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R-optimal Designs for Linear Haar-Wavelet Regression Models

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Abstract

This paper considers the R-optimal design problem for a linear Haar-wavelet regression model. It is proved that the proposed designs are R-optimal by means of the equivalence theorem.

Key words: Haar-wavelet; R-optimal designs; Equivalence theorem.

AMS Subject Classifications: 62K05, 62J05

1. Introduction

An extensive literature review reveals that the wavelet models are gradually becoming popular from theoretical and application point of view. Mention may be made in this regard to Herzberg and Traves (1994), Oyet and Wiens (2000), Oh, Naveau and Lee (2001), Oyet (2002), Xie (2002), Tian and Herzberg (2006, 2007) and Maronge *et al.* (2017) for the novel study of various linear and approximately linear wavelet models and their optimal designs.

Tian and Herzberg (2007) rightly pointed out that wavelets can be considered as a basis for representing square integrable functions in different scales in the same way as polynomials, trigonometric functions, rational functions can be. Haar (1910) pioneered the notion of wavelet system on the real line $\mathbb{R}$. The system is an orthogonal basis in $L_2(\mathbb{R})$ generated by the Haar scaling function $\phi(x)$ and the Haar primary wavelet $\psi(x)$, where

$$\phi(x) = \begin{cases} 1, & 0 \le x < 1, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise}. \end{cases}$$

It is to be noted that Haar-wavelets are piecewise constant functions on the real line $\mathbb{R}$ and can take only three values. Moreover, Haar-wavelets, like the well-known Wash functions (Rao 1983), form an orthogonal and complete set of functions representing discretized functions and piecewise constant functions. Tian and Herzberg (2007) investigated the linear Haar-wavelet models and obtained the D-, A- and E-optimal designs.

The R-optimality criterion was introduced by Dette (1997) and it minimizes the volume of the rectangular confidence region for the regression parameters based on the Bonferroni t-intervals. The R-optimal design problem has been investigated in linear models [see He and Yue (2019) for

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recent reference]. Our aim here is to consider the R-optimal designs for the linear Haar-wavelet regression model.

The rest of this paper is organized as follows. In Section 2, we introduce the notation and preliminaries. Section 3 provides the main result of this paper with an example.

2. Model and Preliminaries

Consider the linear Haar-wavelet model of order m

$$E[y(x)] = \beta_0 + \sum_{j=0}^{m} \sum_{k=0}^{2^j-1} \beta_{jk}\psi_{jk}(x), \quad x \in \mathcal{X} = [0,1], \tag{1}$$

with unknown parameters $\beta_0, \beta_{00}, \cdots, \beta_{m,2^m-1}$, where

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k) = \begin{cases} 2^{j/2}, & \frac{k}{2^j} \le x < \frac{k}{2^j} + \frac{1}{2^{j+1}}, \\ -2^{j/2}, & \frac{k}{2^j} + \frac{1}{2^{j+1}} \le x < \frac{k}{2^j} + \frac{1}{2^j} \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

for $j \in \{0,1,\cdots,m\},\ k \in \{0,1,\cdots,2^j-1\}$.

Throughout the paper we consider approximate designs of the form

$$\xi = \begin{Bmatrix} x_1 & \cdots & x_n \\ w_1 & \cdots & w_n \end{Bmatrix}, \quad x_i \in \mathcal{X}, \quad 0 < w_i < 1, \quad \sum_{i=1}^{n} w_i = 1.$$

Denote the set of all approximate designs with non-singular information matrix on $\mathcal{X}$ by Ξ. For the model (1) the information matrix of $\xi \in \Xi$ is

$$M(\xi) = \int_{\mathcal{X}} \boldsymbol{f}(x)\boldsymbol{f}^T(x)d\xi(x), \tag{3}$$

where $\boldsymbol{f}(x) = (1, \psi_{00}(x), \cdots, \psi_{m,2^m-1}(x))^T$.

The following definition, due to Dette (1997), provides the R-optimality criterion for a design belonging to Ξ.

