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Identifying the Time of a Permanent Shift in the Normal Process Mean with Memory Type Control Chart

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Abstract

Control chart is a valuable statistical process control (SPC) tool used for monitoring the process performance. When control chart gives the out-of-control signal, the search initiates to identify the sources responsible for the special cause of variation. But control chart does not give the exact time when the process change begun. The time when the process change appears first in the process called change point. Knowing the change point in the process helps to identify the special cause of variation. This article discusses the approach based on the maximum likelihood estimator of process change to identify the time of a permanent shift in the normal mean with EWMA and MA control charts.

Key words: Statistical quality control; Change point; maximum Likelihood; EWMA control chart; Moving average control chart; Average run length.

1. Introduction

Control charts distinguish between the special cause of variation and the common cause of the variation in the process. To improve and control the process control charts are widely used in the manufacturing industries. Once control chart issues a signal that the special cause is present in the process. Process professionals should initiate a search for the special cause of the variation which could be quite difficult. The search depends on the professional's knowledge and experience. To quality improvement it is necessary to bring the process back into the statistical control. One essential step would help to quality improvement is that knowing the starting time the special cause of variation appears first in the process. Once it is possible to identify the exact time when the process happens due to special cause of variation appears first in the process, there may not be delay finding the occurrence of the special cause of the variation in the process. As a result, the special cause of variation can be identified more quickly, and the corrective action can be taken to eliminate the sources of the special cause of variation which leads to process improvement.

In recent years change point estimation in control charts has received a great deal of attention, as the change point estimation procedure simplify the effort to search for and identify special causes in statistical process monitoring. Hinkley (1970) considered inference about the point in a sequence of random variables at which the probability distribution changes. They compared asymptotic distribution of the MLE and likelihood ratio statistic with some finite sample distributions. Samuel *et al.* (1998a) proposed a method of maximum likelihood estimator to identify the time of step change in the normal mean with $\overline{X}$ control chart. Samuel *et al.* (1998b) considered the step change in the normal process variance.

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Samuel and Pignatiello (1998) estimated the change point in the rate parameter of the Poisson process. Nedumaran *et al.* (2000) considered the time of the step change in the multivariate process with chi-square control chart. Pignatiello and Samuel (2001a) considered the change point in the normal process mean in SPC applications. Pignatiello and Samuel (2001b) estimated the step change point in the process fraction nonconforming. They have estimated MLE of a change point when a step change occurred in the fraction nonconforming. Park and Park (2004) considered the time of step change in the normal process mean and variance when $\overline{X}$ and S control charts used simultaneously. Khoo (2004) determined the permanent shift in the process mean with CUSUM control chart. Perry *et al.* (2005) estimated the time of step change in the rate parameter of the Poisson distribution with linear trend and monotonic change, respectively. Fahmy and Elsayed (2006) estimated the maximum likelihood estimator of the change point when Shewhart control chart is used under linear trend disturbance. Perry and Pignatiello (2006) estimated the time of a linear trend change in the normal process mean. Perry *et al.* (2007) considered the monotonic change in the non-conformity level p, when the process is modeled by binomial distribution. Gazanfari *et al.* (2008) used clustering approach to identify the time of a step change in Shewhart control charts. Noorossana et al. (2009) estimated the step change point in the process non-conformity proportion when process is modelled by geometric distribution. Dogu and Kocakoc (2011) proposed change point model for generalized variance control chart. Zandi *et al.* (2011) estimated MLE of a change point for a linear trend disturbance in the process non-conformity.

There are many situations in which the sample size used for process monitoring is one (Montgomery (2012)). An individual control chart is usually used to monitor shifts in the process mean when it is not possible to form subgroups. Shewhart individual X chart have been extensively used in monitoring the process mean. The main drawback of Shewhart X chart is that it uses only information of the last sample observation and ignores the information of the process which makes it insensitive to small shifts in process mean. An alternative to detect small shifts is to use the memory type chart as like Cumulative sum (CUSUM) chart, exponentially weighted moving average (EWMA) chart or moving average (MA) chart. These charts consider the past as well as current information about the process, which makes charts very sensitive to small shifts in process parameters. Relative to CUSUM chart, the EWMA and MA charts are quite basic. The EWMA chart uses a weighted average as the chart statistic while the time weighted MA chart is based on simple moving averages. Kapase and Ghute (2018) estimated the time of a step change in the normal process mean with Tukey's control chart and individual X control chart and compared both the control charts in detecting the occurrence of the special cause in the process.

