bearing a stigmatizing character A or its complement A^c, which may or may not also be stigmatizing. Later several authors including Mangat and Singh (1990), Singh and Tarray (2012, 2013, 2014, 2015, 2016, 2017), Tarray and Singh (2015,2016,2017) and Tarray (2017) etc. have modified and suggested alternative randomized response procedures applicable to different situations.

Hong et al. (1994) envisaged RR technique that applied the same randomization device to every stratum. Stratified random sampling is generally obtained by dividing the

Corresponding Author: Tanveer Ahmad Tarray
E-mail: tanveerstat@gmail.com

population into non – overlapping groups called strata and selecting a simple random sample from each stratum. An RR technique using a stratified sampling gives the group characteristics related to each stratum estimator. Also, stratified samples protect a researcher from the possibility of obtaining a poor sample. For the sake of completeness and convenience to the readers, we have given the descriptions of fuzzy sets, fuzzy numbers, Triangular Fuzzy Number (TFN) and Trapezoidal Fuzzy Number (TrFN) which are reproduced here from Bector and Chandra (2005), Mahapatra and Roy (2006), Hassanzadeh et al. (2012), and Aggarwal and Sharma (2013).

Fuzzy sets were introduced by Zadeh (1965) to represent/manipulate data and information possessing non-statistical uncertainties.

It was specifically designed to mathematically represent uncertainty and vagueness and to provide formalized tools for dealing with the imprecision intrinsic to many problems. However, the story of fuzzy logic started much more earlier. To devise a concise theory of logic, and later mathematics, Aristotle posited the so-called "Laws of Thought". One of these, the "Law of the Excluded Middle," states that every proposition must either be True (T) or False (F). Even when Parminedes proposed the first version of this law (around 400 Before Christ) there were strong and immediate objections: for example, Heraclitus proposed that things could be simultaneously True and not True. It was Plato who laid the foundation for what would become fuzzy logic, indicating that there was a third region (beyond T and F) where these opposites "tumbled about." A systematic alternative to the bi-valued logic of Aristotle was first proposed by Łukasiewicz around 1920, when he described a three-valued logic, along with the mathematics to accompany it. The third value he proposed can best be translated as the term "possible," and he assigned it a numeric value between T and F. Eventually, he proposed an entire notation and axiomatic system from which he hoped to derive modern mathematics. Later, he explored four-valued logics, five-valued logics, and then declared that in principle there was nothing to prevent the derivation of an infinite-valued logic. Łukasiewicz felt that three- and infinite-valued logics were the most intriguing, but he ultimately settled on a four-valued logic because it seemed to be the most easily adaptable to Aristotelian logic.

The notion of an infinite-valued logic was introduced in Zadeh's seminal work "Fuzzy Sets" where he described the mathematics of fuzzy set theory, and by extension fuzzy logic. This theory proposed making the membership function (or the values F and T) operate over the range of real numbers [0, 1]. New operations for the calculus of logic were proposed, and showed to be in principle at least a generalization of classic logic. Fuzzy logic provides an inference morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The theory of fuzzy logic provides a mathematical strength to capture the uncertainties associated with human cognitive processes, such as thinking and reasoning. The conventional approaches to knowledge representation lack the means for representating the meaning of fuzzy concepts. As a consequence, the approaches based on first order logic and classical probability theory do not provide an appropriate conceptual framework for dealing with the representation of commonsense knowledge, since such knowledge is by its nature both lexically imprecise and non-categorical.
There are two main characteristics of fuzzy systems that give them better performance for specific applications.

i) Fuzzy systems are suitable for uncertain or approximate reasoning, especially for the system with a mathematical model that is difficult to derive.

ii) Fuzzy logic allows decision making with estimated values under incomplete or uncertain information.

Fuzzy set: A fuzzy set $\tilde{A}$ in a universe of discourse X is defined as the following set of pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$. Here $\mu_{\tilde{A}}(x) : X \to [0,1]$ is a mapping called the membership function of the fuzzy set $\tilde{A}$ and $\mu_{\tilde{A}}$ is called the membership value or degree of membership of $x \in X$ in the fuzzy set $\tilde{A}$. The larger
the value of $\mu_{\tilde{A}}$, the stronger the grade of membership in $\tilde{A}$.

α-Cut: The α-cut for a fuzzy set $\tilde{A}$ is shown by $\tilde{A}_\alpha$ and for $\alpha \in [0,1]$ is defined to be

$$\tilde{A}_\alpha = \{(x \mid \mu_{\tilde{A}}(x) \geq \alpha) : x \in X\} \quad (1)$$

where X is the universal set.

Upper and lower bounds for any α-cut $\tilde{A}_\alpha$ are given by $\tilde{A}_\alpha^U$ and $\tilde{A}_\alpha^L$ respectively.