Definition 1: A design $\xi^* \in \Xi$ is called R-optimal for the model (1) if it minimizes

$$\Psi(\xi) = \prod_{i=1}^{p} \left(M^{-1}(\xi)\right)_{ii} = \prod_{i=1}^{p} e_i^T M^{-1}(\xi) e_i \tag{4}$$

over Ξ, where p is the dimension of the regression vector $\boldsymbol{f}(x)$ and e_i denotes the ith unit vector in $\mathbb{R}^p$.

The following equivalence theorem provides an important tool for the determination of R-optimal designs which has been proved by Dette (1997).

Theorem 1: For the model (1) let

$$\phi(\mathbf{x},\xi) = \boldsymbol{f}^T(\mathbf{x})M^{-1}(\xi)\left(\sum_{i=1}^{p}\frac{\boldsymbol{e}_i\boldsymbol{e}_i^T}{\boldsymbol{e}_i^T M^{-1}(\xi)\boldsymbol{e}_i}\right)M^{-1}(\xi)\boldsymbol{f}(\mathbf{x}). \tag{5}$$

Then a design $\xi^* \in \Xi$ is R-optimal if and only if

$$\sup_{\mathbf{x}\in\mathcal{X}}\phi(\mathbf{x},\xi^*) = p. \tag{6}$$

Moreover, the supremum is achieved at the support points of ξ^*.

3. R-optimal Designs

The following theorem provides R-optimal designs for the linear Haar-wavelet regression model (1).

Theorem 2: For the model (1), let x_i be arbitrary point in

$$\mathcal{X}_i = \left[\frac{i-1}{2^{m+1}}, \frac{i}{2^{m+1}}\right), \quad i = 1,\cdots,2^{m+1}.$$

Then the design ξ^* of the form

$$\xi^* = \begin{pmatrix} x_1 & \cdots & x_{2^{m+1}} \\ \frac{1}{2^{m+1}} & \cdots & \frac{1}{2^{m+1}} \end{pmatrix} \tag{7}$$

is R-optimal.

Proof: It is to be noted that $\psi_{jk}(x)$'s are step functions and for any $x \in \mathcal{X}_i,\ (i = 1,\cdots,2^{m+1})$

$$\psi_{jk}(x) \quad = \quad \psi_{jk}(\mu_i)$$

and

$$\boldsymbol{f}(x) \quad = \quad \boldsymbol{f}(\mu_i),$$

where $\mu_i = (i-1)2^{-(m+1)}$. For any design $\xi^* \in \Xi$ of the form (7), we have

$$M(\xi^*) \quad = \quad \frac{1}{2^{m+1}}\sum_{i=1}^{2^{m+1}}\boldsymbol{f}(\mu_i)\boldsymbol{f}^T(\mu_i) \quad = \quad \frac{1}{2^{m_2+1}}\left(M_{s_1,s_2}\right)_{1\le s_1,s_2\le 2^{m+1}},$$

where $M_{00} = 2^{m+1}$ and for $s_r = 2^{j_r} + k_r$ with $j_r \in \{0,1,\cdots,m\}$, $k_r \in \{0,1,\cdots,2^{j_r}-1\}$ and $r = 1,2$,

$$M_{0s_2} \quad = \quad \sum_{i=1}^{2^{m+1}}\psi_{j_2k_2}(\mu_i), \quad M_{s_10} \quad = \quad \sum_{i=1}^{2^{m+1}}\psi_{j_1k_1}(\mu_i), \quad M_{s_1s_2} \quad = \quad \sum_{i=1}^{2^{m+1}}\psi_{j_1k_1}(\mu_i)\psi_{j_2k_2}(\mu_i).$$