In this paper, we describe the application of change point estimators to memory type control charts namely EWMA and MA control charts based on individual observations using an approach developed by Samuel *et al.* (1998a). The remainder of this paper is organized as follows: In Section 2, change point estimation procedure is given. Section 3 provides the details of EWMA and MA charts. In Sections 4 and 5, we analyze the performance of the change point estimator for EWMA and MA control charts respectively. Section 6 provides numerical examples to demonstrate the use of estimator each with EWMA control chart and MA control chart. It is shown that change point estimator works well with EWMA and MA control charts. Some conclusions are given in Section 7.

2. Change Point Estimator

It is assumed that the process initially is in-control with a known value of mean μ_0 and variance σ_0^2. Following an unknown point in time a change in the process mean occurs from μ_0 to an out-of-control state mean $\mu_1 = \mu_0 + \delta\,\sigma_0/\sqrt{n}$ where n is the subgroup size and δ is the unknown magnitude of the change. Here we consider the case of individual observations (Subgroup size $n = 1$). It is assumed that σ_0 does not change while shift occurs in μ_0. It is also assumed that once this step change in the process mean occurs, the process remains at the new level of μ_1 until special cause has been identified and eliminated.

We will consider the process move to the out-of-control state at observation *T*. This out of signal can be obtained when a point is plotted beyond the control limits. Assuming this is not a false alarm, this is the point at which process professionals should initiate a search for special cause of variation. Let $X_1, X_2, \ldots, X_\tau$ be the observations from the in-control process, while $X_{\tau+1}, X_{\tau+2}, \ldots, X_T$ are the observations when the process changed, so that τ is the point where process change happened. This point τ is point in time when the shift in the process mean appears for first time and then process gets changed. Identifying this point in time when process change appears for first time the change point estimator works uniquely.

The data with subgroup size $n = 1$, the change point estimator is derived based on the method of the maximum likelihood estimator (Samuel *et al.* (1998) and Khoo (2004)).We denote the MLE of the change point τ as $\hat{\tau}$. For given single observations the MLE of τ is the value of τ which maximizes the logarithm of the likelihood function. The probability density function of the observation X, which follows normal distribution with mean μ and variance σ^2.

$$f(x,\mu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma}(x-\mu)^2} \quad , -\infty < x < \infty$$

The likelihood function (apart from constant) is

$$L(\tau, \mu_1 / x) = \prod_{i=1}^{\tau} e^{-(x_i-\mu_0)/2\sigma_0^2} + \prod_{i=\tau+1}^{T} e^{-(x_i-\mu_0)/2\sigma_0^2}$$

$$\log L(\tau, \mu_1 \mid x) = -\frac{1}{2\sigma_0^2}\left[\sum_{i=1}^{\tau}(x_i-\mu_0)^2 + \sum_{i=\tau+1}^{T}(x_i-\mu_1)^2\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\sum_{i=1}^{T}x_i^2 - 2\mu_0\sum_{i=1}^{\tau}x_i + \tau\mu_0^2 - 2\mu_1\sum_{i=\tau+1}^{T}x_i + (T-\tau)\mu_1^2\right] \tag{1}$$

There are two unknown parameters τ and μ_1 in the equation (1). If the change point τ is known the MLE of μ_1 is $\overline{X}_{T,t} = \sum_{i=\tau+1}^{T} \frac{X_i}{T-\tau}$.