Fuzzy Number: A fuzzy set A in R is called a fuzzy number if it satisfies the following conditions:

i) A is convex and normal.

ii) A_α is a closed interval for every $\alpha \in (0, 1]$.

iii) The support of A is bounded.

Triangular Fuzzy Number (TFN): A fuzzy number $\tilde{A} = (p, q, r)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{x-p}{q-p}, & if \quad p \leq x \leq q, \\ \dfrac{r-x}{r-q}, & if \quad q \leq x \leq r, \\ 0, & otherwise \end{cases} \quad (2)$$

Trapezoidal Fuzzy Number (TrFN): A fuzzy set $\tilde{A} = (p, q, r, s)$ on real numbers R is called a trapezoidal fuzzy number with membership function as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & if \quad x \leq p, \\ \dfrac{x-p}{q-p}, & if \quad p \leq x \leq q, \\ \dfrac{s-x}{s-r}, & if \quad r \leq x \leq s, \\ 0, & if \quad s \leq x. \end{cases} \quad (3)$$

2. Problem Formulation

The randomization model requires two randomization devices R_{1i} and R_{2i} with known unrelated attribute. The randomization device R_{2i} is same as used by Greenberg et al. (1969)

model. In the first stage of the survey interview, an individual respondent in the sample from stratum i is instructed to use the randomization device R_{1i} which consists of a sensitive question (S) cards with probability T_i and a "Go to the random device R_{2i} in the second stage" direction card with probability $(1 - T_i)$. The respondents in the second stage of stratum i are instructed to use the randomization device R_{2i} which consists of a sensitive question (S) card with probability P_i and a non – sensitive question (Y) card with probability $(1-P_i)$. The respondents selects randomly one of these statements unobserved by the interviewer and reports "Yes" if he / she possesses statement and "No" otherwise. Let n_i denote the number of units in the sample from stratum i and n denote the total number of units in all strata so that $\sum_{i=1}^{k} n_i = n$. Under the assumption that these "Yes" and "No" reports are made truthfully and P_i and T_i are set by the researcher, the probability X_i of a "Yes" answer in stratum i for this procedure is:

$$X_i = \pi_{Si}T_i + (1-T_i)[\pi_{Si}P_i + (1-P_i)\pi_{yi}] \qquad \text{for } i = 1,2,\cdots,k \tag{4}$$

where π_{Si} is the proportion of people with sensitive traits in i and π_{yi} is the proportion of people with the non-sensitive traits in i.

Under the condition that π_{yi} is known, the unbiased estimator $\hat{\pi}_{Si}$ of π_{Si} is:

$$\hat{\pi}_{Si} = \frac{\left\{\hat{X}_i - (1-T_i)(1-P_i)\pi_{yi}\right\}}{\left\{T_i + P_i(1-T_i)\right\}} \qquad \text{for } i = 1,2,\cdots,k \tag{5}$$

where $\hat{X}_i$ is the proportion of "Yes" answer in the sample from stratum for i.

Since each $\hat{X}_i$ is a binomial distribution $B\left(n_i, \hat{X}_i\right)$, the variance of the estimator $\hat{\pi}_{Si}$ is

$$V\left(\hat{\pi}_{Si} \mid \pi_{yi}\right) = \frac{X_i(1-X_i)}{n_i\left\{T_i + P_i(1-T_i)\right\}^2} \tag{6}$$

Since the selections in different strata are made independently, the estimators for individual strata can be added together to obtain an estimator for the entire population. Thus the unbiased estimator of π_S is

$$\hat{\pi}_S = \sum_{i=1}^{k} w_i \hat{\pi}_{Si} \tag{7}$$

$$= \sum_{i=1}^{k} w_i \frac{\left\{\hat{X}_i - (1-T_i)(1-P_i)\pi_{yi}\right\}}{\left\{T_i + P_i(1-T_i)\right\}} \tag{8}$$

The variance of the unbiased estimator $\hat{\pi}_S$ given π_{yi} is:

$$V\left(\hat{\pi}_S \mid \pi_{yi}\right) = \sum_{i=1}^{k} w_i^2 \frac{X_i(1-X_i)}{n_i\left\{T_i + P_i(1-T_i)\right\}^2} \tag{9}$$

or

$$V\left(\hat{\pi}_S \mid \pi_{yi}\right) = \sum_{i=1}^{k} \frac{w_i^2}{n_i^2}\{A_i\} \tag{10}$$

To find the optimum allocation we either maximize the precision for fixed budget or minimize the cost for fixed precision. A linear cost function which is an adequate approximation of the actual cost incurred will be

The linear cost function is $C = C_0 + \sum_{i=1}^{k} c_i n_i$, (11)

where C_0 is the over head cost, c_i is the per unit cost of measurement in i^{th} stratum, C is the available fixed budget for the survey.

