Statistics and Applications {ISSN 2454-7395 (online)}
Volume 21, No. 2, 2023 (New Series), pp 37-58
http://www.ssca.org.in/journal

Prediction Intervals in ARCH Models Using Sieve Bootstrap Robust Against Outliers

Samir Barman, Ramasubramanian V., Mrinmoy Ray and Ranjit Kumar Paul
ICAR-Indian Agricultural Statistics Research Institute, New Delhi, India

Received: 07 May 2022; Revised: 28 August 2022; Accepted: 30 September 2022

Abstract

One of the primary goals of time series (TS) modeling is to forecast future observations. Although point forecasts are the most common type of prediction, interval forecasts are more informative and are typically obtained as prediction intervals (PIs). For non-linear TS data, the ARCH model is one of the widely used models. The Sieve Bootstrap method is a popular method for constructing PIs in TS models. The TS data are not always free from outliers, whose presence may result in an increase in the length of PIs obtained also with poor coverage. In this study, two new robust Sieve Bootstrap approaches based on weighted least squares estimation have been proposed to deal with the presence of outliers for developing PIs for both returns and volatilities in the ARCH model setup. The performances of the proposed methods viz., Robust Unconditional Sieve Bootstrap (RUSB) and Robust Sieve Bootstrap (RSB) for constructing PIs using both simulated as well as real data sets have been found to be better when compared with their existing counterparts.

Key words: Coverage probability; Innovative outlier; Length of prediction interval; Return; Volatility; Weighted least squares.

1. Introduction

A time series (TS) is an ordered sequence of data points observed over time, typically at equally spaced time intervals. The analysis of TS is essential not only in agriculture but also in other diverse fields such as economics, finance pattern recognition, tourism etc. In all these areas, TS methodologies are used not only to model TS data, but also to forecast future values of such processes. TS predictions can be observed either as point or interval estimates. Point estimation is concerned with predicting a single value from a set of observations, whereas interval estimation provides prediction intervals (PIs), with some probability, within which forecasted future values will lie. There are many reasons for preferring PIs over point estimates. PIs help to assess the future uncertainty in a broad manner for better risk management decisions, plan different strategies for the range of possible outcomes, and explore scenarios based on different assumptions more carefully and so on. A good account

Corresponding Author: Ramasubramanian V.
Email: ram.vaidhyanathan@gmail.com

on PIs in TS can be found in many books, to cite a few, Politis *et al.* (1999), Chatfield (2000) and Lahiri (2003).

In the context of agricultural commodity price or any financial TS data, generally linear TS models with homoscedastic error variance are popularly used until it need to deal with volatile data. Volatility being the sudden unexpected rise or fall in TS, measuring it plays an important role in assigning risk and uncertainty. While modeling TS, a series is said to be volatile when a few error terms are larger than the others and are responsible for the unique behavior of the series, resulting in heteroscedasticity. To deal with volatilities and non-linear dynamics, the Auto-Regressive Conditional Heteroscedastic (ARCH) model proposed by Engle (1982) where the idea is to model volatilities as a linear function of previous returns, is popularly employed. By adding a moving average part, the ARCH model was generalized by Bollerslev (1986) in the form of the Generalized ARCH (GARCH) model for the parsimonious representation of ARCH. In the GARCH model, the conditional variance is also a linear function of its own lags. In this context, the GARCH model became the most popularly used for modeling volatility and obtaining dynamic PIs for returns and volatilities. Many recent studies are found on the non-linear TS processes in modeling volatilities (to cite a few, see, Bhardwaj *et al.*, 2014; Lama *et al.*, 2015; Bentes, 1015; Dyhrberg, 2016).

