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# Regular Group Divisible Designs Using Symmetric Groups

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**Abstract**

Two regular group divisible designs with parameters: $v = 30$, $b = 60$, $r = 8$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = 5$, $n = 6$ and $v = 36$, $b = 90$, $r = 10$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = n = 6$ in the range of $r, k \leq 10$ are obtained from generalized Bhaskar Rao designs over a symmetric group of order 6.

*Key words*: Regular group divisible designs; Generalized Bhaskar Rao designs; Symmetric groups.

**MSC:** 62K10; 05B05

## 1. Introduction

Saurabh and Sinha (2021) obtained a new regular group divisible (*RGD*) design with parameters: $v = b = 39$, $r = k = 9$, $\lambda_1 = 0$, $\lambda_2 = 2$, $m = 13$, $n = 3$ by replacing the group entries of *BGW* (13, 9, 6; $D_3$) by suitable permutation matrices of order 3. Here we have used the method of Gibbons and Mathon (1987) for the construction of group divisible designs. As a particular case we obtain two RGD designs with parameters: $v = 30$, $b = 60$, $r = 8$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = 5$, $n = 6$ and $v = 36$, $b = 90$, $r = 10$, $k = 4$, $\lambda_1 = 0$, $\lambda_2 = 1$, $m = n = 6$ in the range of $r, k \leq 10$. These designs may be considered new as these are not found in the tables of Clatworthy (1973) and Sinha (1991) but included in Saurabh and Sinha (2021).

A generalized Bhaskar Rao design *GBRD* ($v$, $b$, $r$, $k$, $\lambda$; $G$) over a group $G$ is a $v \times b$ array with entries from $G \cup \{0\}$such that:

1. each row has exactly $r$ group element entries;
2. each column has exactly $k$ group element entries;
3. for each pair of distinct rows $(x_1, x_2, \dots, x_b)$ and $(y_1, y_2, \dots, y_b)$, the multi–set $\{x_i y_i^{-1}: i = 1, 2, \dots, b;\ x_i, y_i \neq 0\}$ contains each group element exactly $\lambda/|G|$ times.

A generalized Bhaskar Rao design *GBRD* ($v, b, r, k, \lambda; G$) with $v = b$ and $r = k$ is known as a *balanced generalized Weighing matrix BGW* ($v, k, \lambda; G$).

A *RGD design* is an arrangement of $v = mn$ elements in $b$ blocks such that:

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(i) each block contains $k$ ($< v$) distinct elements;
(ii) each element occurs $r$ times;
(iii) the elements can be divided into $m$ groups each of size $n$, any two distinct elements occurring together in $\lambda_1$ blocks if they belong to the same group, and in $\lambda_2$ blocks if they belong to the different groups;
(iv) $r - \lambda_1 > 0$ and $rk - v\lambda_2 > 0$.

Let $\mathbf{N}$ be the incidence matrix of a RGD design then the structure of $\mathbf{NN}^T$ is given as: $\mathbf{NN}^T = (r - \lambda_1)(\mathbf{I}_m \otimes \mathbf{I}_n) + (\lambda_1 - \lambda_2)(\mathbf{I}_m \otimes \mathbf{J}_n) + \lambda_2(\mathbf{J}_m \otimes \mathbf{J}_n)$ where $\mathbf{A} \otimes \mathbf{B}$ denotes the Kronecker product of two matrices $\mathbf{A}$ and $\mathbf{B}$. For details on RGD designs, see Clatworthy (1973) and Saurabh *et al.* (2021).

*Notations*: $\mathbf{I}_n$ is the identity matrix of order $n$, $\mathbf{J}_v$ is the $v \times v$ matrix all whose entries are 1 and $\mathbf{A}^T$ is the transpose of matrix $\mathbf{A}$. $S_n$ and $D_n$ denote symmetric and dihedral groups with orders $n!$ and $2n$ respectively. For $n = 3$, $S_n$ is isomorphic to the dihedral group $D_n$.

## 2. Two new RGD designs in the range of $r, k \leq 10$

Gibbons and Mathon (1987) gave the following method for the construction of GD designs from GBRD ($v, b, r, k, \lambda; G$):

Replacing the elements of a group $G$ of order $g$ by the corresponding $g$ x $g$ permutation matrices and 0 entry by $g$ x $g$ null matrix in GBRD ($v, b, r, k, \lambda; G$), we obtain a GD design with parameters: $v^* = vg,\ b^* = bg, r^* = r,\ k^* = k, \lambda_1 = 0, \lambda_2 = \lambda/g\,, m = v, n = g.$ (1)

In the above method, Palmer and Seberry (1988) used permutation group of order 6 and dihedral groups of order 8 and 12 while Sarvate and Seberry (1998) used elementary abelian groups for the construction of GD designs.