In view of (4) and (11), the problem of optimum allocation can be formulated as a non linear programming problem (NLPP) for fixed cost as

$$\left.\begin{aligned} &Minimize\ V(\hat{\pi}_S) = \sum_{i=1}^{k} \frac{w_i^2}{n_i} A_i \\ &subject\ to\ \sum_{i=1}^{k} c_i n_i \le c_0 \\ &\qquad 1 \le n_i \le N_i \quad and\ n_i\ integers, i = 1,2,...k \end{aligned}\right\} \tag{12}$$

The restrictions $1 \le n_i$ and $n_i \le N_i$ are placed to have the representation of every stratum in the sample and to avoid the oversampling, respectively.

3. Fuzzy Formulation

Generally, real-world situations involve a lot of parameters such as cost and time, whose values are assigned by the decision makers and in the conventional approach, they are required to fix an exact value to the aforementioned parameters. However decision-makers frequently do not precisely know the value of those parameters. Therefore, in such cases it is better to consider those parameters or coefficients in the decision-making problems as fuzzy numbers. The mathematical modeling of fuzzy concepts was presented by Zadeh (1965). Therefore, the fuzzy formulation of problem (12) with fuzzy cost constraint is given by considering two cases of fuzzy numbers, that is, triangular fuzzy number (TFN) and trapezoidal fuzzy number (TrFN).

For triangular fuzzy number (TFN) we consider

$$\left.\begin{aligned} &Minimize\ \sum_{i=1}^{k} \frac{w_i^2}{n_i} A_i \\ &subject\ to\ \sum_{i=1}^{k} (c_i^1, c_i^2, c_i^3) n_i \le (c_0^1, c_0^2, c_0^3) \\ &\qquad 1 \le n_i \le N_i \quad and\ n_i\ integers, i = 1,2,...k \end{aligned}\right\} \tag{13}$$

where

$$A_i = \frac{X_i(1 - X_i)}{\{T_i + P_i(1 - T_i)\}^2} \tag{14}$$

and $\tilde{C}_i = (c_i^1, c_i^2, c_i^3)$ is triangular fuzzy numbers with membership function

$$\mu_{\tilde{C}_i}(x)=\begin{cases}\dfrac{x-c_i^1}{c_i^2-c_i^1}, & if \quad c_i^1\le x\le c_i^2,\\ \dfrac{c_i^3-x}{c_i^3-c_i^2}, & if \quad c_i^2\le x\le c_i^3,\\ 0, & otherwise\end{cases} \tag{15}$$

Similarly, the membership function for available budget can be expressed as

$$\mu_{\tilde{C}_0}(x)=\begin{cases}\dfrac{x-c_0^1}{c_i^2-c_0^1}, & if \quad c_0^1\le x\le c_0^2,\\ \dfrac{c_0^3-x}{c_0^3-c_0^2}, & if \quad c_0^2\le x\le c_0^3,\\ 0, & otherwise\end{cases} \tag{16}$$

and for trapezoidal fuzzy number (TrFN) we consider

$$\left.\begin{array}{l} Minimize \quad \sum_{i=1}^{k}\dfrac{w_i^{\,2}}{n_i}A_i \\ subject\ to \quad \sum_{i=1}^{k}(c_i^1,c_i^2,c_i^3,c_i^4)n_i\le(c_0^1,c_0^2,c_0^3c_0^4) \\ \qquad 1\le n_i\le N_i \quad and\ n_i\ \text{int}\,egers,\ i=1,2,...k \end{array}\right\} \tag{17}$$

Where $A_i=\dfrac{X_i(1-X_i)}{\{T_i+P_i(1-T_i)\}^2}$ (18)

and $\tilde{C}_i=(c_i^1,c_i^2,c_i^3,c_i^4)$ is trapezoidal fuzzy numbers with membership function

$$\mu_{\tilde{C}_i}(x)=\begin{cases}0, & if \quad x\le c_i^1,\\ \dfrac{x-c_i^1}{c_i^2-c_i^1}, & if \quad c_i^1\le x\le c_i^2,\\ 1, & if \quad c_i^2\le x\le c_i^3,\\ \dfrac{c_i^4-x}{c_i^4-c_i^3}, & if \quad c_i^3\le x\le c_i^4,\\ 0, & if \quad c_i^4\le x\end{cases} \tag{19}$$

Similarly, the membership function for available budget can be expressed as

$$\mu_{\tilde{C}_0}(x)=\begin{cases}0, & if \quad x \le c_0^1, \\ \dfrac{x-c_0^1}{c_0^2-c_0^1}, & if \quad c_0^1 \le x \le c_0^2, \\ 1, & if \quad c_0^2 \le x \le c_0^3, \\ \dfrac{c_0^4-x}{c_0^4-c_0^3}, & if \quad c_0^3 \le x \le c_0^4, \\ 0, & if \quad c_0^4 \le x \end{cases} \tag{20}$$

4. Lagrange Multipliers Formulation

Let us now determine the solution of problems (13) by ignoring upper and lower bounds and integer requirements the NLPP with TFNs is solved by Lagrange multipliers technique (LMT).