Existing literature mainly focused on point forecasts of volatilities and little attention has been given to constructing the PIs (Baillie and Bollerslev, 1992; Andersen and Bollerslev, 1998; Andersen *et al.*, 2001; Poon, 2005). However, the construction of PIs in TS models with finite parameters, requires knowledge of the distribution of the observed data, which is typically unknown in practice. Several studies have shown that when the underlying distributional assumptions are violated the resulting PIs can be adversely affected yielding poor results (Thombs and Schucany, 1990). The construction of PIs in TS models with finite parameters and with known innovative processes has been widely discussed in the literature and it has been found that these PIs are extremely sensitive to the presence of outliers (Tsay, 1988, 2010). Moreover, over time, several distribution-free methods, using resampling techniques using Bootstrap method, have been proposed as an alternative for the construction of PIs. One of the popular and effective Bootstrap procedures is residual-based resampling *i.e.* resampling the residuals from the fitted model on the TS (Bühlmann, 2002; Politis, 2003; Härdle *et al.*, 2003). Miguel and Olave (1999) first proposed a Bootstrap procedure for a non-linear ARCH model for the construction of PIs for return and volatilities by directly adding resampled residuals from the ARCH model to the respective point forecasts. This work was improved by Reeves (2005) by adding an additional step of re-estimating the ARCH parameters for each Bootstrap realization of the returns, which considered the variability of the estimated parameters of the ARCH model. Further, Pascual *et al.* (2006) extended these procedures for the GARCH model in different ways and obtained the PIs for both returns and volatilities which were found to be well-calibrated *i.e.*, the number of observed data falling within PIs coincided with the declared coverage. However, these procedures involve the estimation of ARCH/GARCH parameters by maximum likelihood (ML) estimation and are computationally expensive. Hence as an improvement over these, Chen *et al.* (2011) proposed a computationally efficient and distribution-free resampling technique for developing PIs for both returns and volatilities in ARCH and GARCH processes. Their method was based on the Sieve Bootstrap procedure used in the linear model AR/ARMA representation of the ARCH/GARCH process. In particular, the squared returns from the ARCH/GARCH model is a linear process that follows an AR/ARMA process (Tsay, 2010;

Box *et al.*, 2015). Bose and Mukherjee (2009) proposed a weighted linear estimator (WLE) to estimate the ARCH parameters, and a corresponding Bootstrap weighted linear estimator (BWLE). An alternative WLE method in the context of multivariate ARCH models was proposed by Iqbal (2011) and improved results were reported. Later, Iqbal and Chand (2013) constructed efficient PIs for returns and volatility for ARCH models using a particular version of residual Bootstrap. Further Pan and Politis (2016) proposed a Bootstrap algorithm for developing PIs for ARCH models based on BWLE. However, these above-mentioned approaches including the Sieve Bootstrap procedure are affected by the presence of innovative outliers, resulting in an undesirable increase in the length of the PIs. In recent times, Ulloa *et al.* (2014) and Allende *et al.* (2015) have proposed a residual-based resampling technique for developing robust PIs for returns and volatilities for GARCH models based on the winsorized residuals. Trucíos *et al.* (2017) constructed Bootstrap densities for returns and volatilities using a robust parameter estimator based on variance-targeting implemented together with an adequate modification of the volatility filter in analyzing the effect of additive outliers. Beyaztas and Shang (2020) proposed a robust Bootstrap technique for PI construction in AR models based on weighted likelihood estimates and weighted residuals. The presence of outliers can have an impact on TS analysis, leading to incorrect model identification and parameter estimation and TS forecasts obtained from such models could be erroneous. Hence, there is always a need to develop improved and computationally efficient Bootstrap methods in computing PIs for TS aimed at providing better forecasts. In this study, the focus is on developing models robust against the presence of outliers to get improved PIs. This approach of robust modeling has been applied using the Sieve Bootstrap procedure for developing PIs for both return and volatilities in the ARCH model setup. In addition, instead of applying least square estimation (see, Chen *et al.*, 2011), a weighted least squares (WLS) estimation has been applied. The details of the new WLS method and the proposed Bootstrap procedure have been described in subsequent sections.

Towards this end, two new Bootstrap approaches for constructing PIs have been proposed in this study. The remainder of the article is organized as follows. The next section discusses the two proposed methods by first describing about the ARCH models and the weighted least squares procedure employed. Thereafter Section 3 deals with the results of the simulation study conducted followed by Section 4 which contains a case study on a real data set. The paper is signed off with concluding remarks in Section 5.