Now, we can get

$$
\begin{aligned}
M_{s_1 0} &= \sum_{i=1}^{2^{m+1}} \psi_{j_1 k_1}(\mu_i) = 2^{j_1/2} \sum_{i=1}^{2^{m+1}} \psi(2^{j_1}\mu_i - k_1) = 2^{j_1/2} \sum_{i=1}^{2^{m+1}} \psi((i-1)/2^{m-j_1+1} - k_1) \\
&= 2^{j_1/2} \sum_{i=2^{m-j_1+1}k_1+1}^{2^{m-j_1+1}(k_1+1)} \psi((i-1)/2^{m-j_1+1} - k_1) = 0,
\end{aligned}
$$

and similarly, $M_{0s_2} = 0$. Moreover,

$$
\begin{aligned}
M_{s_1 s_1} &= \sum_{i=1}^{2^{m+1}} (\psi_{j_1 k_1}(\mu_i))^2 = 2^{j_1} \sum_{i=1}^{2^{m+1}} (\psi(2^{j_1}\mu_i - k_1))^2 = 2^{j_1} \sum_{i=1}^{2^{m+1}} (\psi((i-1)/2^{m-j_1+1} - k_1))^2 \\
&= 2^{j_1} \sum_{i=2^{m-j_1+1}k_1+1}^{2^{m-j_1+1}(k_1+1)} (\psi((i-1)/2^{m-j_1+1} - k_1))^2 = 2^{m+1},
\end{aligned}
$$

and

$$
\begin{aligned}
M_{s_1 s_2} &= \sum_{i=1}^{2^{m+1}} \psi_{j_1 k_1}(\mu_i)\psi_{j_2 k_2}(\mu_i) = 2^{(j_1+j_2)/2} \sum_{i=1}^{2^{m+1}} \psi(2^{j_1}\mu_i - k_1)\psi(2^{j_2}\mu_i - k_2) \\
&= 2^{(j_1+j_2)/2} \sum_{i=1}^{2^{m+1}} \psi((i-1)/2^{m-j_1+1} - k_1)\psi((i-1)/2^{m-j_2+1} - k_2) = 0.
\end{aligned}
$$

Therefore, it is clear that $M(\xi^*) = I_{2^{m+1}}$, where I_n is the $n \times n$ identity matrix. It follows that, for any $x \in [0, 1]$,

$$
\begin{aligned}
\phi(x, \xi^*) &= \boldsymbol{f}^T(x) M^{-1}(\xi^*) \left(\sum_{i=1}^{p} \frac{\boldsymbol{e}_i \boldsymbol{e}_i^T}{\boldsymbol{e}_i^T M^{-1}(\xi^*) \boldsymbol{e}_i} \right) M^{-1}(\xi^*) \boldsymbol{f}(x) \\
&= \boldsymbol{f}^T(x)\boldsymbol{f}(x) = 1 + \sum_{j=0}^{m} \sum_{k=0}^{2^j-1} \psi_{jk}^2(x) = 1 + \sum_{j=0}^{m_1} 2^j = 2^{m+1}, \qquad (8)
\end{aligned}
$$

and then ξ^* is R-optimal from Theorem 1.

Example: Consider the R-optimal design for the Haar-wavelet model (1) of order $m = 2$, *i.e.*,

$$
\begin{aligned}
E[y(x)] = \beta_0 + \beta_{00}\psi_{00}(x) + \beta_{10}\psi_{10}(x) + \beta_{11}\psi_{11}(x) + \beta_{20}\psi_{20}(x) \\
+\beta_{21}\psi_{21}(x) + \beta_{22}\psi_{22}(x) + \beta_{23}\psi_{23}(x),
\end{aligned} \qquad (9)
$$

In this case, the design region $\mathcal{X} = [0, 1]$ is divided into $2^{m+1} = 8$ sub-intervals

$$
\mathcal{X}_i = \left[\frac{i-1}{8}, \frac{i}{8} \right), \quad i = 1, \cdots, 8.
$$

From Theorem 2, a R-optimal design for the model (9) is as follows:

$$
\xi^* = \begin{pmatrix} 0/8 & 1/8 & \cdots & 7/8 \\ 1/8 & 1/8 & \cdots & 1/8 \end{pmatrix}. \qquad (10)
$$

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