Substituting this in the equation (1) follows:

$$\log L(\tau,\mu_1 \mid x) = -\frac{1}{2\sigma_0^2}\left[\sum_{i=1}^{T} x_i^2 - 2\,\mu_0 \sum_{i=1}^{\tau} x_i + \tau\,\mu_0^2 - \frac{2(\sum_{i=\tau+1}^{T} x_i)^2}{T-t} + \frac{(\sum_{i=\tau+1}^{T} x_i)^2}{T-t}\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\sum_{i=1}^{T} x_i^2 - 2\,\mu_0 \sum_{i=1}^{\tau} x_i + \tau\,\mu_0^2 - (T-t)\,\overline{X}_{T,t}^2\right]$$

$$= -\frac{1}{2\sigma_0^2}\left[\sum_{i=1}^{T} x_i^2 - 2\,\mu_0 \sum_{i=1}^{T} x_i + T\mu_0^2 - (T-\tau)(\overline{X}_{T,t} - \mu_0)^2\right]$$

It follows that the value of τ which maximizes the log-likelihood function is

$$\hat{\tau} = \underset{0 \le t < \mathrm{T}}{\arg\max}\left\{(T-t)\,(\overline{X}_{T,t} - \mu_0)^2\right\}$$

3. EWMA and MA Control Charts

Assume that $X_1, X_2, X_3, \ldots$ denote independent and identically distributed observations with an in-control mean μ_0 and standard deviation σ_0. We assume that both the parameters are known. In practice, μ_0 and σ_0 are estimated from the observed historical data. The EWMA control statistic for individual observations is defined as

$$Z_i = \lambda\,X_i + (1-\lambda)\,Z_{i-1}\,, \qquad \text{for } i = 1,2,\ldots \text{ and } Z_0 = \mu_0,$$

where X_i is the current observation and $0 < \lambda \le 1$ is the smoothing parameter.
The exact control limits for the EWMA chart are

$$LCL = \mu_0 - L\,\sigma_0\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}]}$$

$$UCL = \mu_0 + L\,\sigma_0\sqrt{\frac{\lambda}{(2-\lambda)}[1-(1-\lambda)^{2i}]}, \quad \text{for } i = 1,2,\ldots$$

where $L > 0$ determines the width of the control limits.

The moving average statistic of span w at time i for a sequence of observations iscomputed as

$$M_i = \frac{X_i + X_{i-1} + \ldots + X_{i-w+1}}{w}, \quad \text{for } i \ge w$$

For periods $i < w$, we compute the average of available observations. In other words, average of all observations up to period i defines moving average.
The control limits for the moving average control chart are as follows:

$$UCL\,/\,LCL = \mu_0 \pm \frac{3\,\sigma_0}{\sqrt{w}} \quad \text{for } i \ge w \text{ and}$$

$$UCL/LCL = \mu_0 \pm \frac{3\sigma_0}{\sqrt{i}} \text{ for } i < w$$

4. Change Point Estimator Used with EWMA Control Chart

We consider EWMA control chart to study the performance of the estimator. We used Monte Carlo simulation to study the performance of the change point estimator. The change point of the process is simulated at observation $\tau = 100$.The first 100 individual observations are randomly generated from standard normal distribution. Then starting from observation 101, the individual observations are randomly generated from changed process with normal distribution with mean δ and standard deviation 1 until the EWMA chart gives an out-of-control signal. At this point $\hat{\tau}$ is computed. This procedure is repeated a total number of $N = 10000$ times for each of the values of $\delta = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ with different values of parameters $L = 2.86, \lambda = 0.2$ which has in-control $ARL_0 = 370.37$, same as the Shewhart control chart and $(L = 2.615, \lambda = 0.05), (L = 2.814, \lambda = 0.1), (L = 2.998, \lambda = 0.25)$ have in-control ARL_0 = 500. The average of the estimates obtained using the estimator from the 10,000 simulation runs is computed with their standard error.

Tables 1-4 show $E(T)$, the expected number of observations at which the control chart signals a change in the process mean that occurred at time 100. Thus, $E(T) = ARL + 100$. We show that $\bar{\hat{\tau}}$ the average change point estimate obtained using MLE change point estimator with their standard error.

Table 1: Average change point estimates for δ and standard errors when used with a EWMA control chart, $ARL_0 = 500, L = 2.615, \lambda = 0.05, \tau = 100$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$E(T)$	123.17	107.16	103.73	102.40	101.75	101.40
$\bar{\hat{\tau}}$	104.10	99.84	99.87	99.86	99.85	99.88
$s.e.(\bar{\hat{\tau}})$	0.2086	0.0532	0.0275	0.0161	0.0111	0.0084

In Table 1, we can see that the expected number of observations required to detect the change in the process mean for the magnitude of the shift $\delta = 1.0$ is 107.16. The average change point estimate is 99.84. Since the change point is simulated at point 100, the average change point estimate should be close to 100. For the magnitude of the shift $\delta = 2.0$ the control chart issues signal at 102.40, that of average change point estimate is 99.86 which is close to 100.