2. Methodology

2.1. ARCH models

A non-linear TS model can be expressed as $y_t = f(\varepsilon_t, \varepsilon_{t-1}, \dots)$ where $f(\cdot)$ is the non-linear function of past and present random shocks. In such a setup, consider a TS $\{y_t\}_{t=1}^n$ following ARCH(p) process, $p \geq 1$ has the following representation:

$$y_t = \sigma_t \varepsilon_t \tag{1}$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 \tag{2}$$

where $\{\varepsilon_t\}_{t=1}^n$ is a sequence of independently and identically distributed (*i.i.d.*) random variables with zero mean and unit variance and $E(\varepsilon_t^4) < \infty$; the volatility process $\{\sigma_t\}_{t=1}^n$ is a stochastic process assumed to be independent of $\{\varepsilon_t\}_{t=1}^n$; α_0, α_i's are unknown parameters satisfying $\alpha_0, \alpha_i \geq 0$, for $i = 1, 2, \ldots, p$. The process is assumed to be weakly stationary (Tsay, 2010) *i.e.* $\sum_{i=1}^p \alpha_i < 1$ is satisfied. Further, it is assumed that the strict stationarity conditions of $\{y_t\}_{t=1}^n$ given in Bougerol and Picard (1992a, 1992b) hold.

Despite the non-linear nature of variance in ARCH models, they can be represented by means of the linear AR model (Tsay, 2010; Box *et al.*, 2015). In particular, the squared returns of an ARCH model is a linear process that can be written as an AR representation. From (1) and (2),

$$y_t^2 = \sigma_t^2 \varepsilon_t^2 \tag{3}$$

$$\alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 = \sigma_t^2 \tag{4}$$

Subtracting equation (4) from equation (3),

$$y_t^2 - \left(\alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2\right) = \sigma_t^2 \varepsilon_t^2 - \sigma_t^2 \tag{5}$$

Let, $\nu_t = \sigma_t^2 \varepsilon_t^2 - \sigma_t^2 = y_t^2 - \sigma_t^2$, and by substituting $\sigma_t^2 = y_t^2 - \nu_t$ in (4) yielding,

$$y_t^2 - \nu_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2$$

$$y_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \nu_t \tag{6}$$

where $\{y_t^2\}_{t=1}^n$is an AR(p) process and $\nu_t = y_t^2 - \sigma_t^2$ is white noise but not *i.i.d.*, in general. Under strict stationarity assumptions of $\{y_t\}_{t=1}^n$, innovations $\{\nu_t\}_{t=1}^n$ are identically distributed.

Let $p = 1$, then, $\{y_t\}_{t=1}^n$ follows ARCH(1):

$$y_t = \sigma_t \varepsilon_t \tag{7}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 \tag{8}$$

Then from equation (6), ARCH(1) can be expressed in AR(1) form:

$$y_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \nu_t \tag{9}$$

Similarly, suppose $\{y_t\}_{t=1}^n$ follows an ARCH(2), then it can be rewritten in AR(2) form as:

$$y_t^2 = \alpha_0 + \alpha_1 y_{t-1}^2 + \alpha_2 y_{t-2}^2 + \nu_t \tag{10}$$

2.2. Weighted least squares (WLS) estimation

In this study, following Chen *et al.* (2011), the AR parameterization of the ARCH model presented in equation (6) has been considered and estimated using WLS estimation for constructing the PIs. Let, $x_t = y_t^2$ and for an ARCH model equation (6) can be written as,

$$x_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i x_{t-i} + \nu_t \tag{11}$$

The Least Squares (LS) estimators of an AR(p) model are obtained by fitting a linear regression of x_t onto $x_{t-1}, x_{t-2}, \ldots, x_{t-m}$. In matrix notation, let $\boldsymbol{z}$ and $\boldsymbol{X}$ as follows:

$$\boldsymbol{z} = \begin{bmatrix} x_{p+1} \\ \vdots \\ x_n \end{bmatrix} \text{ and } \boldsymbol{X} = \begin{bmatrix} 1 & x_p & x_{p-1} & \cdots & x_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-1} & x_{n-2} & \cdots & x_{n-p} \end{bmatrix}$$

The LS estimate of parameters $\widehat{\Phi} = (\widehat{\alpha}_0, \widehat{\alpha}_1, \ldots, \widehat{\alpha}_p)'$ is obtained as

$$\widehat{\Phi} = \left(\boldsymbol{X}'\boldsymbol{X}\right)^{-1} \boldsymbol{X}'\boldsymbol{z} \tag{12}$$

with $\boldsymbol{X}'\boldsymbol{X}$ is non-singular.

It is a known fact that when the TS data are contaminated with outliers, the LS estimates of model parameters are affected *i.e.* they produce biased estimates and the errors computed corresponding to outliers will be large. Thus the Bootstrap PIs based on LS estimates may not provide reliable results in the presence of outliers. Therefore it is proposed to construct robust Bootstrap PIs for the ARCH process based on WLS estimates of parameters, on similar lines to the weighted procedure employed in the case of likelihood estimation by Markatou *et al.* (1998) and Beyaztas and Shang (2020); also in partial least squares estimation by Beyaztas and Shang (2021) to improve the robustness of the estimates.

