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Simple Proof of the Non-existence of Some Affine Resolvable SRGD Designs

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Abstract

The existence of resolvable and affine resolvable block designs has been discussed in the literature (cf. Clatworthy (1973), Raghavarao (1988)). Recently, a necessary condition for the existence of a certain resolvable pairwise balanced design is provided by Kadowaki and Kageyama (2020). In this paper, through the necessary condition, we can provide a simple proof of the non-existence result of some affine resolvable SRGD design, rather than by the usual methods in combinatorics.

Key words: Affine resolvability; Resolvability; PB design;SRGD design.

1. Introduction

A block design BD(v, b, r, k) with v treatments is said to be resolvable if the b blocks of size k each can be grouped into r resolution sets of b/r blocks each such that in each resolution set every treatment occurs exactly once. A resolvable BD is said to be affine resolvable if every two blocks belonging to different resolution sets intersect in the same number of treatments (cf. Raghavarao (1988)).

A BD(v, b, r, k) is called a group divisible (GD) design with parameters $v = mn, b, r, k, \lambda_1, \lambda_2$ if the mn treatments are divided into m groups of n treatments each such that any two treatments in the same group occur together in exactly λ_1 blocks, whereas any two treatments from different groups occur together in exactly λ_2 blocks. The GD designs are further classified into three subclasses: Singular if $r-\lambda_1 = 0$; Semi-Regular (SR) if $r-\lambda_1 > 0$ and $rk - v\lambda_2 = 0$; Regular if $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

A pairwise balanced (PB) design with parameters v, b, r, K, λ is a collection of b subsets (blocks) of a set of v treatments such that the size of each block is an element of a set K, each treatment occurs in exactly r different blocks and any two treatments occur in exactly λ blocks (cf. Colbourn and Dinitz (2007)).

Kadowaki and Kageyama (2009, 2010) produced comprehensive combinatorial findings on affine resolvable partially balanced incomplete block designs including SRGD designs.

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Recently, Kadowaki and Kageyama (2020) derived a necessary condition for the existence of a certain resolvable PB design. In this paper, through the necessary condition, we can simply prove the non-existence of certain affine resolvable SRGD designs, rather than by using the other usual methods in combinatorics.

2. Statements

Kadowaki and Kageyama (2020) have presented a necessary condition for the existence of a resolvable PB design, as the following shows.

Theorem 1: When $v \geq 3$ and $b = 2r$, if there exists a resolvable PB design with parameters v, b, r, K, λ, then $r - \lambda$ is even.

By use of Theorem 1, the following can be provided.

Theorem 2: In a resolvable GD design with parameters $v = mn, b = 2r, r, k, \lambda_1, \lambda_2$, it holds that

(1) when $n \geq 3, r - \lambda_1$ is even;

(2) when $m \geq 3, r - \lambda_2$ is even.

Proof: For the resolvable GD design, by taking n treatments of the first associates in a group of the GD association scheme as new treatments, a resolvable PB design with parameters $v^* = n, b^* = 2r^*, r^* = r, k_j^*, \lambda^* = \lambda_1$ can be obtained. Then Theorem 1 implies (1), because $r^* - \lambda^* = r - \lambda_1$. Similarly, by taking m treatments of the second associates in different groups of the GD association scheme as new treatments, a resolvable PB design with parameters $v^* = m, b^* = 2r^*, r^* = r, k_j^*, \lambda^* = \lambda_2$ can be obtained. Hence (2) is proved by Theorem 1. □

It should be remarked that the presentation of Theorem 2 is simply in terms of design parameters.

Now we look at affine resolvable SRGD designs. Since an affine resolvable design is also a resolvable design, Theorem 2 will be utilized to discuss the existence of affine resolvable SRGD designs.

More refinement of Theorem 2 is given.

Corollary 2.1: In an affine resolvable SRGD design with parameters $v = mn, b = 2r, r, k, \lambda_1, \lambda_2$, it holds that

(3) m and n are both even;

(4) for even m (≥ 2), the necessary condition (1) is always satisfied;

(5) for even n (≥ 2), the necessary condition (2) is satisfied if and only if $m \equiv 0 \pmod 4$. Then when $m \equiv 2 \pmod 4$, the corresponding affine resolvable SRGD design does not exist. In particular, when $n = 2$, the parameters are expressed by $v = b = 2m, r = k = m, \lambda_1 = 0, \lambda_2 = m/2$ for $m \equiv 0 \pmod 4$.