In Table 2, we can see that the control chart gives out of control signal at 108.16 on an average of 10000 simulation trial for the magnitude of the shift $\delta = 1.0$. The average change point estimate is 99.54 which is close to 100. For the magnitude of the shift $\delta = 2.5$ the control chart issues signal at 101.92 and that of average change point estimate is 99.90. For $\delta = 3.0$, $E(T) = 101.50$ and that of the average change point estimate is 99.92 which is again close to 100.

In Table 3, we can see that for the magnitude of the shift $\delta = 1.0$ the expected number of observations at the point when control chart gives out of control signal at 110.36 on an average of 10000 simulation trial. The average change point estimate is 100.09 which are close to 100. For the magnitude of the shift $\delta = 2.5$ the control chart issues signal at 102.08 and that of average change point estimate is 99.94. For $\delta = 3.0$, $E(T) = 101.62$ and that of the average change point estimate is 99.92 which is again close to 100.

In Table 4, we can see that for the magnitude of the shift $\delta = 1.0$ the expected number of observations at the point when control chart gives out of control signal at 108.80 on an average of 10000 simulation trial. The average change point estimate is 99.85 which are close to 100. For the magnitude of the shift $\delta = 1.5$ the control chart issues signal at 104.32 on an average of the 10000 simulation trials and that of average change point estimate is 99.94. For $\delta = 3.0$, $E(T) = 101.53$ and that of the estimated change point is 99.93 on an average of total 10000 simulation trial which is again close to 100.

Table 2: Average change point estimates for δ and standard errors when used with a EWMA control chart, $ARL_0 = 500, L = 2.814, \lambda = 0.1, \tau = 100$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$E(T)$	128.61	108.16	104.13	102.64	101.92	101.50
$\bar{\hat{\tau}}$	105.45	99.54	99.91	99.90	99.90	99.92
$s.e.(\bar{\hat{\tau}})$	0.2471	0.0527	0.0258	0.0158	0.0106	0.0076

Table 3: Average change point estimates for δ and standard errors when used with a EWMA control chart, $ARL_0 = 500,\ L = 2.998,\ \lambda = 0.25,\ \tau = 100$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$E(T)$	147.41	110.36	104.8	102.93	102.08	101.62
$\bar{\hat{\tau}}$	103.03	100.09	99.98	99.91	99.94	99.92
$s.e.(\bar{\hat{\tau}})$	0.2235	0.0506	0.0247	0.0142	0.0092	0.0069

Table 4: Average change point estimates for δ and standard errors when used with a EWMA control chart, $ARL_0 = 370.37, L = 2.86, \lambda = 0.2, \tau = 100$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$E(T)$	135.00	108.80	104.32	102.71	101.95	101.53
$\bar{\hat{\tau}}$	103.99	99.85	99.94	99.88	99.91	99.93
$s.e.(\bar{\hat{\tau}})$	0.2382	0.0521	0.0251	0.0152	0.01	0.0072

The precision of the change point estimator for process mean can be examining the probability that $\hat{\tau}$ within m observations of the exact change point. Table 5 contains the results for the case where $L = 2.86, \lambda = 0.2$. For the magnitude of the shift $\delta = 0.5$, the probability that the change point estimator correctly identified the actual time of change in the process mean 9% of the simulation trials. The change point estimator correctly identified the actual time of the change time of the change in 45% and 55% of the simulation trials within 6 and 9 observations respectively.

It can also be seen that for the value of the shift in the parameter 1.5, the estimator is within 2 (6) observations of the actual process change point in 83% (96%) of the simulation trials. For the magnitude of the shift $\delta = 2.0$, the probability that the change point estimator correctly identified the exact time of change in process mean is 62% of simulation trials. The probability that the estimator correctly identified the time of the change within 3 (7) observations is 95% (99%) of the simulation trials.