Now, from equation (11), let $\nu_t(\Phi) = \nu_t(\Phi|x_t) = x_t - \alpha_0 - \sum_{i=1}^{p} \alpha_i x_{t-i}$ for $t = p+1,\ p+2, \ldots, n$ be the model residuals, where the values of ν_t for $t \leq p$ are taken as zero. Let $f^*(\cdot)$ be the non-parametric kernel density estimator and $m^*(\cdot)$ be the smoothed model density, respectively, defined as follows:

$$f^*\left(\nu_t(\Phi), \hat{F}_\nu(\Phi)\right) = \int k\left(\nu_t(\Phi), r, d\right) d\hat{F}_\nu(r,\Phi) \quad \forall\ t = 1, 2, \ldots, n$$

$$m^*\left(\nu_t(\Phi), \sigma^2\right) = \int k\left(\nu_t(\Phi), r, d\right) dM\left(r, \sigma^2\right)$$

where $\hat{F}_\nu(\Phi)$ is the empirical cumulative distribution function based on $\nu_t(\Phi)$ and $M(\sigma^2)$ is actual assumed model distribution function with variance σ^2, such as general normal distribution with zero mean and variance σ^2. Function $k(\nu_t(\Phi), r, d)$ is the kernel density with bandwidth d. The weight function, say $w(\cdot)$, is defined according to the minimum discrepancy measure, as a measure of agreement between the parametric model of the error and the actual residuals. Following Beyaztas and Shang (2020, 2021), the Pearson residual δ_t is then defined as:

$$\delta_t = \delta\left(\nu_t(\Phi); M\left(\sigma^2\right), \hat{F}_\nu(\Phi)\right) = \frac{f^*\left(\nu_t(\Phi), \hat{F}_\nu(\Phi)\right) - m^*\left(\nu_t(\Phi), \sigma^2\right)}{m^*\left(\nu_t(\Phi), \sigma^2\right)} \quad \forall\ t = 1, 2, \ldots, n \tag{13}$$

and weight function $w(\delta_t)$ is then defined as:

$$w(\delta_t) = w\left(\nu_t(\Phi); M\left(\sigma^2\right), \hat{F}_\nu(\Phi)\right) = \min\left\{1, \frac{[A(\delta_t)+1]^+}{\delta_t + 1}\right\} \tag{14}$$

where $[\cdot]^+$ indicates the positive part and $A(\cdot)$ denotes the residual adjustment function (RAF) of Lindsay (1994) (here in this study, Hellinger RAF $A(\delta) = 2\left[(\delta+1)^{1/2} - 1\right]$ have been used). Then the WLS estimate for Φ is obtained as:

$$\widehat{\Phi}^w = \left(\boldsymbol{X}'\boldsymbol{W}\boldsymbol{X}\right)^{-1}\boldsymbol{X}'\boldsymbol{W}\boldsymbol{z} \tag{15}$$

where $\boldsymbol{W} = diag\,(w(\delta_t))$ and $\widehat{\Phi}^w = \left(\widehat{\alpha}_0^w, \widehat{\alpha}_1^w, \ldots, \widehat{\alpha}_p^w\right)'$. From equations (13) and (14), it can be seen that when the model assumptions are holding good and with no outliers present in the data, δ_t converges to zero and $w(\delta_t)$ converges to 1. Similarly, in the presence of outliers, δ_t will be larger and corresponding $w(\delta_t)$ will be smaller than 1 *i.e.* the outlier observations will get less weight.

2.3. Robust bootstrap procedures

Sieve Bootstrap was first proposed by Buhlmann (1997) as a variation in Bootstrap process where sieves of linear autoregressive processes are used to approximate the underlying process to estimate the distribution of a statistical quantity of the process. The idea of Sieve Bootstrap is that it involves the sampling of the residuals of a fitted autoregressive or AR(p_n) models of order p_n, where $p_n \to \infty$ as $n \to \infty$, and then new Bootstrap realizations are generated from the resampled residuals. In this study, two new Bootstrap methods robust against outliers have been proposed for constructing PIs for an ARCH model. The first one *i.e.* robust unconditional Sieve Bootstrap (RUSB) is an improvement of the unconditional Sieve Bootstrap (USB) method for the ARCH process proposed by Chen *et al.* (2011) and the second one *i.e.* robust Sieve Bootstrap (RSB) is a modification of the SB method described by Tresch (2015). In both the existing methods, the estimation of parameters was done by the ordinary least squares method. This estimation yields poor results in the presence of outliers. To handle such outliers, here the estimations of parameters have been done by the WLS procedure.