The Lagragian function may be

$$\varphi(n_h,\lambda)= \sum_{i=1}^{k}\frac{w_i^{\,2}}{n_i}\{A_i\} + \lambda\left[\sum_{i=1}^{k}\left(c_i^{(1)},c_i^{(2)},c_i^{(3)}\right)n_i-\left(c_0^{(1)},c_0^{(2)},c_0^{(3)}\right)\right] \tag{21}$$

Differentiating (21) with respect to n_i and λ and equating to zero, we get the following sets of equations:

$$\frac{\bar{V}\varphi}{\bar{V}n_i}=0 \Rightarrow n_i=-\frac{w_i^2}{n_h^2}\{A_i\}+\lambda\left(c_i^{(1)},c_i^{(2)},c_i^{(3)}\right) \tag{22}$$

or

$$n_i=\frac{1}{\sqrt{\lambda}}\frac{\sqrt{\{A_i\}}}{\sqrt{\left(c_i^{(1)},c_i^{(2)},c_i^{(3)}\right)}} \tag{23}$$

Also,

$$\frac{\bar{V}\varphi}{\bar{V}\lambda}=\left\{\sum_{i=1}^{k}\left(c_i^{(1)},c_i^{(2)},c_i^{(3)}\right)n_i-\left(c_0^{(1)},c_0^{(2)},c_0^{(3)}\right)\right\}=0 \tag{24}$$

which gives

$$\sum_{i=1}^{k}w_i\left(c_i^{(1)},c_i^{(2)},c_i^{(3)}\right)\sqrt{\frac{\{A_i\}}{\lambda\left(c_i^{(1)},c_i^{(2)},c_i^{(3)}\right)}}-\left(c_0^{(1)},c_0^{(2)},c_0^{(3)}\right)=0 \tag{25}$$

or

$$\frac{1}{\sqrt{\lambda}}=\frac{\left(c_0^{(1)},c_0^{(2)},c_0^{(3)}\right)}{\sum_{i=1}^{k}w_i\sqrt{\{A_i\}\left(c_i^{(1)},c_i^{(2)},c_i^{(3)}\right)}} \tag{26}$$

Substituting (23) in (26), we have

$$n_i^* = \frac{\left(c_0^{(1)}, c_0^{(2)}, c_0^{(3)}\right) w_i \sqrt{\frac{\{A_i\}}{\left(c_i^{(1)}, c_i^{(2)}, c_i^{(3)}\right)}}}{\sum_{i=1}^{k} w_i \sqrt{\{A_i\}\left(c_i^{(1)}, c_i^{(2)}, c_i^{(3)}\right)}} \quad . \tag{27}$$

In similar manner, the optimum allocation of NLPP (17) with trapezoidal fuzzy number can be obtained as follows

$$n_i^* = \frac{\left(c_0^{(1)}, c_0^{(2)}, c_0^{(3)}, c_0^{(4)}\right) w_i \sqrt{\frac{\{A_i\}}{\left(c_i^{(1)}, c_i^{(2)}, c_i^{(3)}, c_i^{(4)}\right)}}}{\sum_{i=1}^{k} w_i \sqrt{\{A_i\}\left(c_i^{(1)}, c_i^{(2)}, c_i^{(3)}, c_i^{(4)}\right)}} \tag{28}$$

To convert fuzzy allocations into a crisp allocation by – cut method.

5. Procedure for conversation of Fuzzy Numbers

The fuzzy allocations into a crisp allocation by α– cut method let $\tilde{A} = (p, q, r)$ be a TFN. An α-cut for $\tilde{A}$, $\tilde{A}_\alpha$ computed as

$$\alpha = \frac{x-p}{q-p} \Rightarrow \tilde{A}_\alpha^{\ L} = x = (q-p)\alpha + p$$

and

$$\alpha = \frac{r-x}{r-q} \Rightarrow \tilde{A}_\alpha^{\ U} = x = r - (r-q)\alpha \tag{29}$$

where $\tilde{A}_\alpha = [\tilde{A}_\alpha^{\ L}, \tilde{A}_\alpha^{\ U}]$ is the corresponding α-cut as shown in figure 1. The allocation obtained in (27) is in the form of triangular fuzzy number, therefore by using (29) the equivalent crisp allocation is given by