Proof: In an affine resolvable SRGD design with $v = mn$ and $b = 2r$, it is known that $b = v - m + r, rk = v\lambda_2$ and $r(k-1) = (n-1)\lambda_1 + n(m-1)\lambda_2$, which imply that $v = mn, b = 2m(n-1), r = m(n-1), k = mn/2, \lambda_1 = k - m$ and $\lambda_2 = k - k/n$. In an SRGD design, it is known that k is divisible by m. By use of these relations, Corollary 2.1 is proved. In fact, since k/m and k/n with $k = mn/2$ are integers, (3) is obtained. (4) For $m = 2l_1 (l_1 \geq 1)$, since $k/m = k/(2l_1)$ is an integer, k is even. Then $r - \lambda_1 = m(n-1) - k + m = 2l_1 n - k$ is even, i.e., (1) of Theorem 2. (5) For $n = 2l_2 (l_2 \geq 1)$, we have $v = 2l_2 m, r = m(2l_2 - 1), k = l_2 m$. Then $\lambda_2 = k - k/n = l_2 m - m/2$ which implies that m is even (=$2l_3$, say). Now $r - \lambda_2 = m(2l_2 - 1) - l_2 m + m/2 = l_2 m - m/2 = l_3(2l_2 - 1)$ is even if and only if l_3 is even, i.e., (2) of Theorem 2. Hence $m \equiv 0 \pmod 4$. □

Remark 1: In general, an affine resolvable SRGD design with $b = 2r$ and $\lambda_1 = 0$ has the parameters as $v = mn, b = 2m(n-1), r = m(n-1), k = m, \lambda_1 = 0, \lambda_2 = m - m/n$.

Some existence on affine resolvable SRGD designs with admissible parameters within the scope of $v = mn \leq 100$ and $r, k \leq 20$ has been investigated and tabulated by Kadowaki and Kageyama (2009, 2010). They showed the non-existence of 10 designs of Nos. 14, 16, 17, 18, 25, 27, 32, 34, 35, 38 in their Table 3.4. Such results are given by use of calculation of the Hilbert norm residue symbol (as in Theorem 12.6.2 of Raghavarao (1988)) and the application of existence on some Hadamard matrix and difference matrix (as in Remark 3.3.1 and Corollary 3.3.4 of Kadowaki and Kageyama (2009)). The derivation of these results is slightly complicated.

Among such 10 non-existence results in the Table 3.4, another proof of the non-existence on designs of Nos. 14, 16, 25, 32, 38 can be given by simply applying Theorem 2 or Corollary 2.1 as follows.

No	m	n	v	b	r	k	λ_1	λ_2	non-existence
14	6	2	12	12	6	6	0	3	Corollary 2.1
16	6	4	24	36	18	12	6	9	Theorem 2
25	10	2	20	20	10	10	0	5	Corollary 2.1
32	14	2	28	28	14	14	0	7	Corollary 2.1
38	18	2	36	36	18	18	0	9	Corollary 2.1

If we extend the investigation of affine resolvable SRGD designs of $b = 2r$ over the scope of $v \leq 200$ and $k \leq 100$, then a large number of admissible parameters of the design are found. Some are constructed and some are unknown on existence. Also some are non-existent.

Among such non-existent designs, the following parameters as

$$v = 136, m = 34, n = 4, b = 204, r = 102, k = 68, \lambda_1 = 34, \lambda_2 = 51$$

cannot be applied for Remark 3.3.1 and Corollary 3.3.4 of Kadowaki and Kageyama (2009), and also for Theorem 12.6.2 of Raghavarao (1988). The present Corollary 2.1 (5) can be only applied to show the non-existence.

The above observation reveals that Theorem 2 or Corollary 2.1 is powerful to show the non-existence of designs, rather than by the usual methods in combinatorics.

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