Following Palmer and Seberry (1988): The existence of a GBRD ($v, b, r, k, \lambda; S_3$) implies the existence of a GD design with parameters:

$$v^* = 6v,\ b^* = 6b, r^* = r,\ k^* = k, \lambda_1 = 0, \lambda_2 = \lambda/6\,, m = v, n = 6. \qquad (2)$$

The above construction procedure may be generalized for any symmetric / dihedral groups but no series of GBRD ($v, b, r, k, \lambda; S_n/ D_n$) is available for $n > 3$. Using GBRD (5, 10, 8, 4, 6; $S_3$) and GBRD (6, 15, 10, 4, 6; $S_3$) from Abel *et al.* (2004) in (2), we obtain the following RGD designs:

**Design 1:** Consider a symmetric group $S_3 = \langle r, s: r^3 = s^2 = e, sr = r^2 s\rangle = \{e, r, r^2, s, sr, sr^2\}$.

The following is a GBRD (5, 10, 8, 4, 6; $S_3$):

$$\mathbf{A} = \begin{bmatrix} e & s & r & 0 & e & e & r^2 & e & 0 & r^2s \\ e & e & s & r & 0 & r^2s & e & r^2 & e & 0 \\ 0 & e & e & s & r & 0 & r^2s & e & r^2 & e \\ r & 0 & e & e & s & e & 0 & r^2s & e & r^2 \\ s & r & 0 & e & e & r^2 & e & 0 & r^2s & e \end{bmatrix}.$$

Replacing 0 by a null matrix of order 6 and the group elements $e, r, r^2, s, sr = r^2s, sr^2 = rs$ by

the 6 $\times$ 6 permutation matrices $\mathbf{I}_6$, $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$,

$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$ respectively in $A$,

we obtain a (0, 1) – matrix **N** of order $30 \times 60$. Then $\mathbf{NN}^T = \text{circ}\,(8I_6, J_6, J_6, J_6, J_6) = 8\mathbf{I}_{30} - \mathbf{I}_5 \otimes \mathbf{J}_6 + \mathbf{J}_5 \otimes \mathbf{J}_6$. Also each column sum of **N** is 4. Hence **N** represents a RGD design with parameters: $v$ = 30, $b$ = 60, $r$ = 8, $k$ = 4, $\lambda_1$ = 0, $\lambda_2$ = 1, $m$ = 5, $n$ = 6.

**Design 2:** Further consider the following GBRD (6, 15, 10, 4, 6; $S_3$):

$$\mathbf{B} = \begin{bmatrix} e & e & e & e & e & e & e & e & e & e & 0 & 0 & 0 & 0 & 0 \\ rs & s & e & r & r^2 & r^2s & 0 & 0 & o & 0 & e & e & e & e & 0 \\ r & s & rs & 0 & 0 & 0 & r^2 & e & r^2s & 0 & s & r^2 & r & 0 & e \\ e & 0 & 0 & r & rs & 0 & r^2 & s & 0 & r^2s & r^2 & r^2s & 0 & r & r \\ 0 & e & 0 & rs & 0 & r^2 & r^2s & 0 & s & r & r^2 & 0 & r & e & r^2 \\ 0 & 0 & r & 0 & s & r^2 & 0 & e & rs & r^2s & 0 & e & s & r^2 & s \end{bmatrix}.$$

Replacing the group elements $e, r, r^2, s, sr = r^2s, sr^2 = rs$ by $6 \times 6$ matrices given as above and 0 by a null matrix of order 6 in **B**, we obtain a (0, 1) – matrix **N** of order $36 \times 90$. Then $\mathbf{NN}^T = \text{circ}\,(10\mathbf{I}_6, \mathbf{J}_6, \mathbf{J}_6, \mathbf{J}_6, \mathbf{J}_6, \mathbf{J}_6) = 10\mathbf{I}_{36} - \mathbf{I}_6 \otimes \mathbf{J}_6 + \mathbf{J}_6 \otimes \mathbf{J}_6$. Also each column sum of **N** is 4. Hence **N** represents a RGD design with parameters: $v$ = 36, $b$ = 90, $r$ = 10, $k$ = 4, $\lambda_1$ = 0, $\lambda_2$ = 1, $m$ = $n$ = 6.

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