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R-optimal Designs for Gamma Regression Model with Two Parameters

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Abstract

This article finds locally R-optimal designs for the gamma regression model having two parameters using the inverse link function. The R-optimality criterion has been proposed in the literature as an alternative criterion to the well-known D-optimality criterion when the target is to minimize the volume of the confidence region for unknown parameters based on the Bonferroni t-intervals. The optimality of the proposed designs is confirmed using the corresponding equivalence theorem.

Key words: Locally R-optimal design; Gamma regression model; Inverse link function; Bonferroni t-intervals; Equivalence theorem.

AMS Subject Classifications: 62K05

1. Introduction

The Generalized Linear Model (GLM), introduced by Nelder and Wedderburn (1972) is a generalized version of the ordinary linear regression model. The GLM has extensive applications in various disciplines of science such as clinical trials, engineering, reliability, survival analysis, image analysis, bioinformatics, economics, insurance, agriculture, and industry. For more details on the applications of GLM, one can refer to the articles of Bailey *et al.* (1960), Myers and Montgomery (1997), de Jong and Heller (2008), Fox (2015), and Goldburd (2016).

The Gamma regression model is a particular form of GLM. This model is useful when the responses are continuous, non-negative, and right-skewed type. There are many instances in the literature where the gamma model with an appropriate link function has been used to analyze the real data. The data analysis of car insurance claims (*pg.* 296, McCullagh and Nelder, 1989) and clotting times of blood (*pg.* 300, McCullagh and Nelder, 1989) was carried out by fitting a first-order Gamma model with the natural link function. Anderson *et al.* (2010) used a first-order gamma model with a natural link function to analyze the reaction time taken by the elders to recognize words on a computer monitor.

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In experimental design, the target for constructing an optimal design is to make the predicted response closer to the mean response over a certain region of interest based on a specific criterion of interest. For the seminal work on optimal designs, one can refer to the work of Kiefer and Wolfowitz (1959), and Kiefer (1959). In the case of GLM, finding the optimal designs becomes a very difficult task because the optimal design depends on the unknown values of the model parameters. In this context, Chernoff (1953) proposed an alternative way of finding optimal design by starting with an initial guess value of parameter values that can lead to locally optimal designs.

Ford *et al.* (1992) obtained a locally D-optimal design for the Gamma regression model that involves a single factor. Subsequently, Burridge and Sebastiani(1992) found the locally D-optimal design for the Gamma model with two factors but without an intercept. Burridge and Sebastiani (1994) obtained the same D-optimal design for the Gamma regression model which involves multiple factors. Aminenjad and Jafari (2017) found Bayesian A- and D-optimal designs for the Gamma model with inverse link function by considering various prior distributions such as Normal, Half-normal, Gamma, and Uniform distributions. Gaffke *et al.* (2019) provided analytical solutions to derive locally D- and A-optimal designs for the Gamma models that involve intercept terms. They also established that the derived designs are essentially a complete class of designs. Idais and Schwabe (2021) found locally D- and A-optimal designs for the Gamma models having linear predictors without intercept. Idais (2021) obtained D-, A-, and Kiefer's Φ_k -criteria optimality for vertex-type designs.

In experimental design, the D-optimality criterion is the most widely used optimal design criterion. The geometrical interpretation of the D-optimality criterion is to minimize the volume of the confidence ellipsoid region of the unknown parameters (*see* Silvey, 1980). However, computation of the D-optimal design for a regression model becomes simple if the number of parameters associated with the given model is small, let's say 2 or 3. In this perspective, an alternative design known as the R-optimal design was introduced by Dette (1997). This design aims at minimizing the volume of the Bonferroni t-intervals. Recently, many authors have obtained R-optimal designs for different types of regression models *e.g.*, second-order response surface models (Liu *et al.*, 2016), multi-factor models with heteroscedastic errors (He and Yue, 2017), multi-response regression models with multiple factors (Liu *et al.*, 2022), and models with mixture experiments (Panda, 2021; Panda and Sahoo, 2024). To the best of our knowledge, the construction of R-optimal designs for GLM has not been discussed yet in the literature except for the work of Panda and Biswal (2024). In this context, the present article aims to construct locally R-optimal designs for the Gamma Model with two parameters including the intercept parameter.

The rest of the article is organized as follows. Section 2 provides the model specification as well as brief details on locally R-optimal designs. In Section 3, we obtain R-optimal designs for the Gamma model with two parameters. Finally, the article is concluded with some discussions and conclusions in Section 4.

2. Model specification and locally R-optimal designs

Let the response variables $Y_1, Y_2, \ldots, Y_n$ are assumed to be independent gamma-distributed random variables *i.e.,* the probability density function (*p.d.f.*) of each Y_i

$$p(y_i; \nu) = \frac{1}{\Gamma(\nu)} y_i^{\nu-1} e^{-y_i},\ y_i,\ \nu > 0,\ i = 1, 2, \ldots, n\,. \tag{1}$$

Here ν is the shape parameter associated with the $p.d.f$ as specified in equation (1). It is assumed to be known and the same for all y_i. However the expected value $i.e.$ μ_i depends on the values of $\boldsymbol{x_i}$ the covariate of $\boldsymbol{x}$. The canonical link for the Gamma distribution given by Equation (1) is the inverse link function defined as

$$\eta_i = \frac{\nu}{\mu_i},\ \ where\ \ \eta_i = \boldsymbol{g'(x_i)\beta},\ i = 1, 2, \ldots, n \tag{2}$$

is the linear predictor. In Equation (2), $\boldsymbol{g} = [g_1, g_2, \ldots, g_p]'$ is a p-dimensional vector valued function defined on a domain set $\Xi \subset \mathbb{R}^t, t \geq 1$. Here the component functions $g_1, g_2, \ldots, g_p$ are assumed to be linearly independent, and $\boldsymbol{\beta} \in \mathbb{R}^p$ are assumed to be a p-dimensional vector consisting of unknown parameters associated with the model Equation (2).

In this case, the variance function of the gamma distribution is $Var(Y) = \nu^{-1}\mu^2$ therefore the intensity function at a particular point $\boldsymbol{x} \in \Xi$ (*see* Atkison and Woods, 2015) can be defined as

$$u_0(\boldsymbol{x}, \boldsymbol{\beta}) = \left(Var(Y) \left(\frac{d\eta}{d\mu} \right)^2 \right)^{-1} = \nu(\boldsymbol{g'(x)\beta})^{-2}. \tag{3}$$

As the gamma-distributed responses are continuous and non-negative and thus for a given experimental region Ξ we assume throughout that the parameter vector $\boldsymbol{\beta}$ satisfies

$$\boldsymbol{g'(x)\beta} > 0\ \ for\ all\ \ \boldsymbol{x} \in \Xi\,. \tag{4}$$

For the model Equation (2), the Fisher information matrix at $\boldsymbol{x}$ and $\boldsymbol{\beta}$ can be defined as

$$\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{\beta}) = u(\boldsymbol{x}, \boldsymbol{\beta})\boldsymbol{g(x)g'(x)} \quad where \quad u(\boldsymbol{x}, \beta) = (\boldsymbol{g'(x)\beta})^{-2}. \tag{5}$$

For more details about the assumption made in Equation (4) and the information matrix defined in Equation (5), one can refer to the articles of Gaffke *et al.* (2019) and Idais *et al.* (2021).

For a given parameter value, let us define $\boldsymbol{g}_\beta$ as the local regression function then

$$\boldsymbol{g}_\beta(\boldsymbol{x}) = (\boldsymbol{g'(x)\beta})^{-1}\boldsymbol{g(x)}\ \ forall\ \ \boldsymbol{x} \in \Xi\,. \tag{6}$$

Using Equation (6), the Fisher information matrix in model Equation (5) can be rewritten as

$$\boldsymbol{M}(\boldsymbol{x}, \boldsymbol{\beta}) = \boldsymbol{g(x)g'(x)}. \tag{7}$$

To obtain the R-optimal design for the model Equation (2), we consider the approximate design $\xi \in \Omega$ (Ω the set of all approximate designs) of the form

$$\xi = \begin{Bmatrix} \boldsymbol{x}_1 & \dots & \boldsymbol{x}_s \\ w_1 & \dots & w_s \end{Bmatrix}, \ \ w_i(>0) \ \ and \ \ \sum_{i=1}^{s} w_i = 1 \tag{8}$$

where $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_s \in \Xi$ are the 's' distinct points and w_i is the weight associated with the point $\boldsymbol{x_i}$ for $i = 1, 2, \dots, s$. For the model Equation (2), the Fisher information matrix of a design ξ at parameter vector $\boldsymbol{\beta}$ is defined as

$$\boldsymbol{M}(\xi, \boldsymbol{\beta}) = \sum_{i=1}^{s} w_i \boldsymbol{M}(\boldsymbol{x}_i, \boldsymbol{\beta}). \tag{9}$$

R-optimal design: A design $\xi \in \Omega$ with a non-singular information matrix $\boldsymbol{M}(\xi)$ is called R-optimal for the model Equation (2) if it minimizes

$$\phi(\xi) = \prod_{i=1}^{p} (\boldsymbol{M}^{-1}(\xi))_{ii} = \prod_{i=1}^{p} \boldsymbol{e_i'} \boldsymbol{M}^{-1}(\xi) \boldsymbol{e_i} \tag{10}$$

for all $\xi \in \Omega$. Here $\boldsymbol{e_i}$ denotes the i^{th} unit vector in $\mathbb{R}^p$, where p is the number of unknown parameters associated with the model Equation (2). The necessary and sufficient conditions for the R-optimality will be examined using the following equivalence theorem. For further details, one can refer to the article of Dette (1997).

Theorem 1: For model Equation (2), let

$$\varphi(\boldsymbol{x}, \xi) = \boldsymbol{g'(x)} \boldsymbol{M}^{-1}(\xi) \left(\sum_{i=1}^{p} \frac{\boldsymbol{e_i e_i'}}{\boldsymbol{e_i} \boldsymbol{M}^{-1}(\xi^*) \boldsymbol{e_i'}} \right) \boldsymbol{M}^{-1}(\xi) \boldsymbol{g(x)}. \tag{11}$$

A design $\xi^* \in \Omega$ is R-optimal if and only if

$$\sup_{\boldsymbol{x} \in \Xi} \ \varphi(\boldsymbol{x}, \xi^*) = p$$

with equality attained at the support points of ξ^*.