Table 5: Precision of estimator for δ when used with EWMA control chart $\tau = 100$ and $L = 2.86$, $\lambda = 0.2$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$P[\hat{\tau} - \tau = 0]$	0.09	0.27	0.44	0.62	0.75	0.84
$P[\lvert \hat{\tau} - \tau \rvert \le 1]$	0.19	0.48	0.71	0.80	0.86	0.96
$P[\lvert \hat{\tau} - \tau \rvert \le 2]$	0.26	0.58	0.83	0.85	0.95	0.98
$P[\lvert \hat{\tau} - \tau \rvert \le 3]$	0.32	0.67	0.89	0.95	0.97	0.995
$P[\lvert \hat{\tau} - \tau \rvert \le 4]$	0.37	0.73	0.93	0.97	0.98	0.997
$P[\lvert \hat{\tau} - \tau \rvert \le 5]$	0.41	0.80	0.95	0.98	0.989	0.9987
$P[\lvert \hat{\tau} - \tau \rvert \le 6]$	0.45	0.83	0.96	0.987	0.993	0.9989
$P[\lvert \hat{\tau} - \tau \rvert \le 7]$	0.49	0.86	0.97	0.991	0.994	0.999
$P[\lvert \hat{\tau} - \tau \rvert \le 8]$	0.52	0.88	0.98	0.994	0.995	0.999
$P[\lvert \hat{\tau} - \tau \rvert \le 9]$	0.55	0.91	0.987	0.995	0.997	0.999
$P[\lvert \hat{\tau} - \tau \rvert \le 10]$	0.59	0.92	0.99	0.997	0.997	1
$P[\lvert \hat{\tau} - \tau \rvert \le 11]$	0.61	0.94	0.996	0.999	0.999	1

5. Change Point Estimator Used with MA Control Chart

In this section, we consider MA control chart to study how well the estimator performs. As with the EWMA control chart, we used Monte Carlo simulation to study the performance of the change point estimator. The change point of the process is simulated at observation $\tau = 100$. The first 100 in-control individual observations are randomly generated from standard normal distribution. Then starting from observation 101, the individual observations are randomly generated from normal distribution with mean δ and standard deviation 1 until the moving average chart gives an out-of-control signal. At this point $\hat{\tau}$ is computed. This procedure is repeated a total number of $N = 10000$ times for each of the values of $\delta = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0$ with different values of moving average span $w = 1, 2, 3$. The average of the estimates obtained using the estimator from the 10000 simulation runs is computed with their standard error.

Table 6-8 shows $E(T)$, the expected number of observations at which the control chart signals a change in the process mean that occurred at time 100. Thus, $E(T) = ARL + 100$. We show that $\bar{\hat{\tau}}$ the average change point estimate obtained using MLE change point estimator with their standard error.

Table 6: Average change point estimates for δ and standard errors when used with MA control chart, $w = 1$, $\tau = 100$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$E(T)$	256.11	143.70	115.10	106.34	103.23	102.02
$\bar{\hat{\tau}}$	108.5	100.85	100.18	99.95	99.85	99.82
$s.e.(\bar{\hat{\tau}})$	0.251	0.057	0.027	0.018	0.015	0.013

In Table 6, we can see that for the magnitude of the shift $\delta = 1.0$ the control chart issues signal at 143.70 on an average of 10000 simulation trial. The average change point estimate is 100.85 which are close to 100. For the magnitude of the shift $\delta = 1.5$ the control chart issues signal at 115.10 and that of average change point estimate is 100.18. For $\delta = 3.0$, the expected number to issue a signal from control chart is 102.02. The average change point estimate is 99.82.

Table 7: Average change point estimates for δ and standard errors when used with MA control chart, $w = 2$, $\tau = 100$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$E(T)$	203.58	122.63	107.58	103.62	102.23	101.65
$\bar{\hat{\tau}}$	109.59	100.93	99.99	99.79	99.73	99.75
$s.e.(\bar{\hat{\tau}})$	0.252	0.057	0.031	0.021	0.018	0.016

In Table 7, it is seen that for the magnitude of the shift $\delta = 1.5$ the control chart issues signal at 107.58. The average change point estimate is 99.99 which are close to 100. For the magnitude of the shift $\delta = 2.0$ the control chart issues signal at 103.62. The average change point estimate is 99.79. For $\delta = 3.0$, the expected number to issue a signal from control chart is 101.65. The average change point estimate is 99.75 which are again close to 100.