Let $\{y_t\}_{t=1}^n$ follows the realization of an ARCH(p) process and it has the model representation given in equation (1), equation (2) and its AR representation in equation (6). Further letting $x_t = y_t^2$ for $t = 1, 2, \ldots n$, it can be easily presented by equation (11).

2.3.1. Robust unconditional sieve bootstrap (RUSB) method

The steps involved in this proposed algorithm are as follows:

1. Considering the model representation of equation (11), estimate the ARMA coefficients $\widehat{\Phi}^w = \left(\widehat{\alpha}_0^w, \widehat{\alpha}_1^w, \ldots, \widehat{\alpha}_p^w\right)'$ using the WLS method as in equation (15).

2. Estimate the residuals $\{\widehat{\nu}_t\}_{t=p+1}^n$ as

$$\widetilde{\nu}_t = x_t - \widehat{\alpha}_0^w - \sum_{i=1}^{p} \widehat{\alpha}_i^w x_{t-i} \tag{16}$$

where $\widetilde{\nu}_t = 0,$ for $t = 1, 2, \ldots, p$.

3. Center the estimated residuals $\widehat{\nu}_t = \widetilde{\nu}_t - (n-p)^{-1}\sum_{t=p+1}^{n}\widetilde{\nu}_t$ and then calculate the empirical distribution of the centered residuals as

$$\hat{F}_{\widehat{\nu}_t}(x) = (n-p)^{-1}\sum_{t=p+1}^{n} I_{(-\infty,x]}(\widehat{\nu}_t) \tag{17}$$

4. Resample with replacement, Bootstrap innovations $\{\nu_t^*\}$ from $\hat{F}_{\widehat{\nu}_t}(x)$.

5. Generate the Bootstrap sample of squared return x_t^*, where $x_t^* = y_t^{2*}$, by the recursion

$$x_t^* = \widehat{\alpha}_0^w + \sum_{i=1}^{p}\widehat{\alpha}_i^w x_{t-i}^* + \nu_t^* \tag{18}$$

   where $x_t^* = \widehat{\alpha}_0^w/\{1-\sum_{i=1}^{p}\widehat{\alpha}_i^w\}$ and $\nu_t^* = 0$ for $t \le p$. Generate $(n+200)$ values of x_t^* and then drop the first 200 "burn-in" observations to reduce the effect of the starting values as asymptotically negligible. (Kreiss and Franke, 1992).

6. Now given $\{x_t^*\}_{t=1}^n$ from Step 5, fit the model given by equation (11) then estimate the coefficients by the WLS method, and let the resultant estimated coefficients be $\widehat{\Phi}^{w*} = \left(\widehat{\alpha}_0^{w*}, \widehat{\alpha}_1^{w*}, \ldots, \widehat{\alpha}_p^{w*}\right)'$.

7. Then Bootstrap sample of volatility $\{\sigma_t^{2*}\}_{t=1}^n$ is obtained as

$$\sigma_t^{2*} = \widehat{\alpha}_0^{w*} + \sum_{i=1}^{p}\widehat{\alpha}_i^{w*} x_{t-i}^* \quad \text{for } t = p+1, p+2, \ldots, n. \tag{19}$$

   where $\sigma_t^{2*} = \widehat{\alpha}_0^w/\{1-\sum_{i=1}^{p}\widehat{\alpha}_i^w\}$ for $t = 1, \ldots, p$.

8. Again sample with replacement, Bootstrap innovations $\left\{\nu_{n+h}^*\right\}_{h=1}^{s}$, $s > 0$, from $\hat{F}_{\widehat{\nu}_t}(x)$ to obtain future Bootstrap observations.