$$n_i^* = \frac{\left(c_0^{(3)} - (c_0^{(3)} - c_0^{(2)})\right) w_i \sqrt{\frac{\{A_i\}}{\left(c_h^{(1)} + (c_h^{(2)} - c_h^{(1)})\right)}}}{\sum_{i=1}^{k} w_i \sqrt{\{A_i\}\left(c_h^{(1)} + (c_h^{(2)} - c_h^{(1)})\right)}} \tag{30}$$

similarly, let $\tilde{A} = (p, q, r, s)$ be a TrFN. An α-cut for $\tilde{A}$, $\tilde{A}_\alpha$ computed as

$$\alpha = \frac{x-p}{q-p} \Rightarrow \tilde{A}_\alpha^{\ L} = x = (q-p)\alpha + p$$

and

$$\alpha = \frac{s-x}{s-q} \Rightarrow \tilde{A}_\alpha^{\ U} = x = s - (s-q)\alpha \tag{31}$$

where $\tilde{A}_\alpha = [\tilde{A}_\alpha^{\ L}, \tilde{A}_\alpha^{\ U}]$ is the corresponding α-cut as shown in figure 2. The allocation obtained in (30) is in the form of triangular fuzzy number, therefore by using (31) the equivalent crisp allocation is given by

$$n_i^* = \frac{\left(c_0^{(4)} - (c_0^{(4)} - c_0^{(3)})\right) w_i \sqrt{\dfrac{\{A_i\}}{\left(c_h^{(1)} + (c_h^{(2)} - c_h^{(1)})\right)}}}{\sum_{i=1}^{k} w_i \sqrt{\{A_i\}\left(c_h^{(1)} + (c_h^{(2)} - c_h^{(1)})\right)}} \tag{32}$$

The allocations obtained by (30) and (32) provide the solution to NLPP (13) and (17) if it satisfies the restriction $1 \le n_i \le N_h$, $i = 1, 2, \ldots, k$. The allocations obtained in (30) and (32) may not be integer allocations, so to get integer allocations, round off the allocations to the nearest integer values. After rounding off we have to be careful in rechecking that the round-off values satisfy the cost constraint. Now we further discuss equal and proportional allocations as follows:

Equal Allocation. In this method, the total sample size n is divided equally among all the strata, that is, for the ith Stratum

$$n_i = \frac{n}{k} \tag{33}$$

where n can be obtained from the cost constraint equation as follows:

$$\sum_{i=1}^{k} w_i \sqrt{\{A_i\}\left(c_h^{(1)} + (c_h^{(2)} - c_h^{(1)})\right)}\, n_i = \left(c_0^{(4)} - (c_0^{(4)} - c_0^{(3)})\right) \tag{34}$$

$$n_i \propto w_i$$

$$\text{or } n_i = n w_i \tag{35}$$

Now substituting the value of n_i in (34), we get

$$n_i = \frac{N\left(c_0^{(4)} - (c_0^{(4)} - c_0^{(3)})\right)}{\sum_{i=1}^{k} \sqrt{\left(c_h^{(1)} + (c_h^{(2)} - c_h^{(1)})\right)} N_h} \tag{36}$$

Proportional Allocation. This allocation was originally proposed by Bowley (1926).This procedure of allocation is very common in practice because of its simplicity. When no other information except N_i , the total number of units in the ith stratum, is available, the allocation of a given sample of size n to different strata is done in proportion to their sizes, that is, in the ith stratum

$$n_i = n \frac{N_i}{N} \tag{37}$$

6. Numerical Illustration

A hypothetical example is given to illustrate the computational details of the proposed problem. Let us suppose the population size is 1000 with total available budget of the survey as TFNs and TrFNS are (3500, 4000, 4800) and (3500, 4000, 4400, 4600) units, respectively. The other required relevant information is given in Table 1. By using the value of Table 1, we compute the values of A_i which is given in Table 2.

After substituting all the values from Tables 1 and 2 in (13), the required FNLLP is given as

$$\left.\begin{aligned} &\text{Minimize } V(\hat{\pi}_S) = \frac{0.02495672}{n_1} + \frac{0.13331246}{n_2} \\ &\text{sujbect to } \ (1,2,4)n_1 + (18,20,24)n_2 \le (3500,4000,4800) \\ &\qquad 1 \le n_1 \le 300 \\ &\qquad 1 \le n_2 \le 700 \end{aligned}\right\} \quad (38)$$