Table 8: Average change point estimates for δ and standard errors when used with MA control chart, $w = 3$, $\tau = 100$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$E(T)$	183.63	116.52	105.95	103.15	102.11	101.62
$\bar{\hat{\tau}}$	109.66	100.79	99.89	99.69	99.71	99.74
$s.e.(\bar{\hat{\tau}})$	0.2413	0.0575	0.0319	0.024	0.019	0.017

In Table 8, we can see that for the magnitude of the shift $\delta = 1.5$ the expected number of observations at the point when control chart gives out of control signal at 105.95 on an average of 10000 simulation trial. The average change point estimate is 99.89 which are close to 100. For the magnitude of the shift $\delta = 2.5$ the control chart issues signal at 102.11 on an average of the 10000 simulation trials and that of average change point estimate is 99.71. For $\delta = 3.0$, $E(T) = 101.62$ and that of the estimated change point is 99.74 on an average of total 10000 simulation trial.

We next consider the observed frequency with which the estimator of the time of step change is within *m* observations of the exact change point, for $m = 0, 1, \ldots, 11$. This indicates the precision of the proposed estimator. This table contains the results for the case where $w = 2$. For the magnitude of the shift $\delta = 0.5$, the precision that the change point estimator correctly identified the actual time of change in the process mean 9% of the simulation trials, same as EWMA control chart. The change point estimator correctly identified the actual time of the change within the 2(6) is 26% and 47% of the simulation trials within the respectively.

It can also be seen that for the value of the shift in the parameter 1.0, the estimator is within 3(9) observations of the actual process change point in 68% (91%) of the simulation trials. For the value of the shift 2.0 the estimator correctly identified the actual change point is 61%. The precision of the estimator within the 4(8) observations of the actual change point is 97% (99%) of the total simulation trials. The precision of estimated time of the change within *m* observations of the actual change point should increase as *m* increases.

Table 9: Precision of estimator for δ when used with MA control chart $\tau = 100$ and $w = 2$ and $N = 10000$ independent simulation trials

δ	0.5	1.0	1.5	2.0	2.5	3.0
$P[\lvert\hat{\tau}-\tau\rvert = 0]$	0.09	0.27	0.45	0.61	0.75	0.84
$P[\lvert\hat{\tau}-\tau\rvert \le 1]$	0.19	0.47	0.70	0.83	0.90	0.93
$P[\lvert\hat{\tau}-\tau\rvert \le 2]$	0.26	0.60	0.82	0.92	0.95	0.97
$P[\lvert\hat{\tau}-\tau\rvert \le 3]$	0.33	0.68	0.88	0.95	0.97	0.98
$P[\lvert\hat{\tau}-\tau\rvert \le 4]$	0.38	0.75	0.92	0.97	0.98	0.984
$P[\lvert\hat{\tau}-\tau\rvert \le 5]$	0.42	0.80	0.94	0.98	0.988	0.988
$P[\lvert\hat{\tau}-\tau\rvert \le 6]$	0.47	0.84	0.96	0.983	0.990	0.990
$P[\lvert\hat{\tau}-\tau\rvert \le 7]$	0.50	0.87	0.97	0.989	0.992	0.992
$P[\lvert\hat{\tau}-\tau\rvert \le 8]$	0.53	0.89	0.98	0.991	0.995	0.994
$P[\lvert\hat{\tau}-\tau\rvert \le 9]$	0.56	0.91	0.983	0.993	0.996	0.997
$P[\lvert\hat{\tau}-\tau\rvert \le 10]$	0.59	0.92	0.986	0.997	0.999	0.999
$P[\lvert\hat{\tau}-\tau\rvert \le 11]$	0.61	0.94	0.99	0.999	1	1

6. Examples of Application

This section provides numerical examples to demonstrate the use of estimator each with EWMA control chart and MA control chart. The change point estimator works well with EWMA and MA control charts.