3. R-optimal designs

In this section, we obtain locally R-optimal designs for the model Equation (2) that involves two unknown parameters including the intercept parameter. Thus the assumption in Equation (4) becomes

$$\boldsymbol{g'}(x)\boldsymbol{\beta} = \beta_0 + \beta_1 x > 0$$

for all $x \in R$. Here, we restrict our search to two-, three-, and four-support points design by considering discrete values of β_0 and β_1 in the arbitrarily chosen intervals $[0, 10]$ and $[0, 100]$ respectively.

3.1. Design based on two support points

Let us consider a 2-point design ξ of the form

$$\xi = \begin{Bmatrix} a & b \\ w & 1-w \end{Bmatrix}, \quad \text{where} \quad 0 < w < 1. \tag{12}$$

Theorem 2: The design ξ^* that assigns a weight of w^* to the point a^* and $1-w^*$ to the point b^* in $\mathbb{R}$ [the numerical values of a^*, b^* and w^* are given in Table 1 (Appendix-I)] is an R-optimal design.

Proof: Using Equation (9), the information matrix for the model Equation (2) at the two-point design ξ will be

$$\boldsymbol{M}(\xi) = \begin{bmatrix} \dfrac{1-w}{(\beta_0+b\beta_1)^2} + \dfrac{w}{(\beta_0+a\beta_1)^2} & \dfrac{b(1-w)}{(\beta_0+b\beta_1)^2} + \dfrac{aw}{(\beta_0+a\beta_1)^2} \\ \dfrac{b(1-w)}{(\beta_0+b\beta_1)^2} + \dfrac{aw}{(\beta_0+a\beta_1)^2} & \dfrac{b^2(1-w)}{(\beta_0+b\beta_1)^2} + \dfrac{a^2w}{(\beta_0+a\beta_1)^2} \end{bmatrix}.$$

The inverse of the information matrix $\boldsymbol{M}(\xi)$ is given by

$$\boldsymbol{M}^{-1}(\xi) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \tag{13}$$

where

$$m_{11} = \frac{-b^2(\beta_0+a\beta_1)^2 + (b-a)\beta_0((a+b)\beta_0 + 2ab\beta_1)w}{(a-b)^2(-1+w)w},$$

$$m_{12} = m_{21} = \frac{b(\beta_0+a\beta_1)^2 + (b-a)(-\beta_0^2 + ab\beta_1^2)w}{(a-b)^2(-1+w)w},$$

and $$m_{22} = \frac{-(\beta_0+a\beta_1)^2 + (a-b)\beta_1(2\beta_0 + (a+b)\beta_1)w}{(a-b)^2(-1+w)w}.$$

Using Equation (10), we obtain

$$\phi(\xi) = \frac{\begin{aligned}\Big[\Big\{-b^2(\beta_0+a\beta_1)^2 + (b-a)\beta_0((a+b)\beta_0 + 2ab\beta_1)w\Big\} \\ \times \Big\{-(\beta_0+a\beta_1)^2 + (a-b)\beta_1(2\beta_0 + (a+b)\beta_1)w\Big\}\Big]\end{aligned}}{(a-b)^4(-1+w)^2w^2}. \tag{14}$$

Now, the problem is to minimize the function $\phi(\xi)$ $w.r.t$ a, b and w for given values of β_0 and β_1. This is done using the "fminsearch" function of Matlab software and getting the optimal values a^*, b^* and w^* by discrete values of β_0 and β_1 in the arbitrarily chosen intervals $[0, 10]$ and $[0, 100]$ respectively. The numerical values a^*, b^* and w^* are given in Table 1 (Appendix-I).

Next, by using Equation (13) we derive the quadratic form as specified in Equation (11) which is as follows:

$$\varphi(\boldsymbol{x},\xi) = \frac{1}{(\beta_0+\beta_1 x)^2}\Big\{m_{11}+m_{12}x+\frac{(b(\beta_0+a\beta_1)^2+(b-a)(-\beta_0^2+ab\beta_1^2)w)(m_{12}+m_{22}x)}{-(\beta_0+a\beta_1)^2+(a-b)\beta_1(2\beta_0+(a+b)\beta_1)w} + x\left(m_{12}+m_{22}x+\frac{(b(\beta_0+a\beta_1)^2+(b-a)(-\beta_0^2+ab\beta_1^2)w)(m_{11}+m_{12}x)}{-b^2(\beta_0+a\beta_1)^2+(b-a)\beta_0((a+b)\beta_0+2ab\beta_1)w}\right)\Big\}. \tag{15}$$

Replacing the numerical values of a^*, b^* and w^* in Equation (15) and using the "fminsearch" function of Matlab software we find $\sup_{\boldsymbol{x}\in\boldsymbol{R}} \varphi(\boldsymbol{x},\xi^*) = 2$. Thus the necessary and sufficient condition of the equivalence theorem is established. This proves Theorem 2. □

3.2. Design based on three support points

Let us consider a 3-point design ξ of the form

$$\xi = \begin{Bmatrix} a & b & c \\ w/2 & 1-w & w/2 \end{Bmatrix}, \quad where \;\; 0 < w < 1. \tag{16}$$

Theorem 3: The design ξ^* that assigns a weight of $w^*/2$ to the point a^*, $1-w^*$ to the point b^*, and $w^*/2$ to the point c^* in $\mathbb{R}$ [the numerical values of a^*, b^*, c^* and w^* are given in Table 2 (Appendix-I)] is an R-optimal design.

Proof: Using Equation (9), the information matrix for the model Equation (2) at the three-point design ξ will be

$$\boldsymbol{M}(\xi) = \begin{bmatrix} \frac{1-w}{(\beta_0+b\beta_1)^2}+\frac{w}{2(\beta_0+a\beta_1)^2}+\frac{w}{2(\beta_0+c\beta_1)^2} & \frac{b(1-w)}{(\beta_0+b\beta_1)^2}+\frac{aw}{2(\beta_0+a\beta_1)^2}+\frac{cw}{2(\beta_0+c\beta_1)^2} \\ \frac{b(1-w)}{(\beta_0+b\beta_1)^2}+\frac{aw}{2(\beta_0+a\beta_1)^2}+\frac{cw}{2(\beta_0+c\beta_1)^2} & \frac{b^2(1-w)}{(\beta_0+b\beta_1)^2}+\frac{a^2w}{2(\beta_0+a\beta_1)^2}+\frac{c^2w}{2(\beta_0+c\beta_1)^2} \end{bmatrix}.$$

The inverse of the information matrix $\boldsymbol{M}(\xi)$ is given by

$$\boldsymbol{M}^{-1}(\xi) = \begin{bmatrix} m_{11}^* & m_{12}^* \\ m_{21}^* & m_{22}^* \end{bmatrix} \tag{17}$$

where

$$m_{11}^* = \frac{\alpha_1}{\alpha_2+(\alpha_3\times\alpha_4)},$$

$$m_{12}^* = m_{21}^* = \frac{\alpha_5}{\alpha_2+(\alpha_3\times\alpha_4)},$$

and $m_{22}^* = \dfrac{2\alpha_6}{\alpha_2+(\alpha_3\times\alpha_4)}$,

with

$$\alpha_1 = 2\left(-\frac{2b^2(1-w)}{(\beta_0+b\beta_1)^2} + \frac{a^2w}{(\beta_0+a\beta_1)^2} + \frac{c^2w}{(\beta_0+c\beta_1)^2}\right),$$

$$\alpha_2 = -\left(-\frac{2b(w-1)}{(\beta_0+b\beta_1)^2} + \frac{aw}{(\beta_0+a\beta_1)^2} + \frac{cw}{(\beta_0+c\beta_1)^2}\right)^2,$$

$$\alpha_3 = \left(-\frac{2(w-1)}{(\beta_0+b\beta_1)^2} + \frac{w}{(\beta_0+a\beta_1)^2} + \frac{w}{(\beta_0+c\beta_1)^2}\right),$$

$$\alpha_4 = \left(-\frac{2b^2(w-1)}{(\beta_0+b\beta_1)^2} + \frac{a^2w}{(\beta_0+a\beta_1)^2} + \frac{c^2w}{(\beta_0+c\beta_1)^2}\right),$$

$$\alpha_5 = 4\left(\frac{b(w-1)}{(\beta_0+b\beta_1)^2} + \frac{1}{2}\left(-\frac{a}{(\beta_0+a\beta_1)^2} - \frac{c}{(\beta_0+c\beta_1)^2}\right)w\right),$$

and $$\alpha_6 = \left(-\frac{2(1-w)}{(\beta_0+b\beta_1)^2} + \frac{w}{(\beta_0+a\beta_1)^2} + \frac{w}{(\beta_0+c\beta_1)^2}\right).$$

Using Equation (10), we obtain the function

$$\phi(\xi) = \frac{4(\alpha_3 \times \alpha_4)}{\{\alpha_2 - (\alpha_3 \times \alpha_4)\}^2}\,. \tag{18}$$

Next, we need to minimize the function $\phi(\xi)$ *w.r.t* a, b, c and w for given values of β_0 and β_1. This is achieved by using the "fminsearch" function of Matlab software and getting the optimal values a^*, b^*, c^* and w^* by discrete values of β_0 and β_1 in the arbitrarily chosen intervals $[0, 10]$ and $[0, 100]$ respectively. The numerical values a^*, b^*, c^* and w^* are given in Table 2 (Appendix-I).