The required optimum allocations for problem (13) obtained by substituting the values from tables 1 and 2 in (30) at α = 0.5 will be

$$n_1 = \frac{(4800-800\alpha)0.3\sqrt{(0.2772969)/(\alpha+1)}}{0.3\sqrt{(0.2772969)(\alpha+1)} + 0.7\sqrt{(0.2716829)(2\alpha+18)}}$$

$$n_2 = \frac{(4800-800\alpha)0.7\sqrt{(0.2716829)/(2\alpha+18)}}{0.3\sqrt{(0.2772969)(\alpha+1)} + 0.7\sqrt{(0.2716829)(2\alpha+18)}}$$

In similar manner, optimum allocation for problem (17) obtained by substituting the values from tables 1 and 2 in (32) at α = 0.55 will be

$$n_1 = \frac{(3750)0.3\sqrt{(0.2772969)/(\alpha+1)}}{0.3\sqrt{(0.2772969)(\alpha+1)} + 0.7\sqrt{(0.2716829)/(2\alpha+18)}}$$

$$n_2 = \frac{(4400-200\alpha)0.7\sqrt{(0.2716829)/(2\alpha+18)}}{0.3\sqrt{(0.2772969)(\alpha+1)} + 0.7\sqrt{(0.2716829)(2\alpha+18)}}$$

Table 1: The stratified population with two strata

Stratum (i)	T_i	w_i	π_Y	π_{Si}	P_i	(c_h^1, c_h^2, c_h^3)	(c_0^1, c_0^2, c_0^3)
1	0.495	0.3	0.91	0.48	0.9	(1,2,4)	(1,2,4,7)
2	0.95	0.7	0.91	0.53	0.1	(18,20,24)	(18,20,24,26)

The values of X_i, A_i and $A_i w_i^2$ are calculated as given in table below.

Table 2: Calculated values of A_i and $A_i w_i^2$

Stratum (i)	X_i	A_i	$A_i w_i^2$
1	0.501715	0.272969	0.0249572
2	0.5471	0.2716829	0.1331246

Applying the α – cut and LMT, the optimum allocation after is obtained and summarized in Table 3 for both the cases i.e. case of TFN and case of TrFN with variance as:

Table 3: Calculated values of optimum allocation and variance

	Case of	n_1	n_2	*Variance*
LMT(optimum allocation)	TFN	318.15	271.00	0.000569678
	TrFN	206.4617	201.76	0.000780694

Case – I:

$$\left.\begin{array}{l} Minimize\ V(\hat{\pi}_S) = \dfrac{0.02495672}{n_1} + \dfrac{0.1331246}{n_2} \\ subject\ to\ \ (1)n_1 + (18)n_2 \le (3500) \\ \qquad 1 \le n_1 \le 300 \\ \qquad 1 \le n_2 \le 700 \end{array}\right\}$$

Using the above minimization problem, we get optimal solution as n_1 = 300, n_2 = 177.778 and optimal value is Minimize $V(\hat{\pi}_S)$= 0.0008320149.

Since n_1 and n_2 are required to be the integers, we branch problem R_1 into two sub problems R_2 and R_3 by introducing the constraints $n_2 \le 177$ and $n_2 \ge 178$ respectively indicated by the value n_1 =300 and n_2 =177 and n_1 =296 and n_2 =178. Hence the solution is treated as optimal. The optimal value is n_1 =296 and n_2 = 178 and optimal solution is to Minimize $V(\hat{\pi}_S)$= 0.0008320149. It may be noted that the optimal integer values are same as obtained by rounding the n_i to the nearest integer. Let us suppose $V(\hat{\pi}_S) = Z$, the various nodes for the NLPP utilizing case - I, are presented below in figure (III).

Case – II:

$$\left.\begin{array}{l} Minimize\ V(\hat{\pi}_S) = \dfrac{0.02495672}{n_1} + \dfrac{0.1331246}{n_2} \\ subject\ to\ \ (2)n_1 + (20)n_2 \le (4000) \\ \qquad 1 \le n_1 \le 300 \\ \qquad 1 \le n_2 \le 700 \end{array}\right\}$$

Using the above minimization problem, we get optimal solution as n_1 = 240.86, n_2 = 175.91 and optimal value is Minimize $V(\hat{\pi}_S)$= 0.0008603746.

Since n_1 and n_2 are required to be the integers, so problem R_1 is further branched into sub problems R_2; R_2; R_4 and R_5 with additional constraints as $n_1 \le 240$; $n_1 \ge 241$; $n_2 \le 175$ and $n_2 \ge 176$; respectively. Problems R_2, R_4 and R_5 stand fathomed as the optimal solution in each case is integral in n_1 and n_2. Problem R_3 has been further branched into sub problems R_4 and R_5 with additional constraints as $n_1 \le 175$ and $n_1 \ge 176$; respectively which suggests that R_6 is fathomed and R_7 has no feasible solution. The optimal value is n_1 =240 and n_2 = 136 and optimal solution is to Minimize $V(\hat{\pi}_S)$= 0.0008603761. Let us suppose $V(\hat{\pi}_S) = Z$, the various nodes for the NLPP utilizing case - II, are presented below in figure (IV).