Example-1: EWMA Control Chart

In this example, we consider the data of a production process for forged piston rings used in the illustrative example of (Samuel *et al*., 1998a). The in-control process follows a normal distribution with mean 100 and standard deviation 5. Each subgroup has n = 4 observations. The EWMA control chart with λ = 0.1 and L = 2.703 is considered. From the original data of 27 subgroups given in (Samuel et al., 1998a), only the first 20 subgroups are required before the EWMA chart signals an out-of-control, since $Z_i > UCL$. Table 10 summarizes the 20 subgroup averages and the corresponding EWMA statistics.

Table 10: Subgroup averages and the corresponding EWMA statistics

Subgroup (i)	$\overline{X}_i$	EWMA Z_i	UCL	LCL
1	100.45	100.045	101.351	98.648
2	97.45	100.15	101.818	98.181
3	102.45	97.95	102.122	97.877
4	100.675	102.272	102.339	97.660
5	98.550	100.462	102.502	97.497
6	102.95	98.990	102.626	97.373
7	98.825	102.537	102.722	97.277
8	101.325	99.075	102.798	97.201
9	103.075	101.5	102.858	97.141
10	99.600	102.727	102.900	97.094
11	98.825	99.522	102.943	97.056
12	97.950	98.737	102.974	97.025
13	100.425	98.197	102.998	97.001
14	96.075	99.99	103.018	96.981
15	101.225	96.59	103.034	96.965
16	103.075	101.41	103.046	96.953
17	101.925	102.96	103.057	96.942
18	101.350	101.867	103.065	96.934
19	103.575	101.572	103.072	96.927
20	102.925	**103.509**	**103.077**	96.922

Table 11 summarizes the reverse cumulative subgroup averages and C_t values for $t = 1, 2, \ldots, 19$ and T = 20. The value of t which gives the maximum C_t value is the estimator of the last subgroup from the in-control process. From the results in Table 11, we observed that the maximum value of C_t is 32.998 and it happens at t = 15. Thus, it is estimated that subgroup 16 is the first subgroup obtained from the shifted process and that subgroup 15 is the last subgroup from the in-control process.

However, the EWMA chart enables an out-of-control signal to be detected earlier, *i.e.*, at subgroup 20 compared to the time when the $\overline{X}$ chart first detected an out-of-control signal, *i.e.*, at subgroup 27 (see Samuel *et al*., 1998a).

Table 11: The computed $\bar{\bar{X}}_{20,t}$ and the corresponding C_t values

Subgroup (i)	$\bar{X}_i$	T	$\bar{\bar{X}}_{20,t}$	*Ct*
1	100.45	0	100.634	8.058
2	97.45	1	100.644	7.891
3	102.45	2	100.821	12.160
4	100.675	3	100.726	8.964
5	98.55	4	100.729	8.511
6	102.95	5	100.874	11.475
7	98.825	6	100.726	7.387
8	101.325	7	100.872	9.900
9	103.075	8	100.835	8.366
10	99.6	9	100.631	4.384
11	98.825	10	100.734	5.394
12	97.95	11	100.946	8.065
13	100.425	12	101.321	13.965
14	96.075	13	101.449	14.703
15	101.225	14	102.345	32.994
16	**103.075**	**15**	**102.569**	**32.998**
17	101.925	16	102.442	23.863
18	101.35	17	102.615	20.514
19	103.575	18	103.2475	21.09251
20	102.92	19	102.92	8.5264

Example 2: MA Control Chart

In this example, we consider the data of 20 observations coming from normal distribution with mean 10 and last 10 observations with mean 11 with common standard deviation 1. This data is used in the example of (Montgomery, 2012). The moving average control chart with subgroup size $n = 1$is considered. For purpose of the use of the change point estimator with given data we take large value of $w = 6$. From the original data of 30observations given in (Montgomery, 2012), only the first 28 observations are required before the moving average chart signals an out-of-control, since $M_i>UCL$. Table A.1 (Appendix) summarizes the 28 observations and the corresponding moving average statistics.