Next, by using Equation (17) we derive the quadratic form as specified in Equation (11) which is as follows:

$$\begin{aligned}\varphi(\boldsymbol{x},\xi) = \frac{1}{(\beta_0+b\beta_1)^2}\Bigg\{ & m_{11}^* + m_{21}^*x + \left(\frac{2(\alpha_7)(m_{21}^*+m_{22}^*x)}{(\alpha_6)}\right) \\ & + x\left(m_{21}^* + m_{22}^*x + \left(\frac{2(\alpha_7)(m_{11}^*+m_{21}^*x)}{(\alpha_4)}\right)\right)\Bigg\}\end{aligned} \tag{19}$$

with

$$\alpha_7 = \frac{b(w-1)}{(\beta_0+b\beta_1)^2} + \frac{1}{2}\left(-\frac{a}{(\beta_0+a\beta_1)^2} - \frac{c}{(\beta_0+c\beta_1)^2}\right)w.$$

Replacing the numerical values of a^*, b^*, c^* and w^* in Equation (19) and using the "fminsearch" function in Matlab software we find $\sup_{\boldsymbol{x}\in\boldsymbol{R}} \varphi(\boldsymbol{x},\xi^*) = 2$. Thus the necessary and sufficient condition of the equivalence theorem is established. This proves Theorem 3. $\square$

3.3. Design based on four support points

Let us consider a 4-point design ξ of the form

$$\xi = \begin{Bmatrix} a & b & c & d \\ w & \left(\frac{1}{2} - w\right) & \left(\frac{1}{2} - w\right) & w \end{Bmatrix}, \quad where \quad 0 < w < 1. \tag{20}$$

Theorem 4: The design ξ^* that assigns a weight of w^* to the point a^*, $(1/2) - w^*$ to the point b^*, $(1/2) - w^*$ to the point c^* and w^* to the point d^* in $\mathbb{R}$ [the numerical values of a^*, b^*, c^*, d^* and w^* are given in Table 3 (Appendix-I)] is an R-optimal design.

Proof: Using Equation (9), the information matrix for the model Equation (2) at the four-point design ξ will be

$$\boldsymbol{M}(\xi) = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_2 & \lambda_3 \end{bmatrix}$$

where

$$\lambda_1 = \frac{\frac{1}{2} - w}{(\beta_0 + b\beta_1)^2} + \frac{\frac{1}{2} - w}{(\beta_0 + c\beta_1)^2} + \frac{w}{(\beta_0 + a\beta_1)^2} + \frac{w}{(\beta_0 + d\beta_1)^2},$$

$$\lambda_2 = \frac{b\left(\frac{1}{2} - w\right)}{(\beta_0 + b\beta_1)^2} + \frac{c\left(\frac{1}{2} - w\right)}{(\beta_0 + c\beta_1)^2} + \frac{aw}{(\beta_0 + a\beta_1)^2} + \frac{dw}{(\beta_0 + d\beta_1)^2},$$

and $$\lambda_3 = \frac{b^2\left(\frac{1}{2} - w\right)}{(\beta_0 + b\beta_1)^2} + \frac{c^2\left(\frac{1}{2} - w\right)}{(\beta_0 + c\beta_1)^2} + \frac{a^2w}{(\beta_0 + a\beta_1)^2} + \frac{d^2w}{(\beta_0 + d\beta_1)^2}.$$

The inverse of the information matrix $\boldsymbol{M}(\xi)$ is given by

$$\boldsymbol{M}^{-1}(\xi) = \begin{bmatrix} m_{11}^+ & m_{12}^+ \\ m_{21}^+ & m_{22}^+ \end{bmatrix} \tag{21}$$

with

$$m_{11}^+ = \frac{\lambda_3}{-(\lambda_2)^2 + (\lambda_1 \times \lambda_3)},$$

$$m_{12}^+ = m_{21}^+ = \frac{\lambda_4}{2\{-(\lambda_2)^2 + (\lambda_1 \times \lambda_3)\}},$$

and $$m_{22}^+ = \frac{\lambda_1}{-(\lambda_2)^2 + (\lambda_1 \times \lambda_3)} .$$

Using Equation (10), we obtain the function

$$\phi(\xi) = \frac{\lambda_1 \times \lambda_3}{\{(\lambda_2)^2 - (\lambda_1 \times \lambda_3)\}^2} . \tag{22}$$

Now, the problem reduces to minimizing the function $\phi(\xi)$ *w.r.t* a, b, c, d and w for given values of β_0 and β_1. This is achieved by using the "fminsearch" function of Matlab software and getting the optimal values a^*, b^*, c^*, d^* and w^* by discrete values of β_0 and β_1 in the arbitrarily chosen intervals $[0, 10]$ and $[0, 100]$ respectively. The numerical values a^*, b^*, c^*, d^* and w^* are given in Table 3 (Appendix-I).

Next, by using Equation (21) we derive the quadratic form as specified in Equation (11) which is as follows:

$$\begin{aligned}\varphi(\boldsymbol{x}, \xi) = \frac{1}{(\beta_0 + \beta_1 x)^2}\bigg\{ & m_{11}^{+} + m_{12}^{+}x + \frac{\lambda_4(m_{12}^{+} + m_{22}^{+}x)}{2\lambda_1} \\ & + x\Big(m_{12}^{+} + m_{22}^{+}x + \frac{\lambda_4(m_{11}^{+} + m_{12}^{+}x)}{2\lambda_3}\Big)\bigg\}\end{aligned} \tag{23}$$

with $\lambda_4 = \dfrac{-2aw}{(\beta_0 + a\beta_1)^2} + \dfrac{-2dw}{(\beta_0 + d\beta_1)^2} + \dfrac{b(2w-1)}{(\beta_0 + b\beta_1)^2} + \dfrac{c(2w-1)}{(\beta_0 + c\beta_1)^2}$.

Replacing the numerical values of a^*, b^*, c^*, d^* and w^* in Equation (23) and using the "fminsearch" function of Matlab software we find $\underset{\boldsymbol{x}\in\boldsymbol{R}}{sup}\,\varphi(\boldsymbol{x}, \xi^*) = 2$. Thus the necessary and sufficient condition of the equivalence theorem is established. This proves Theorem 4. □

4. Discussion and conclusion

This article finds locally R-optimal designs for two parameters Gamma regression model when the model is associated with inverse link function based on two-, three-, and four-support point designs. The support points of the optimal designs and the weights assigned to these points are calculated numerically using the "fminsearch" function of Matlab software whereas the necessary and sufficient condition of R-optimality *i.e.* the equivalence theorem is also established at the support points of the R-optimal design using "fminsearch" function of Matlab software. A catalog of support points and the weight assigned to each of the support points corresponding to R-optimal designs are listed in Table 1, Table 2, and Table 3 (Appendix I). These Tables provide the solutions for only those values of β_0 and β_1 when the equivalence theorem is satisfied.

The present work considers three types of designs : (i) two-point designs (ii) three-point designs where equal weights are assigned to one-pair of support points (iii) four-point designs where equal weights are assigned to two-pair of support points. In all these cases, we observe that the equivalence theorem does not hold for many discrete values of the unknown parameters which indicates that the proposed designs are sensitive towards the R-optimality criterion with the varying parameter choices. However, when we relax the assumption of equal weights the optimal search criterion does not converge to any solution as the problem becomes complicated with an increase in the number of unknown entities (support points and weights). Therefore, more research work is required especially to propose an alternative optimal search criterion that can converge to a finite solution that satisfies the weight restriction as well. Nevertheless, the present work provides the necessary motivation to find the solution of local R-optimal designs for GLM when the parameters take continuous values.

For the two-support points design, we find that the support points lie in the third quadrant of the two-dimensional space. The values of the first coordinate and second coordinate of the support points are approximately equal.

One can extend this idea to obtain R-optimal designs for the Gamma model with more than two parameters.

Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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Appendix-I

Table 1: R-optimal design for Gamma regression model with two parameters (Two support points)