9. Compute the h-step ahead, $h = 1, 2, \ldots, s$, future Bootstrap observations for squared returns x_{n+h}^* and volatility σ_{n+h}^{2*} by the recursions

$$x_{n+h}^* = \widehat{\alpha}_0^{w*} + \sum_{i=1}^{p}\widehat{\alpha}_i^{w*} x_{n+h-i}^* + \nu_{n+h}^* \tag{20}$$

$$\sigma_{n+h}^{2*} = \widehat{\alpha}_0^{w*} + \sum_{i=1}^{p}\widehat{\alpha}_i^{w*} x_{n+h-i}^* \tag{21}$$

   where $x_{n+h}^* = x_{n+h}$ for $h \le 0$.

10. Repeat Steps 4 to 9 B times to generate B Bootstrap replicates.

11. Obtain the empirical Bootstrap distribution function $\hat{F}^*_{x_{n+h}^*}$ of x_{n+h}^*, where $x_{n+h}^* = y_{n+h}^{2*}$, to approximate the unknown distribution of x_{n+k} given the observed sample and $\hat{F}^*_{\sigma_{n+h}^{2*}}$

of σ^{2*}_{n+h} to approximate the unknown distribution σ^2_{n+h}.
The $(1-\alpha)\,100\%$ PIs for future returns y_{n+h} is given by

$$\left[Q^*_{n+h}\left(\alpha/2\right),\ Q^*_{n+h}(1-\alpha/2)\right] \tag{22}$$

where $Q^*_{n+h}\left(\alpha/2\right) = -\sqrt{H^*_{n+h}\left(1-\alpha\right)}$ and $Q^*_{n+h}\left(1-\alpha/2\right) = \sqrt{H^*_{n+h}\left(1-\alpha\right)}$ where $H^*_{n+h}\left(1-\alpha\right)$ is the $(1-\alpha)$ quantile of $\hat{F}^*_{x^*_{n+h}}$.
Similarly, the $(1-\alpha)\,100\%$ PIs for σ^2_{n+h} is given by

$$\left[0,\ K^*_{n+h}(1-\alpha)\right] \tag{23}$$

where $K^*_{n+h}(1-\alpha)$ is the $(1-\alpha)$ quantile of $\hat{F}^*_{\sigma^{2*}_{n+h}}$.

2.3.2. Robust sieve bootstrap (RSB) method

It is possible to write that an AR process of $\{x_t\}_{t=1}^n$, as in equation (11), in the form of an infinite AR representation:

$$\sum_{j=0}^{\infty}\varphi_j\left(x_{t-j}-\mu_x\right)=\nu_t\ ,\ \varphi_0=1,\quad for\quad t\in\mathbb{Z} \tag{24}$$

with coefficients satisfying the condition $\sum_{j=0}^{\infty}\varphi_j^2<\infty$. Let the parameter μ_x be estimated by its empirical mean $\overline{x}=\frac{1}{n}\sum_{t=1}^{n}x_t$, as has been done by Alonso *et al.* (2002, 2003, 2004). The steps involved in the proposed algorithm are as follows:

1. For the given realization of squared return series, $\{x_t\}_{t=1}^n$, select the maximum order $p_{\max}=p\left(n\right)$ of the AR approximation and using AICC criteria, obtain the optimum order. The optimum order has been considered as $\hat{p}=p_{AICC}+1$ for the order of the AR model to be fitted to the observed data. In the Monte Carlo simulation, $p_{\max}=p\left(n\right)$ was taken as $(n/10)$, as recommended by Bhansali (1983) where n is the sample size.

2. Estimate the coefficients of AR($\hat{p}$) process using the WLS method described in equation (15). Let the estimates be $\widehat{\varphi}^w_1,\widehat{\varphi}^w_2,\ldots,\widehat{\varphi}^w_{\hat{p}}$ in place of the Yule-Walker method used for coefficient estimation within Tresch (2015).

3. Compute the $(n-\hat{p})$ residuals as

$$\widetilde{\nu}_t=\sum_{j=0}^{\hat{p}}\widehat{\varphi}^w_j\left(x_{t-j}-\overline{x}\right);\quad \widehat{\varphi}^w_0=1,\ t\in\left(\hat{p}+1,\hat{p}+2,\ldots,n\right) \tag{25}$$

   where $\overline{x}$ is the mean of $\{x_t\}_{t=1}^n$.

4. Center the residuals as $\widehat{\nu}_t=\widetilde{\nu}_t-\overline{\widetilde{\nu}}_t$, where $\overline{\widetilde{\nu}}_t=\left(n-\hat{p}\right)^{-1}\sum_{t=\hat{p}+1}^{n}\widetilde{\nu}_t$. Then compute the empirical distribution function of the centered residuals $\hat{F}_{\widehat{\nu}}\left(x\right)=\left(n-\hat{p}\right)^{-1}\sum_{t=\hat{p}+1}^{n}I_{(-\infty,x]}\left(\widehat{\nu}_t\right)$.