Case – III:

$$\left.\begin{aligned} &Minimize\ V(\hat{\pi}_S) = \frac{0.02495672}{n_1} + \frac{0.1331246}{n_2} \\ &subject\ to\ \ (4)n_1 + (24)n_2 \le (4800) \\ &\qquad\qquad 1 \le n_1 \le 300 \\ &\qquad\qquad 1 \le n_2 \le 700 \end{aligned}\right\}$$

Using the above minimization problem, we get optimal solution as n_1 = 180.25, n_2 = 170 and optimal value is Minimize $V(\hat{\pi}_S)$= 0.0009217340.

Since n_1 and n_2 are required to be the integers, so problem R_1 is further branched into sub problems R_2 and R_3 with additional constraints as $n_1 \le 180$; $n_1 \ge$ 181respectively. Problems R_2 stand fathomed as the optimal solution in each case is integral in n_1 and n_2. Problem R_3 has been further branched into sub problems R_4 and R_5 with additional constraints as $n_2 \le 169$ and $n_2 \ge 170$; respectively. R_4 is fathomed and R_5 has no feasible solution. Hence the solution is treated as optimal. The optimal value is n_1 =180.25 and n_2 = 170 and optimal solution is to Minimize $V(\hat{\pi}_S)$= 0.0009217340. Let us suppose $V(\hat{\pi}_S) = Z$, the various nodes for the NLPP utilizing case - III, are presented below in figure (V).
In both the three cases we find that the optimal value n_1 =296 and n_2 = 178 and optimal solution is to Minimize $V(\hat{\pi}_S)$= 0.0008320149.

7. Discussion

A stratified randomized response method assists to solve the limitations of randomized response that is the loss of individual characteristics of the respondents. The optimum allocation problem for two-stage stratified random sampling based on Singh and Tarray (2014) model with fuzzy costs is formulated as a problem of fuzzy nonlinear programming problem. The problem is then solved by using Lagrange multipliers technique for obtaining optimum allocation. The optimum allocation obtained in the form of fuzzy numbers is converted into an equivalent crisp number by using α-cut method at a prescribed value of α.

For practical purposes we need integer sample sizes. Therefore, in instead of rounding off the continuous solution, we have obtained integer solution, by formulating the problem as fuzzy integer nonlinear programming problem and obtained the integer solution by LINGO software.

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Figure (1): Triangular fuzzy number with an α – cut