$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 2$	$\beta_0 = 1,\ \beta_1 = 3$	$\beta_0 = 1,\ \beta_1 = 4$	$\beta_0 = 1,\ \beta_1 = 5$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.5000 & -0.4999 \\ 0.1057 & 0.8943 \end{pmatrix}$	$\begin{pmatrix} -0.3333 & -0.3333 \\ 0.3042 & 0.6958 \end{pmatrix}$	$\begin{pmatrix} -0.2500 & -0.2499 \\ 0.5512 & 0.4488 \end{pmatrix}$	$\begin{pmatrix} -0.2000 & -0.1999 \\ 0.6098 & 0.3092 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 6$	$\beta_0 = 1,\ \beta_1 = 7$	$\beta_0 = 1,\ \beta_1 = 8$	$\beta_0 = 1,\ \beta_1 = 9$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1666 & -0.1666 \\ 0.6084 & 0.3916 \end{pmatrix}$	$\begin{pmatrix} -0.1428 & -0.1428 \\ 0.6067 & 0.3933 \end{pmatrix}$	$\begin{pmatrix} -0.1250 & -0.1250 \\ 0.5672 & 0.4328 \end{pmatrix}$	$\begin{pmatrix} -0.1111 & -0.1111 \\ 0.5824 & 0.4176 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 10$	$\beta_0 = 1,\ \beta_1 = 11$	$\beta_0 = 1,\ \beta_1 = 12$	$\beta_0 = 1,\ \beta_1 = 13$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1000 & -0.0999 \\ 0.6891 & 0.3109 \end{pmatrix}$	$\begin{pmatrix} -0.0909 & -0.0909 \\ 0.6094 & 0.3906 \end{pmatrix}$	$\begin{pmatrix} -0.0833 & -0.0833 \\ 0.6905 & 0.3095 \end{pmatrix}$	$\begin{pmatrix} -0.0769 & -0.0769 \\ 0.6137 & 0.3863 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 14$	$\beta_0 = 1,\ \beta_1 = 15$	$\beta_0 = 2,\ \beta_1 = 4$	$\beta_0 = 2,\ \beta_1 = 5$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.0714 & -0.0714 \\ 0.6344 & 0.3656 \end{pmatrix}$	$\begin{pmatrix} -0.0666 & -0.0666 \\ 0.6135 & 0.3865 \end{pmatrix}$	$\begin{pmatrix} -0.5000 & -0.4998 \\ 0.1058 & 0.8942 \end{pmatrix}$	$\begin{pmatrix} -0.4000 & -0.3999 \\ 0.2226 & 0.7774 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 6$	$\beta_0 = 2,\ \beta_1 = 7$	$\beta_0 = 2,\ \beta_1 = 8$	$\beta_0 = 2,\ \beta_1 = 9$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3333 & -0.3333 \\ 0.3043 & 0.6957 \end{pmatrix}$	$\begin{pmatrix} -0.2857 & -0.2856 \\ 0.1965 & 0.8035 \end{pmatrix}$	$\begin{pmatrix} -0.2500 & -0.2499 \\ 0.5512 & 0.4488 \end{pmatrix}$	$\begin{pmatrix} -0.2222 & -0.2222 \\ 0.5871 & 0.4129 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 10$	$\beta_0 = 2,\ \beta_1 = 11$	$\beta_0 = 2,\ \beta_1 = 12$	$\beta_0 = 2,\ \beta_1 = 13$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.2000 & -0.1999 \\ 0.6098 & 0.3902 \end{pmatrix}$	$\begin{pmatrix} -0.1818 & -0.1818 \\ 0.6003 & 0.3997 \end{pmatrix}$	$\begin{pmatrix} -0.1666 & -0.1666 \\ 0.6148 & 0.3852 \end{pmatrix}$	$\begin{pmatrix} -0.1538 & -0.1538 \\ 0.6397 & 0.3603 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 14$	$\beta_0 = 2,\ \beta_1 = 15$	$\beta_0 = 3,\ \beta_1 = 6$	$\beta_0 = 3,\ \beta_1 = 7$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1428 & -0.1428 \\ 0.6067 & 0.3933 \end{pmatrix}$	$\begin{pmatrix} -0.1333 & -0.1333 \\ 0.6100 & 0.3900 \end{pmatrix}$	$\begin{pmatrix} -0.5000 & -0.4997 \\ 0.1058 & 0.8942 \end{pmatrix}$	$\begin{pmatrix} -0.4285 & -0.4285 \\ 0.1702 & 0.8298 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 8$	$\beta_0 = 3,\ \beta_1 = 9$	$\beta_0 = 3,\ \beta_1 = 10$	$\beta_0 = 3,\ \beta_1 = 11$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3750 & -0.3749 \\ 0.2920 & 0.7080 \end{pmatrix}$	$\begin{pmatrix} -0.3333 & -0.3333 \\ 0.3043 & 0.6957 \end{pmatrix}$	$\begin{pmatrix} -0.3000 & -0.2999 \\ 0.3204 & 0.6796 \end{pmatrix}$	$\begin{pmatrix} -0.2727 & -0.2727 \\ 0.6455 & 0.3545 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 12$	$\beta_0 = 3,\ \beta_1 = 13$	$\beta_0 = 3,\ \beta_1 = 14$	$\beta_0 = 3,\ \beta_1 = 15$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.2500 & -0.2499 \\ 0.5512 & 0.4488 \end{pmatrix}$	$\begin{pmatrix} -0.2307 & -0.2307 \\ 0.5830 & 0.4170 \end{pmatrix}$	$\begin{pmatrix} -0.2142 & -0.2142 \\ 0.6218 & 0.3782 \end{pmatrix}$	$\begin{pmatrix} -0.2000 & -0.1999 \\ 0.6098 & 0.3902 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 7$	$\beta_0 = 4,\ \beta_1 = 8$	$\beta_0 = 4,\ \beta_1 = 10$	$\beta_0 = 4,\ \beta_1 = 11$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.5714 & -0.5715 \\ 0.0837 & 0.9163 \end{pmatrix}$	$\begin{pmatrix} -0.4999 & -0.5000 \\ 0.1061 & 0.8939 \end{pmatrix}$	$\begin{pmatrix} -0.4000 & -0.3999 \\ 0.2226 & 0.7774 \end{pmatrix}$	$\begin{pmatrix} -0.3636 & -0.3636 \\ 0.2373 & 0.7627 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 12$	$\beta_0 = 4,\ \beta_1 = 13$	$\beta_0 = 4,\ \beta_1 = 14$	$\beta_0 = 4,\ \beta_1 = 15$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3333 & -0.3333 \\ 0.3043 & 0.6957 \end{pmatrix}$	$\begin{pmatrix} -0.3076 & -0.3076 \\ 0.3441 & 0.6559 \end{pmatrix}$	$\begin{pmatrix} -0.2857 & -0.2856 \\ 0.1965 & 0.8035 \end{pmatrix}$	$\begin{pmatrix} -0.2666 & -0.2666 \\ 0.5384 & 0.4616 \end{pmatrix}$

Table 1: Continued

$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 10$	$\beta_0 = 5,\ \beta_1 = 11$	$\beta_0 = 5,\ \beta_1 = 12$	$\beta_0 = 5,\ \beta_1 = 13$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.5000 & -0.4997 \\ 0.1058 & 0.8942 \end{pmatrix}$	$\begin{pmatrix} -0.4545 & -0.4544 \\ 0.0894 & 0.9106 \end{pmatrix}$	$\begin{pmatrix} -0.4166 & -0.4167 \\ 0.1951 & 0.8049 \end{pmatrix}$	$\begin{pmatrix} -0.3846 & -0.3846 \\ 0.3048 & 0.6952 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 14$	$\beta_0 = 5,\ \beta_1 = 15$	$\beta_0 = 5,\ \beta_1 = 16$	$\beta_0 = 5,\ \beta_1 = 17$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3571 & -0.3571 \\ 0.3731 & 0.6269 \end{pmatrix}$	$\begin{pmatrix} -0.3333 & -0.3333 \\ 0.3043 & 0.6957 \end{pmatrix}$	$\begin{pmatrix} -0.3125 & -0.3124 \\ 0.3168 & 0.6832 \end{pmatrix}$	$\begin{pmatrix} -0.2941 & -0.2941 \\ 0.3197 & 0.6803 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 18$	$\beta_0 = 5,\ \beta_1 = 19$	$\beta_0 = 5,\ \beta_1 = 20$	$\beta_0 = 6,\ \beta_1 = 11$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.2777 & -0.2777 \\ 0.5890 & 0.4110 \end{pmatrix}$	$\begin{pmatrix} -0.2631 & -0.2631 \\ 0.6598 & 0.3402 \end{pmatrix}$	$\begin{pmatrix} -0.2500 & -0.2499 \\ 0.5512 & 0.4488 \end{pmatrix}$	$\begin{pmatrix} -0.5454 & -0.5454 \\ 0.0680 & 0.9320 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 12$	$\beta_0 = 6,\ \beta_1 = 13$	$\beta_0 = 6,\ \beta_1 = 14$	$\beta_0 = 6,\ \beta_1 = 15$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.5000 & -0.4997 \\ 0.1058 & 0.8942 \end{pmatrix}$	$\begin{pmatrix} -0.4615 & -0.4615 \\ 0.1378 & 0.8622 \end{pmatrix}$	$\begin{pmatrix} -0.4285 & -0.4285 \\ 0.1702 & 0.8298 \end{pmatrix}$	$\begin{pmatrix} -0.3999 & -0.4000 \\ 0.2226 & 0.7774 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 16$	$\beta_0 = 6,\ \beta_1 = 17$	$\beta_0 = 6,\ \beta_1 = 18$	$\beta_0 = 6,\ \beta_1 = 19$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3750 & -0.3749 \\ 0.2920 & 0.7080 \end{pmatrix}$	$\begin{pmatrix} -0.3529 & -0.3528 \\ 0.2523 & 0.7477 \end{pmatrix}$	$\begin{pmatrix} -0.3333 & -0.3333 \\ 0.3043 & 0.6957 \end{pmatrix}$	$\begin{pmatrix} -0.3158 & -0.3157 \\ 0.3279 & 0.6721 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 20$	$\beta_0 = 7,\ \beta_1 = 12$	$\beta_0 = 7,\ \beta_1 = 13$	$\beta_0 = 7,\ \beta_1 = 14$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3000 & -0.2999 \\ 0.3204 & 0.6796 \end{pmatrix}$	$\begin{pmatrix} -0.5833 & -0.5820 \\ 0.0341 & 0.9659 \end{pmatrix}$	$\begin{pmatrix} -0.5384 & -0.5384 \\ 0.0428 & 0.9572 \end{pmatrix}$	$\begin{pmatrix} -0.5000 & -0.4997 \\ 0.1058 & 0.8942 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 7,\ \beta_1 = 16$	$\beta_0 = 7,\ \beta_1 = 17$	$\beta_0 = 7,\ \beta_1 = 18$	$\beta_0 = 7,\ \beta_1 = 19$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.4375 & -0.4374 \\ 0.2406 & 0.7594 \end{pmatrix}$	$\begin{pmatrix} -0.4117 & -0.4117 \\ 0.2248 & 0.7752 \end{pmatrix}$	$\begin{pmatrix} -0.3888 & -0.3888 \\ 0.2329 & 0.7671 \end{pmatrix}$	$\begin{pmatrix} -0.3684 & -0.3684 \\ 0.2640 & 0.7360 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 7,\ \beta_1 = 20$	$\beta_0 = 8,\ \beta_1 = 14$	$\beta_0 = 8,\ \beta_1 = 15$	$\beta_0 = 8,\ \beta_1 = 16$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3499 & -0.3500 \\ 0.3458 & 0.6542 \end{pmatrix}$	$\begin{pmatrix} -0.5714 & -0.5715 \\ 0.0837 & 0.9163 \end{pmatrix}$	$\begin{pmatrix} -0.5333 & -0.5332 \\ 0.0423 & 0.9577 \end{pmatrix}$	$\begin{pmatrix} -0.4999 & -0.5000 \\ 0.1061 & 0.8939 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 19$	$\beta_0 = 8,\ \beta_1 = 20$	$\beta_0 = 9,\ \beta_1 = 16$	$\beta_0 = 9,\ \beta_1 = 17$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.4210 & -0.4211 \\ 0.1934 & 0.8066 \end{pmatrix}$	$\begin{pmatrix} -0.4000 & -0.3999 \\ 0.2226 & 0.7774 \end{pmatrix}$	$\begin{pmatrix} -0.5625 & -0.5624 \\ 0.1161 & 0.8839 \end{pmatrix}$	$\begin{pmatrix} -0.5294 & -0.5294 \\ 0.0694 & 0.9306 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 18$	$\beta_0 = 9,\ \beta_1 = 19$	$\beta_0 = 9,\ \beta_1 = 20$	$\beta_0 = 10,\ \beta_1 = 19$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.5000 & -0.4997 \\ 0.1058 & 0.8942 \end{pmatrix}$	$\begin{pmatrix} -0.4737 & -0.4733 \\ 0.1565 & 0.8435 \end{pmatrix}$	$\begin{pmatrix} -0.4497 & -0.4525 \\ 0.8914 & 0.1086 \end{pmatrix}$	$\begin{pmatrix} -0.5263 & -0.5261 \\ 0.0598 & 0.9402 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 20$	-	-	-
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.5000 & -0.4997 \\ 0.1058 & 0.8942 \end{pmatrix}$	-	-	-

Table 2: R-optimal design for Gamma regression model with two parameters (Three support points)