Table A.2 shows the reverse cumulative averages and corresponding values of C_t for $t = 1, 2, \ldots, 27$ and $T = 28$. The value of t which maximizes the value of C_t is the estimator of the last observation from the in-control process. From Table A.2, we can observe that at observation $t = 22$ the maximum value of C_t is **6.025.** Thus, it is estimated that observation 23 is the first observation obtained from the shifted process and that observation 22 is the last observation from the in-control process.

From this example we can observe the moving average control chart signals out-of-control at $t = 28$ and that the change point estimator identified the change in the process at

$t = 22$. This shows that the change point estimator fairly works with moving average control chart to identify the out-of-control signal earlier.

7. Conclusion

Control charts are used to detect whether or not a process has changed. When a control chart detects the shift in a process, process professionals initiate a search to find the special causes of variation in the process. When the process gets changed, the process change is not usually known to the process professionals. However, the process professionals knew when the change started in the process; it will help to provide the scope of the search window at what time the process gets changed. Subsequently, it helps to eliminate the sources of the special causes.

In this paper, an estimator based on the maximum likelihood estimator method is used with EWMA control chart and MA chart to find the step change that occurred in the normal process mean. The results show that the change point estimator is helpful to detect the change in the process. The EWMA and MA control charts are effective to detect the small shifts in the process mean. The change point estimator also performs well to detect the small changes with EWMA and moving average control chart.

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APPENDIX

Table A.1: Averages and the corresponding moving average statistics

Subgroup (i)	$\overline{X}_i$	Moving Average M_i	UCL	LCL
1	9.45	9.45	13	7
2	7.99	8.72	12.12	7.88
3	9.29	8.91	11.73	8.27
4	11.66	9.722	11.50	8.5
5	12.16	9.814	11.34	8.66
6	10.18	9.518	11.22	8.78
7	8.04	9.853	11.22	8.78
8	11.46	10.055	11.22	8.78
9	9.20	10.23	11.22	8.78
10	10.34	9.708	11.22	8.78
11	9.03	9.923	11.22	8.78
12	11.47	10.335	11.22	8.78
13	10.51	9.991	11.22	8.78
14	9.40	10.138	11.22	8.78
15	10.08	9.976	11.22	8.78

16	9.37	10.241	11.22	8.78
17	10.62	10.04	11.22	8.78
18	10.31	9.716	11.22	8.78
19	8.52	9.956	11.22	8.78
20	10.84	10.083	11.22	8.78
21	10.90	10.338	11.22	8.78
22	9.33	10.123	11.22	8.78
23	12.29	10.453	11.22	8.78
24	11.50	10.95	11.22	8.78
25	10.60	10.91	11.22	8.78
26	11.08	10.95	11.22	8.78
27	10.38	10.863	11.22	8.78
28	11.62	**11.245**	**11.22**	8.78

Source: Montgomery D. C. (2012). *Introduction to Statistical Quality Control*

Table A.2: The computed $\bar{\bar{X}}_{28,t}$ and the corresponding C_t values

Subgroup (i)	$\bar{X}_i$	t	$\bar{\bar{X}}_{28,t}$	C_t
1	9.45	0	10.242	0
2	7.99	1	10.272	0.023
3	9.29	2	10.36	0.356
4	11.66	3	10.402	0.639
5	12.16	4	10.329	0.180
6	10.18	5	10.336	0.199
7	8.04	6	10.440	0.858
8	11.46	7	10.391	0.466
9	9.20	8	10.451	0.870
10	10.34	9	10.457	0.874
11	9.03	10	10.536	1.553
12	11.47	11	10.481	0.970
13	10.51	12	10.480	0.899
14	9.40	13	10.552	1.433
15	10.08	14	10.585	1.645
16	9.37	15	10.679	2.475
17	10.62	16	10.684	2.337
18	10.31	17	10.718	2.485
19	8.52	18	10.938	4.832
20	10.84	19	10.948	4.486
21	10.90	20	10.962	4.143
22	9.33	21	10.971	3.715
23	**12.29**	**22**	**11.245**	**6.025**
24	11.50	23	11.036	3.145
25	10.60	24	10.920	1.834
26	11.08	25	11.026	1.843
27	10.38	26	11.000	1.146
28	11.62	27	11.620	1.896