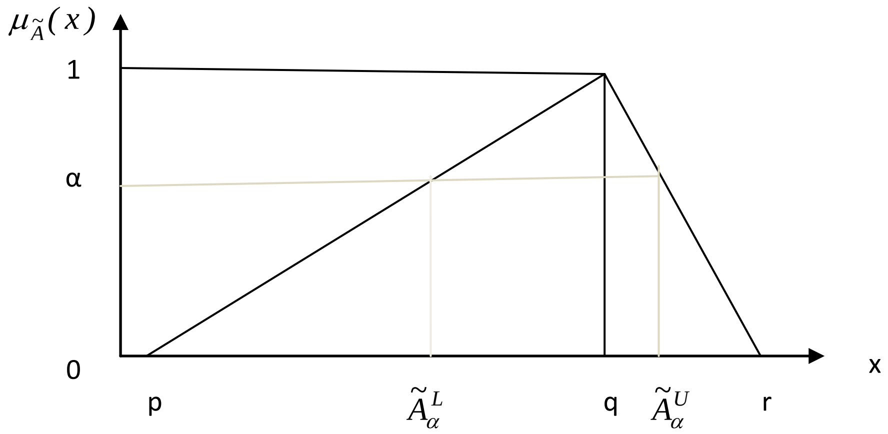

Figure (II): Trapezoidal fuzzy number with an α – cut

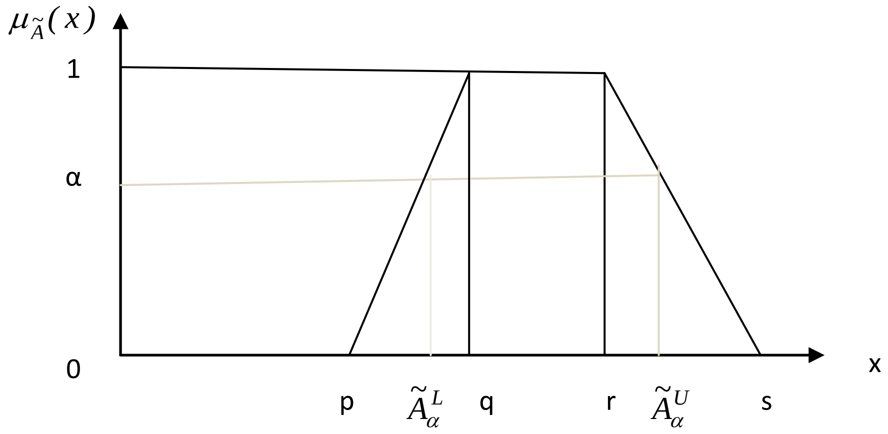

Figure (III) : Various nodes of NLPP

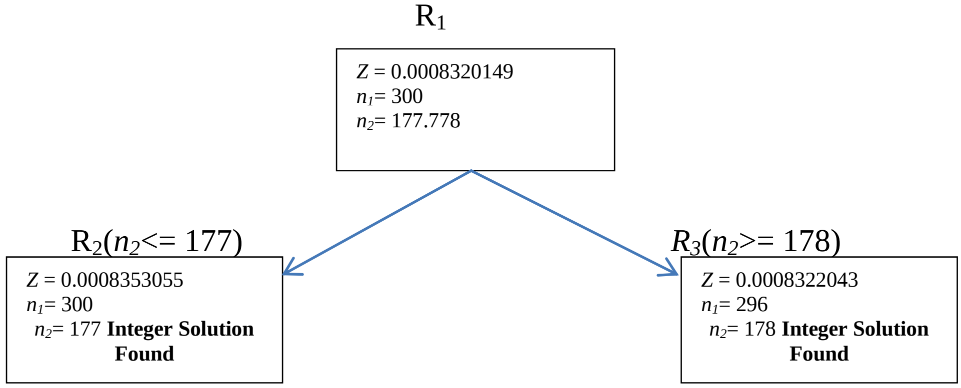

Figure (IV) : Various nodes of NLPP

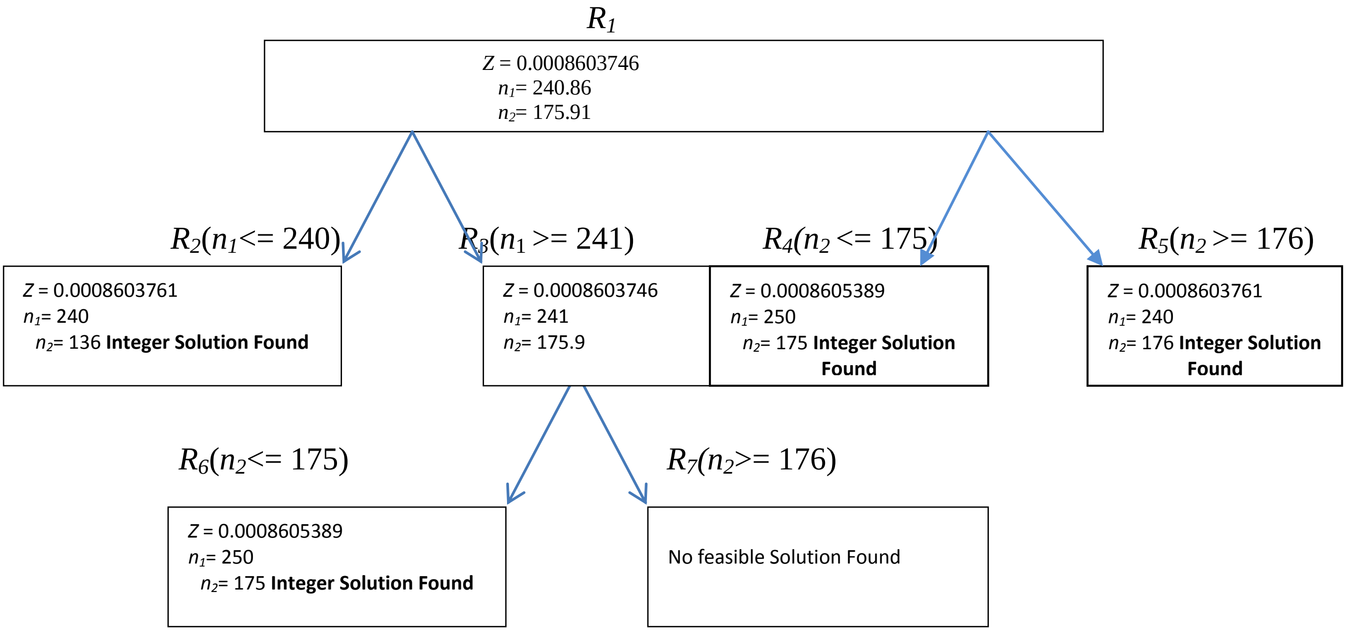

Figure (V) : Various nodes of NLPP