$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 1$	$\beta_0 = 1,\ \beta_1 = 3$	$\beta_0 = 1,\ \beta_1 = 4$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.9998 & -0.9791 & -1.0001 \\ 0.4588 & 0.0824 & 0.4588 \end{pmatrix}$	$\begin{pmatrix} -0.2966 & -0.3332 & -0.3333 \\ 0.3704 & 0.2592 & 0.3704 \end{pmatrix}$	$\begin{pmatrix} 2.3673 & -0.2498 & -0.2500 \\ 0.1827 & 0.3646 & 0.1827 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 5$	$\beta_0 = 1,\ \beta_1 = 6$	$\beta_0 = 1,\ \beta_1 = 7$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1997 & -0.2000 & -0.5475 \\ 0.4745 & 0.0510 & 0.4745 \end{pmatrix}$	$\begin{pmatrix} -0.1664 & -0.1667 & -3.0330 \\ 0.4474 & 0.1052 & 0.4474 \end{pmatrix}$	$\begin{pmatrix} -0.1427 & -0.1428 & -1.9328 \\ 0.4364 & 0.1272 & 0.4364 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 14$	$\beta_0 = 1,\ \beta_1 = 15$	$\beta_0 = 2,\ \beta_1 = 1$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.9991 & -0.9072 & -1.0008 \\ 0.3528 & 0.2944 & 0.3528 \end{pmatrix}$	$\begin{pmatrix} -0.0666 & -0.0666 & -0.4346 \\ 0.3194 & 0.3403 & 0.3194 \end{pmatrix}$	$\begin{pmatrix} 4.1382 & -1.9996 & -2.0001 \\ 0.2301 & 0.5398 & 0.2301 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 2$	$\beta_0 = 2,\ \beta_1 = 3$	$\beta_0 = 2,\ \beta_1 = 6$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.9998 & -0.9791 & -1.0001 \\ 0.4588 & 0.0824 & 0.4588 \end{pmatrix}$	$\begin{pmatrix} -0.3689 & -0.6666 & -0.6667 \\ 0.2737 & 0.4526 & 0.2737 \end{pmatrix}$	$\begin{pmatrix} -0.3027 & -0.3333 & -0.3333 \\ 0.3641 & 0.2718 & 0.3641 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 7$	$\beta_0 = 2,\ \beta_1 = 8$	$\beta_0 = 2,\ \beta_1 = 9$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 1.2836 & -0.2857 & -0.2857 \\ 0.2417 & 0.5166 & 0.2417 \end{pmatrix}$	$\begin{pmatrix} 2.3673 & -0.2498 & -0.2500 \\ 0.2827 & 0.4346 & 0.2827 \end{pmatrix}$	$\begin{pmatrix} 2.4172 & -0.2221 & -0.2222 \\ 0.1870 & 0.6260 & 0.1870 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 10$	$\beta_0 = 2,\ \beta_1 = 12$	$\beta_0 = 2,\ \beta_1 = 14$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1999 & -0.2000 & -2.2390 \\ 0.4746 & 0.0508 & 0.4746 \end{pmatrix}$	$\begin{pmatrix} -0.1666 & -0.1666 & -2.1437 \\ 0.4169 & 0.1662 & 0.4169 \end{pmatrix}$	$\begin{pmatrix} -0.1428 & -0.1428 & -1.9329 \\ 0.4364 & 0.1272 & 0.4364 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 1$	$\beta_0 = 3,\ \beta_1 = 2$	$\beta_0 = 3,\ \beta_1 = 3$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.9973 & -2.9995 & -3.0002 \\ 0.2611 & 0.4778 & 0.2611 \end{pmatrix}$	$\begin{pmatrix} -1.4997 & -0.3850 & -1.5002 \\ 0.1797 & 0.6406 & 0.1797 \end{pmatrix}$	$\begin{pmatrix} -0.9991 & -0.9073 & -1.0008 \\ 0.3528 & 0.2944 & 0.3528 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 4$	$\beta_0 = 3,\ \beta_1 = 5$	$\beta_0 = 3,\ \beta_1 = 8$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.7501 & -0.4096 & -0.7498 \\ 0.4127 & 0.1746 & 0.4127 \end{pmatrix}$	$\begin{pmatrix} -0.6063 & -0.5995 & -0.6004 \\ 0.3365 & 0.3270 & 0.3365 \end{pmatrix}$	$\begin{pmatrix} 3.8301 & -0.3707 & -0.3753 \\ 0.0732 & 0.8536 & 0.0732 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 9$	$\beta_0 = 3,\ \beta_1 = 11$	$\beta_0 = 3,\ \beta_1 = 12$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.2966 & -0.3332 & -0.3334 \\ 0.3704 & 0.2592 & 0.3704 \end{pmatrix}$	$\begin{pmatrix} 1.4888 & -0.2727 & -0.2727 \\ 0.2294 & 0.5412 & 0.2294 \end{pmatrix}$	$\begin{pmatrix} 2.3673 & -0.2499 & -0.2500 \\ 0.1827 & 0.6346 & 0.1827 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 13$	$\beta_0 = 3,\ \beta_1 = 14$	$\beta_0 = 3,\ \beta_1 = 15$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 2.0636 & -0.2306 & -0.2308 \\ 0.2044 & 0.5912 & 0.2044 \end{pmatrix}$	$\begin{pmatrix} 2.4503 & -0.2142 & -0.2143 \\ 0.1891 & 0.6218 & 0.1891 \end{pmatrix}$	$\begin{pmatrix} -0.1999 & -0.2000 & -2.239 \\ 0.4746 & 0.0508 & 0.4746 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 1$	$\beta_0 = 4,\ \beta_1 = 2$	$\beta_0 = 4,\ \beta_1 = 3$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -3.1435 & -4.0012 & -3.9995 \\ 0.2141 & 0.5718 & 0.2141 \end{pmatrix}$	$\begin{pmatrix} 13.4070 & -1.9995 & -2.0001 \\ 0.2284 & 0.5432 & 0.2284 \end{pmatrix}$	$\begin{pmatrix} -1.3334 & -0.2232 & -1.3332 \\ 0.2226 & 0.5480 & 0.2226 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 4$	$\beta_0 = 4,\ \beta_1 = 5$	$\beta_0 = 4,\ \beta_1 = 6$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.9986 & -0.9514 & -1.0013 \\ 0.3523 & 0.2954 & 0.3523 \end{pmatrix}$	$\begin{pmatrix} -0.7998 & -0.1396 & -0.8001 \\ 0.3622 & 0.2756 & 0.3622 \end{pmatrix}$	$\begin{pmatrix} -0.3687 & -0.6669 & -0.6664 \\ 0.2770 & 0.4460 & 0.2770 \end{pmatrix}$

Table 2: Continued

$\boldsymbol{\beta}$	$\beta_0=4,\ \beta_1=10$	$\beta_0=4,\ \beta_1=11$	$\beta_0=4,\ \beta_1=12$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}1.4528 & -0.3996 & -0.4002\\ 0.2719 & 0.4562 & 0.2719\end{pmatrix}$	$\begin{pmatrix}1.1967 & -0.3632 & -0.3639\\ 0.3046 & 0.3908 & 0.3046\end{pmatrix}$	$\begin{pmatrix}-0.2966 & -0.3328 & -0.3339\\ 0.3704 & 0.2592 & 0.3704\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=4,\ \beta_1=13$	$\beta_0=4,\ \beta_1=14$	$\beta_0=4,\ \beta_1=15$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}1.2290 & -0.3076 & -0.3077\\ 0.2352 & 0.5296 & 0.2352\end{pmatrix}$	$\begin{pmatrix}1.2836 & -0.2857 & -0.2857\\ 0.2917 & 0.5166 & 0.2917\end{pmatrix}$	$\begin{pmatrix}1.1672 & -0.2662 & -0.2668\\ 0.2263 & 0.5474 & 0.2263\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\ \beta_1=1$	$\beta_0=5,\ \beta_1=2$	$\beta_0=5,\ \beta_1=3$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}-4.9990 & 8.9742 & -5.0010\\ 0.4527 & 0.0946 & 0.4527\end{pmatrix}$	$\begin{pmatrix}4.3431 & -2.5042 & -2.4998\\ 0.0265 & 0.9470 & 0.0265\end{pmatrix}$	$\begin{pmatrix}-1.6668 & 0.4464 & -1.6664\\ 0.2989 & 0.4022 & 0.2989\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\ \beta_1=4$	$\beta_0=5,\ \beta_1=5$	$\beta_0=5,\ \beta_1=6$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}-1.2500 & -1.2498 & -2.2318\\ 0.1785 & 0.6430 & 0.1785\end{pmatrix}$	$\begin{pmatrix}-0.9985 & -0.9465 & -1.0001\\ 0.3517 & 0.2966 & 0.3517\end{pmatrix}$	$\begin{pmatrix}-0.7045 & -0.8334 & -0.8332\\ 0.3373 & 0.3254 & 0.3373\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\ \beta_1=7$	$\beta_0=5,\ \beta_1=8$	$\beta_0=5,\ \beta_1=13$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}-0.7141 & 0.3424 & -0.7141\\ 0.4088 & 0.1824 & 0.4088\end{pmatrix}$	$\begin{pmatrix}0.3565 & -0.6072 & -0.6256\\ 0.0349 & 0.9301 & 0.0349\end{pmatrix}$	$\begin{pmatrix}1.3649 & -0.3770 & -0.3848\\ 0.0288 & 0.9423 & 0.02888\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=5,\ \beta_1=14$	$\beta_0=5,\ \beta_1=15$	$\beta_0=6,\ \beta_1=1$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}1.3238 & -0.3570 & -0.3571\\ 0.2666 & 0.4668 & 0.2666\end{pmatrix}$	$\begin{pmatrix}-0.3030 & -0.3332 & -0.3334\\ 0.3660 & 0.3680 & 0.3660\end{pmatrix}$	$\begin{pmatrix}-5.9967 & 8.2075 & -6.0032\\ 0.4825 & 0.0350 & 0.4825\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=6,\ \beta_1=2$	$\beta_0=6,\ \beta_1=3$	$\beta_0=6,\ \beta_1=4$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}-0.9973 & -2.9996 & -3.0001\\ 0.2611 & 0.4778 & 0.2611\end{pmatrix}$	$\begin{pmatrix}4.1383 & -1.9994 & -2.0002\\ 0.2289 & 0.5422 & 0.2289\end{pmatrix}$	$\begin{pmatrix}-1.4997 & -0.3850 & -1.5002\\ 0.1796 & 0.6408 & 0.1796\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=6,\ \beta_1=6$	$\beta_0=6,\ \beta_1=7$	$\beta_0=6,\ \beta_1=8$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}-0.9994 & -0.9073 & -1.0005\\ 0.3528 & 0.2944 & 0.3528\end{pmatrix}$	$\begin{pmatrix}-0.8572 & -0.8570 & -1.7236\\ 0.3553 & 0.2894 & 0.3553\end{pmatrix}$	$\begin{pmatrix}-0.7501 & 0.4096 & -0.7498\\ 0.4127 & 0.1746 & 0.4127\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=6,\ \beta_1=9$	$\beta_0=6,\ \beta_1=10$	$\beta_0=6,\ \beta_1=13$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}-0.3687 & -0.6664 & -0.6668\\ 0.2770 & 0.4460 & 0.2770\end{pmatrix}$	$\begin{pmatrix}0.6068 & -0.5995 & -0.6004\\ 0.3365 & 0.3270 & 0.3365\end{pmatrix}$	$\begin{pmatrix}1.9172 & -0.4613 & -0.4615\\ 0.1120 & 0.7760 & 0.1120\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=6,\ \beta_1=15$	$\beta_0=7,\ \beta_1=1$	$\beta_0=7,\ \beta_1=2$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}1.4524 & -0.3994 & -0.4003\\ 0.2719 & 0.4562 & 0.2719\end{pmatrix}$	$\begin{pmatrix}-6.9988 & 0.7763 & -7.0011\\ 0.4372 & 0.1256 & 0.4372\end{pmatrix}$	$\begin{pmatrix}0.9603 & -3.5000 & -3.5000\\ 0.3192 & 0.3616 & 0.3192\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=7,\ \beta_1=3$	$\beta_0=7,\ \beta_1=4$	$\beta_0=7,\ \beta_1=5$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}4.2596 & -2.3290 & -2.3333\\ 0.0538 & 0.8924 & 0.0538\end{pmatrix}$	$\begin{pmatrix}-0.2754 & -1.6012 & -1.7518\\ 0.0119 & 0.9762 & 0.0119\end{pmatrix}$	$\begin{pmatrix}-1.3999 & 1.2023 & -1.4000\\ 0.4997 & 0.0006 & 0.4997\end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0=7,\ \beta_1=6$	$\beta_0=7,\ \beta_1=7$	$\beta_0=7,\ \beta_1=8$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix}-0.2672 & -1.1666 & -1.1666\\ 0.1937 & 0.6126 & 0.1937\end{pmatrix}$	$\begin{pmatrix}-0.9993 & -0.9072 & -1.0006\\ 0.3528 & 0.2944 & 0.3528\end{pmatrix}$	$\begin{pmatrix}-0.8048 & -0.8752 & -1.3642\\ 0.2562 & 0.4848 & 0.2562\end{pmatrix}$

Table 2: Continued

$\boldsymbol{\beta}$	$\beta_0 = 7,\ \beta_1 = 9$	$\beta_0 = 7,\ \beta_1 = 10$	$\beta_0 = 7,\ \beta_1 = 11$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.7778 & -0.1762 & -0.7776 \\ 0.3684 & 0.2632 & 0.3684 \end{pmatrix}$	$\begin{pmatrix} -0.6997 & 0.1970 & -0.7002 \\ 0.3791 & 0.2418 & 0.3791 \end{pmatrix}$	$\begin{pmatrix} 1.0681 & -0.6361 & -0.6364 \\ 0.1999 & 0.6002 & 0.1999 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 7,\ \beta_1 = 13$	$\beta_0 = 7,\ \beta_1 = 15$	$\beta_0 = 8,\ \beta_1 = 1$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 1.8182 & -0.2227 & -0.5385 \\ 0.0001 & 0.9998 & 0.0001 \end{pmatrix}$	$\begin{pmatrix} -0.4666 & -0.3031 & -0.4666 \\ 0.3535 & 0.2930 & 0.3535 \end{pmatrix}$	$\begin{pmatrix} -7.9959 & 4.9336 & -8.0040 \\ 0.2500 & 0.5000 & 0.2500 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 2$	$\beta_0 = 8,\ \beta_1 = 3$	$\beta_0 = 8,\ \beta_1 = 4$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -3.1435 & -3.9975 & -4.0009 \\ 0.2142 & 0.5716 & 0.2142 \end{pmatrix}$	$\begin{pmatrix} -2.6585 & -1.2272 & -2.6747 \\ 0.1341 & 0.7318 & 0.1341 \end{pmatrix}$	$\begin{pmatrix} 4.1382 & -2.0003 & -1.9998 \\ 0.2289 & 0.5422 & 0.2289 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 5$	$\beta_0 = 8,\ \beta_1 = 6$	$\beta_0 = 8,\ \beta_1 = 8$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -1.5998 & -1.0041 & -1.6001 \\ 0.2402 & 0.5196 & 0.2402 \end{pmatrix}$	$\begin{pmatrix} -1.3334 & -0.2232 & -1.3332 \\ 0.2260 & 0.5480 & 0.2260 \end{pmatrix}$	$\begin{pmatrix} -0.9983 & -0.9380 & -1.0016 \\ 0.3460 & 0.3080 & 0.3460 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 9$	$\beta_0 = 8,\ \beta_1 = 10$	$\beta_0 = 8,\ \beta_1 = 11$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.8892 & -0.8883 & -1.7513 \\ 0.2902 & 0.4196 & 0.2902 \end{pmatrix}$	$\begin{pmatrix} -0.7998 & 0.1396 & -0.8001 \\ 0.3621 & 0.2758 & 0.3621 \end{pmatrix}$	$\begin{pmatrix} -0.7271 & 0.1861 & -0.7274 \\ 0.3672 & 0.2256 & 0.3672 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 12$	$\beta_0 = 8,\ \beta_1 = 13$	$\beta_0 = 9,\ \beta_1 = 1$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3687 & -0.6669 & -0.6664 \\ 0.2770 & 0.4460 & 0.2770 \end{pmatrix}$	$\begin{pmatrix} 1.3602 & -0.6118 & -0.6158 \\ 0.1059 & 0.7882 & 0.1059 \end{pmatrix}$	$\begin{pmatrix} -9.0008 & 3.9228 & -8.9991 \\ 0.4933 & 0.0135 & 0.4933 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 2$	$\beta_0 = 9,\ \beta_1 = 3$	$\beta_0 = 9,\ \beta_1 = 4$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.6459 & -4.4982 & -4.5003 \\ 0.1327 & 0.7346 & 0.1327 \end{pmatrix}$	$\begin{pmatrix} -0.9973 & -2.9958 & -3.0022 \\ 0.2617 & 0.4766 & 0.2617 \end{pmatrix}$	$\begin{pmatrix} 4.6884 & -2.2488 & -2.2501 \\ 0.1033 & 0.7934 & 0.1033 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 5$	$\beta_0 = 9,\ \beta_1 = 6$	$\beta_0 = 9,\ \beta_1 = 7$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.5318 & -1.7996 & -1.8001 \\ 0.2556 & 0.4888 & 0.2556 \end{pmatrix}$	$\begin{pmatrix} -1.4998 & -1.4731 & -1.5001 \\ 0.1908 & 0.6184 & 0.1908 \end{pmatrix}$	$\begin{pmatrix} -0.8256 & -1.2855 & -1.2858 \\ 0.2492 & 0.5016 & 0.2492 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 8$	$\beta_0 = 9,\ \beta_1 = 9$	$\beta_0 = 9,\ \beta_1 = 10$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.9108 & -1.1170 & -1.1263 \\ 0.1460 & 0.7080 & 0.1460 \end{pmatrix}$	$\begin{pmatrix} -0.9988 & -0.9072 & -1.0011 \\ 0.3528 & 0.2944 & 0.3528 \end{pmatrix}$	$\begin{pmatrix} -0.9000 & -0.8999 & -1.1354 \\ 0.2253 & 0.5494 & 0.2253 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 11$	$\beta_0 = 9,\ \beta_1 = 12$	$\beta_0 = 9,\ \beta_1 = 13$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.8180 & 1.3889 & -0.8183 \\ 0.3722 & 0.2552 & 0.3722 \end{pmatrix}$	$\begin{pmatrix} -0.7501 & 0.4096 & -0.7498 \\ 0.4127 & 0.1746 & 0.4127 \end{pmatrix}$	$\begin{pmatrix} -0.6116 & -0.6921 & -0.6924 \\ 0.3537 & 0.2926 & 0.3537 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 15$	$\beta_0 = 10,\ \beta_1 = 1$	$\beta_0 = 10,\ \beta_1 = 2$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.6063 & -0.5999 & -0.6000 \\ 0.3364 & 0.3272 & 0.3364 \end{pmatrix}$	$\begin{pmatrix} -9.9992 & -0.8220 & -10.0002 \\ 0.2986 & 0.4028 & 0.2986 \end{pmatrix}$	$\begin{pmatrix} -4.9990 & 8.9742 & -5.0009 \\ 0.4528 & 0.0944 & 0.4528 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 3$	$\beta_0 = 10,\ \beta_1 = 4$	$\beta_0 = 10,\ \beta_1 = 5$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -3.3219 & -4.5466 & -3.3447 \\ 0.4872 & 0.0256 & 0.4872 \end{pmatrix}$	$\begin{pmatrix} 3.8099 & -2.4494 & -2.5000 \\ 0.0001 & 0.9998 & 0.0001 \end{pmatrix}$	$\begin{pmatrix} 4.1382 & -1.9993 & -2.0002 \\ 0.2296 & 0.5408 & 0.2296 \end{pmatrix}$

Table 2: Continued

$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 6$	$\beta_0 = 10,\ \beta_1 = 7$	$\beta_0 = 10,\ \beta_1 = 8$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -1.6668 & -0.4464 & -1.6665 \\ 0.2989 & 0.4022 & 0.2989 \end{pmatrix}$	$\begin{pmatrix} -1.1727 & -1.4280 & -1.4296 \\ 0.3997 & 0.2006 & 0.3997 \end{pmatrix}$	$\begin{pmatrix} -1.2500 & -1.2498 & -2.2318 \\ 0.1785 & 0.6430 & 0.1785 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 9$	$\beta_0 = 10,\ \beta_1 = 10$	$\beta_0 = 10,\ \beta_1 = 11$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.6269 & -1.1046 & -1.1122 \\ 0.1294 & 0.7412 & 0.1294 \end{pmatrix}$	$\begin{pmatrix} -0.9985 & -0.9561 & -1.0013 \\ 0.3678 & 0.2644 & 0.3678 \end{pmatrix}$	$\begin{pmatrix} -0.4497 & -0.9093 & -0.9080 \\ 0.2630 & 0.4740 & 0.2630 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 12$	$\beta_0 = 10,\ \beta_1 = 13$	$\beta_0 = 10,\ \beta_1 = 14$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.7045 & -0.8334 & -0.8332 \\ 0.3373 & 0.3254 & 0.3373 \end{pmatrix}$	$\begin{pmatrix} -0.7690 & 0.2041 & -0.7693 \\ 0.3732 & 0.2536 & 0.3732 \end{pmatrix}$	$\begin{pmatrix} -0.7141 & 0.3424 & -0.7144 \\ 0.4088 & 0.1824 & 0.4088 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 15$	$\beta_0 = 10,\ \beta_1 = 16$	$\beta_0 = 10,\ \beta_1 = 17$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.3687 & -0.6665 & -0.6667 \\ 0.2759 & 0.4482 & 0.2759 \end{pmatrix}$	$\begin{pmatrix} -0.9998 & -0.9791 & -1.0001 \\ 0.4588 & 0.0824 & 0.4588 \end{pmatrix}$	$\begin{pmatrix} -0.9998 & -0.9791 & -1.0001 \\ 0.4588 & 0.0824 & 0.4588 \end{pmatrix}$

Table 3: R-optimal design for Gamma regression model with two parameters (Four support points)

$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 8$	$\beta_0 = 1,\ \beta_1 = 10$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1251 & 1.4191 & -0.1249 & 0.4328 \\ 0.1328 & 0.3672 & 0.3672 & 0.1328 \end{pmatrix}$	$\begin{pmatrix} -0.1000 & 0.6989 & -0.0999 & 0.4254 \\ 0.3494 & 0.1506 & 0.1506 & 0.3494 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 1,\ \beta_1 = 11$	$\beta_0 = 2,\ \beta_1 = 3$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.1024 & 1.2526 & -0.0909 & -0.0909 \\ 0.2028 & 0.2972 & 0.2972 & 0.2028 \end{pmatrix}$	$\begin{pmatrix} -0.6668 & 1.5448 & -0.6663 & 1.0896 \\ 0.1767 & 0.3233 & 0.3233 & 0.1767 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 9$	$\beta_0 = 2,\ \beta_1 = 15$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.3407 & 1.3105 & -0.2221 & -0.2222 \\ 0.2319 & 0.2681 & 0.2681 & 0.2319 \end{pmatrix}$	$\begin{pmatrix} -0.1333 & 0.9882 & -0.1333 & 0.0713 \\ 0.1449 & 0.3551 & 0.3551 & 0.1449 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 16$	$\beta_0 = 2,\ \beta_1 = 17$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1250 & 1.4191 & -0.1249 & 0.4328 \\ 0.1327 & 0.3673 & 0.3673 & 0.1327 \end{pmatrix}$	$\begin{pmatrix} -0.1176 & 1.0332 & -0.1176 & 0.3824 \\ 0.2506 & 0.2494 & 0.2494 & 0.2506 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 2,\ \beta_1 = 20$	$\beta_0 = 3,\ \beta_1 = 18$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1000 & 0.6989 & -0.0999 & 0.4254 \\ 0.3494 & 0.1506 & 0.1506 & 0.3494 \end{pmatrix}$	$\begin{pmatrix} -0.1450 & 0.7336 & -0.1576 & -0.1669 \\ 0.0159 & 0.4841 & 0.4841 & 0.0159 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 26$	$\beta_0 = 3,\ \beta_1 = 30$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1154 & 0.8000 & -0.1153 & 0.3216 \\ 0.2307 & 0.2693 & 0.2693 & 0.2307 \end{pmatrix}$	$\begin{pmatrix} -0.0999 & 0.6989 & -0.1000 & 0.4254 \\ 0.3495 & 0.1505 & 0.1505 & 0.3495 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 3,\ \beta_1 = 32$	$\beta_0 = 4,\ \beta_1 = 6$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.0937 & 1.0108 & -0.0937 & 0.6547 \\ 0.4206 & 0.0794 & 0.0794 & 0.4206 \end{pmatrix}$	$\begin{pmatrix} -0.6668 & 1.5449 & -0.6663 & 1.0897 \\ 0.1767 & 0.3233 & 0.3233 & 0.1767 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 15$	$\beta_0 = 4,\ \beta_1 = 20$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.2664 & 0.8092 & -0.2668 & 0.4288 \\ 0.2419 & 0.2581 & 0.2581 & 0.2419 \end{pmatrix}$	$\begin{pmatrix} 0.0882 & 0.5674 & -0.1999 & -0.2001 \\ 0.0008 & 0.4992 & 0.4992 & 0.0008 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 22$	$\beta_0 = 4,\ \beta_1 = 30$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.1844 & 0.6942 & -0.1817 & -0.1818 \\ 0.1322 & 0.3678 & 0.3678 & 0.1322 \end{pmatrix}$	$\begin{pmatrix} -0.1333 & 0.9608 & -0.1333 & 0.4805 \\ 0.1575 & 0.3425 & 0.3425 & 0.1575 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 31$	$\beta_0 = 4,\ \beta_1 = 32$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1283 & 1.4199 & -0.1317 & 0.5303 \\ 0.1067 & 0.3933 & 0.3933 & 0.1067 \end{pmatrix}$	$\begin{pmatrix} -0.1250 & 1.4191 & -0.1249 & 0.4328 \\ 0.1327 & 0.3673 & 0.3673 & 0.1327 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 33$	$\beta_0 = 4,\ \beta_1 = 34$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1212 & 1.5383 & -0.1211 & 0.4779 \\ 0.1514 & 0.3486 & 0.3486 & 0.1514 \end{pmatrix}$	$\begin{pmatrix} -0.1176 & 1.0332 & -0.1176 & 0.3824 \\ 0.2506 & 0.2494 & 0.2494 & 0.2506 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 4,\ \beta_1 = 35$	$\beta_0 = 5,\ \beta_1 = 19$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1143 & 0.8657 & -0.1142 & 0.4394 \\ 0.3047 & 0.1953 & 0.1953 & 0.3047 \end{pmatrix}$	$\begin{pmatrix} -0.2630 & 1.0310 & -0.2632 & 0.4691 \\ 0.2334 & 0.2666 & 0.2666 & 0.2334 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 26$	$\beta_0 = 5,\ \beta_1 = 27$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.2964 & 0.6382 & -0.1922 & -0.1923 \\ 0.1173 & 0.3827 & 0.3827 & 0.1173 \end{pmatrix}$	$\begin{pmatrix} 0.1441 & 0.6884 & -0.1848 & -0.1851 \\ 0.0072 & 0.4928 & 0.4928 & 0.0072 \end{pmatrix}$

Table 3: Continued

$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 28$	$\beta_0 = 5,\ \beta_1 = 29$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.0525 & 0.5365 & -0.1770 & -0.1785 \\ 0.0011 & 0.4988 & 0.4988 & 0.0011 \end{pmatrix}$	$\begin{pmatrix} 0.2664 & 0.5590 & -0.1723 & 0.1724 \\ 0.0871 & 0.4129 & 0.4129 & 0.0871 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 30$	$\beta_0 = 5,\ \beta_1 = 31$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.1447 & 0.7737 & -0.1664 & -0.1666 \\ 0.0450 & 0.4550 & 0.4550 & 0.0450 \end{pmatrix}$	$\begin{pmatrix} -0.1612 & 0.7699 & -0.1613 & 0.1804 \\ 0.0968 & 0.4032 & 0.4032 & 0.0968 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 34$	$\beta_0 = 5,\ \beta_1 = 38$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1471 & 0.7057 & -0.1469 & 0.1660 \\ 0.1272 & 0.3728 & 0.3728 & 0.1272 \end{pmatrix}$	$\begin{pmatrix} -0.1315 & 0.5356 & -0.1126 & -0.0003 \\ 0.0002 & 0.4997 & 0.4997 & 0.0002 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 39$	$\beta_0 = 5,\ \beta_1 = 40$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1282 & 1.6957 & -0.1282 & -0.4058 \\ 0.0359 & 0.4641 & 0.4641 & 0.0359 \end{pmatrix}$	$\begin{pmatrix} -0.1250 & 1.4191 & -0.1249 & -0.4328 \\ 0.1327 & 0.3670 & 0.3670 & 0.1327 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 41$	$\beta_0 = 5,\ \beta_1 = 42$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1219 & 1.5634 & -0.1218 & 0.4439 \\ 0.1339 & 0.3661 & 0.3661 & 0.1339 \end{pmatrix}$	$\begin{pmatrix} -0.1190 & 1.5748 & -0.1190 & 0.4092 \\ 0.0836 & 0.4164 & 0.4164 & 0.0836 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 5,\ \beta_1 = 43$	$\beta_0 = 5,\ \beta_1 = 44$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1162 & 1.0502 & -0.1162 & 0.4671 \\ 0.2928 & 0.2072 & 0.2072 & 0.2928 \end{pmatrix}$	$\begin{pmatrix} -0.1136 & 0.9680 & -0.1136 & 0.3829 \\ 0.2390 & 0.2610 & 0.2610 & 0.2390 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 9$	$\beta_0 = 6,\ \beta_1 = 23$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.6668 & 1.5449 & -0.6663 & 1.0897 \\ 0.1767 & 0.3233 & 0.3233 & 0.1767 \end{pmatrix}$	$\begin{pmatrix} -0.2608 & 0.9196 & -0.2608 & 0.3851 \\ 0.2032 & 0.2968 & 0.2968 & 0.2032 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 27$	$\beta_0 = 6,\ \beta_1 = 30$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.2864 & 1.4903 & -0.2222 & -0.2221 \\ 0.3334 & 0.1666 & 0.1666 & 0.3334 \end{pmatrix}$	$\begin{pmatrix} -0.0270 & 0.3240 & -0.1986 & -0.2000 \\ 0.0007 & 0.4992 & 0.4992 & 0.0007 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 37$	$\beta_0 = 6,\ \beta_1 = 45$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1622 & 0.6520 & -0.1619 & -0.2367 \\ 0.1578 & 0.3422 & 0.3422 & 0.1578 \end{pmatrix}$	$\begin{pmatrix} -0.1333 & 1.0400 & -0.1332 & 0.0001 \\ 0.1556 & 0.3444 & 0.3444 & 0.1556 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 48$	$\beta_0 = 6,\ \beta_1 = 51$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1249 & 1.4193 & -0.1250 & 0.4328 \\ 0.1327 & 0.3673 & 0.3673 & 0.1327 \end{pmatrix}$	$\begin{pmatrix} -0.1176 & 1.0332 & -0.1176 & 0.3824 \\ 0.2506 & 0.2494 & 0.2494 & 0.2506 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 6,\ \beta_1 = 60$	$\beta_0 = 7,\ \beta_1 = 39$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.0999 & 0.6989 & -0.1000 & 0.4254 \\ 0.3495 & 0.1505 & 0.1505 & 0.3495 \end{pmatrix}$	$\begin{pmatrix} -0.0421 & 0.3061 & -0.1764 & -0.1794 \\ 0.0002 & 0.4997 & 0.4997 & 0.0002 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 7,\ \beta_1 = 40$	$\beta_0 = 7,\ \beta_1 = 50$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.3548 & 0.6450 & -0.1749 & -0.1750 \\ 0.1389 & 0.3611 & 0.3611 & 0.1389 \end{pmatrix}$	$\begin{pmatrix} -0.1400 & 0.9105 & -0.1399 & 0.1282 \\ 0.0461 & 0.4539 & 0.4539 & 0.0461 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 7,\ \beta_1 = 51$	$\beta_0 = 7,\ \beta_1 = 52$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1376 & 0.9959 & -0.1267 & 0.1245 \\ 0.0174 & 0.4826 & 0.4826 & 0.0174 \end{pmatrix}$	$\begin{pmatrix} -0.1346 & 1.2422 & -0.1345 & 0.2680 \\ 0.0185 & 0.4815 & 0.4815 & 0.0185 \end{pmatrix}$

Table 3: Continued

$\boldsymbol{\beta}$	$\beta_0 = 7,\ \beta_1 = 56$	$\beta_0 = 7,\ \beta_1 = 70$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1250 & 1.4203 & -0.1249 & 0.4340 \\ 0.1310 & 0.3690 & 0.3690 & 0.1310 \end{pmatrix}$	$\begin{pmatrix} -0.1000 & 0.6989 & -0.0999 & 0.4254 \\ 0.3493 & 0.1506 & 0.1506 & 0.3493 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 12$	$\beta_0 = 8,\ \beta_1 = 30$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.6668 & 1.5449 & -0.6663 & 1.0897 \\ 0.1767 & 0.3233 & 0.3233 & 0.1767 \end{pmatrix}$	$\begin{pmatrix} -0.2664 & 0.8092 & -0.2668 & 0.4228 \\ 0.2419 & 0.2581 & 0.2581 & 0.2419 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 36$	$\beta_0 = 8,\ \beta_1 = 40$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.3996 & 1.4467 & -0.2221 & -0.2221 \\ 0.3056 & 0.1944 & 0.1944 & 0.3056 \end{pmatrix}$	$\begin{pmatrix} 0.0882 & 0.5674 & -0.1993 & -0.2000 \\ 0.0007 & 0.4992 & 0.4992 & 0.0007 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 44$	$\beta_0 = 8,\ \beta_1 = 60$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.0286 & 0.2663 & -0.1811 & 0.1818 \\ 0.0003 & 0.4996 & 0.4996 & 0.0003 \end{pmatrix}$	$\begin{pmatrix} -0.1333 & 1.0420 & -0.1332 & 0.1441 \\ 0.1554 & 0.3446 & 0.3446 & 0.1554 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 63$	$\beta_0 = 8,\ \beta_1 = 64$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1269 & 1.1873 & -0.1275 & 0.3266 \\ 0.0005 & 0.4994 & 0.4994 & 0.0005 \end{pmatrix}$	$\begin{pmatrix} -0.1250 & 1.4191 & -0.1249 & 0.4328 \\ 0.1327 & 0.3673 & 0.3673 & 0.1327 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 65$	$\beta_0 = 8,\ \beta_1 = 66$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1232 & 1.4500 & -0.1221 & 0.3714 \\ 0.0731 & 0.4269 & 0.4269 & 0.0731 \end{pmatrix}$	$\begin{pmatrix} -0.1212 & 1.5383 & -0.1211 & 0.4779 \\ 0.1514 & 0.3486 & 0.3486 & 0.1514 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 68$	$\beta_0 = 8,\ \beta_1 = 70$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1176 & 1.0332 & -0.1176 & 0.3824 \\ 0.2506 & 0.2494 & 0.2494 & 0.2506 \end{pmatrix}$	$\begin{pmatrix} -0.1143 & 0.8657 & -0.1142 & 0.4394 \\ 0.3047 & 0.1953 & 0.1953 & 0.3047 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 8,\ \beta_1 = 80$	$\beta_0 = 8,\ \beta_1 = 88$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1000 & 0.6989 & -0.0999 & 0.4254 \\ 0.3494 & 0.1506 & 0.1506 & 0.3494 \end{pmatrix}$	$\begin{pmatrix} -0.1043 & 0.2520 & -0.0908 & -0.0909 \\ 0.1951 & 0.3049 & 0.3049 & 0.1951 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 54$	$\beta_0 = 9,\ \beta_1 = 72$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.1447 & 0.7737 & -0.1663 & -0.1666 \\ 0.0450 & 0.4550 & 0.4550 & 0.0450 \end{pmatrix}$	$\begin{pmatrix} -0.1250 & 1.4191 & -0.1249 & 0.4328 \\ 0.1327 & 0.3673 & 0.3673 & 0.1327 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 78$	$\beta_0 = 9,\ \beta_1 = 90$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1153 & 0.8000 & -0.1153 & 0.3216 \\ 0.2359 & 0.2641 & 0.2641 & 0.2359 \end{pmatrix}$	$\begin{pmatrix} -0.1000 & 0.6989 & -0.0999 & 0.4254 \\ 0.3493 & 0.1507 & 0.1507 & 0.3493 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 9,\ \beta_1 = 96$	$\beta_0 = 10,\ \beta_1 = 15$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.0939 & 1.0108 & -0.0937 & 0.6547 \\ 0.4199 & 0.0801 & 0.0801 & 0.4199 \end{pmatrix}$	$\begin{pmatrix} -0.6666 & 1.3241 & -0.6666 & 0.9085 \\ 0.1786 & 0.3214 & 0.3214 & 0.1786 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 38$	$\beta_0 = 10,\ \beta_1 = 52$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.2630 & 1.0310 & -0.2632 & 0.4691 \\ 0.2334 & 0.2666 & 0.2666 & 0.2334 \end{pmatrix}$	$\begin{pmatrix} 0.1965 & 0.6549 & -0.1922 & -0.1923 \\ 0.1199 & 0.3801 & 0.3801 & 0.1199 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 58$	$\beta_0 = 10,\ \beta_1 = 68$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} 0.2664 & 0.5590 & -0.1724 & -0.1724 \\ 0.0871 & 0.4129 & 0.4129 & 0.0871 \end{pmatrix}$	$\begin{pmatrix} -0.1471 & 0.7057 & -0.1468 & -0.1660 \\ 0.3494 & 0.1506 & 0.1506 & 0.3494 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 75$	$\beta_0 = 10,\ \beta_1 = 78$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1333 & 0.9059 & -0.1332 & -0.0098 \\ 0.1466 & 0.3534 & 0.3534 & 0.1466 \end{pmatrix}$	$\begin{pmatrix} -0.1282 & 1.6985 & -0.1280 & 0.4065 \\ 0.0290 & 0.4710 & 0.4710 & 0.0290 \end{pmatrix}$

Table 3: Continued

$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 81$	$\beta_0 = 10,\ \beta_1 = 82$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1234 & 1.7453 & -0.1233 & -0.5493 \\ 0.1206 & 0.3794 & 0.3794 & 0.1206 \end{pmatrix}$	$\begin{pmatrix} -0.1219 & 1.5634 & -0.1219 & 0.4439 \\ 0.1339 & 0.3661 & 0.3661 & 0.1339 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 85$	$\beta_0 = 10,\ \beta_1 = 86$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1176 & 1.0332 & -0.1176 & 0.3824 \\ 0.2506 & 0.2494 & 0.2494 & 0.2506 \end{pmatrix}$	$\begin{pmatrix} -0.1162 & 0.8101 & -0.1162 & 0.3439 \\ 0.2934 & 0.2066 & 0.2066 & 0.2934 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 88$	$\beta_0 = 10,\ \beta_1 = 92$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1136 & 0.9680 & -0.1136 & 0.3829 \\ 0.2346 & 0.2654 & 0.2654 & 0.2346 \end{pmatrix}$	$\begin{pmatrix} -0.1087 & 0.5971 & -0.1086 & 0.4081 \\ 0.3679 & 0.1321 & 0.1321 & 0.3679 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 93$	$\beta_0 = 10,\ \beta_1 = 96$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1074 & 0.6330 & -0.1075 & 0.4220 \\ 0.3676 & 0.1324 & 0.1324 & 0.3676 \end{pmatrix}$	$\begin{pmatrix} -0.1041 & 0.7818 & -0.1041 & 0.4055 \\ 0.3092 & 0.1908 & 0.1908 & 0.3092 \end{pmatrix}$
$\boldsymbol{\beta}$	$\beta_0 = 10,\ \beta_1 = 98$	$\beta_0 = 10,\ \beta_1 = 100$
$\boldsymbol{x}$ $\boldsymbol{w}$	$\begin{pmatrix} -0.1020 & 0.5326 & -0.1020 & 0.4335 \\ 0.4111 & 0.0889 & 0.0889 & 0.4111 \end{pmatrix}$	$\begin{pmatrix} -1.0000 & 0.6989 & -0.0999 & 0.4254 \\ 0.3487 & 0.1513 & 0.1513 & 0.3487 \end{pmatrix}$