5. Resample with replacement, Bootstrap innovations ν_t^* from this distribution $\hat{F}_{\widehat{\nu}}(x)$ for $t = -199, -198, \ldots, 0, 1, \ldots, n$.

6. Generate the Bootstrap series x_t^*, $t = -199, -198, \ldots, 0, 1, \ldots, n$ by the recursion as:

$$\sum_{j=0}^{\hat{p}} \widehat{\varphi}_j^w \left(x_{t-j}^* - \overline{x} \right) = \nu_t^* \tag{26}$$

where the first $\hat{p}$ values are taken as $x_t^* = \overline{x}$. Then drop the first 200 "burn-in" observations to reduce the effect of the starting values as asymptotically negligible.

7. Fit an $AR\left(\hat{p}\right)$ model to the pseudo-data $\{x_1^*, x_2^*, \ldots, x_n^*\}$, re-estimate the coefficients using the WLS method and let the estimated coefficients be $\widehat{\varphi}_1^{w*}, \widehat{\varphi}_2^{w*}, \ldots, \widehat{\varphi}_{\hat{p}}^{w*}$.

8. Using the new coefficients $\widehat{\varphi}_1^{w*}, \widehat{\varphi}_2^{w*}, \ldots, \widehat{\varphi}_{\hat{p}}^{w*}$, compute the h-step ahead future Bootstrap observations by the recursion as:

$$x_{n+h}^* - \overline{x} = -\sum_{j=1}^{\hat{p}} \widehat{\varphi}_j^{w*} \left(x_{n+h-j}^* - \overline{x} \right) + \nu_{n+h}^{**} \tag{27}$$

where $x_t^* = x_t$ when $t \leq n$ with ν_{n+h}^{**} for $h = 1, 2, \ldots, s$, resampled from $\hat{F}_{\widehat{\nu}}(x)$. Also, instead of employing fixed $\overline{x}$, here the mean of the Bootstrap series $\overline{x}^*$ has been employed as an estimate of the mean μ_x at individual Bootstrap prediction, following Mukhopadhyay and Samaranayake (2010), since it includes sampling variability. So to account for the sampling variability due to the estimate of the mean μ_x of the TS, add $(\overline{x}^* - \overline{x})$ to predict future observations x_{n+h}^*. Thus the future Bootstrap squared return is then $\hat{x}_{n+h}^* = x_{n+h}^* + \overline{x}^* - \overline{x}$ for $h = 1, 2, \ldots, s$.

9. Using the future values x_{n+h}^* and the relationship for AR and ARCH/GARCH process, the future volatility can be calculated by the following recursion:

$$\sigma_{n+h}^{2*} = \overline{x}^* - \sum_{j=1}^{\hat{p}} \widehat{\varphi}_j^{w*} \left(x_{n+h-j}^* - \overline{x} \right) \tag{28}$$

where $x_{n+h-j}^* = x_{n+h}$for $h \leq 0$.

10. Repeat steps 4 to 9 B times to generate B Bootstrap replicates. Then obtain the empirical Bootstrap distribution function $\hat{F}_{x_{n+h}^*}^*$ of x_{n+h}^*, where $x_{n+h}^* = y_{n+h}^{2*}$, to approximate the unknown distribution of x_{n+k} given the observed sample and $\hat{F}_{\sigma_{n+h}^{2*}}^*$ of σ_{n+h}^{2*} to approximate the unknown distribution σ_{n+h}^2.

11. The $(1-\alpha)\,100\%$ PIs for future return y_{n+h} is given by:

$$\left[Q_{n+h}^*\left(\alpha/2\right),\ Q_{n+h}^*(1-\alpha/2) \right] \tag{29}$$

where $Q_{n+h}^*\left(\alpha/2\right) = -\sqrt{H_{n+h}^*\left(1-\alpha\right)}$ and $Q_{n+h}^*\left(1-\alpha/2\right) = \sqrt{H_{n+h}^*\left(1-\alpha\right)}$ where $H_{n+h}^*\left(1-\alpha\right)$ is the $(1-\alpha)$ quantile of $\hat{F}_{x_{n+h}^*}^*$.
Similarly, the $(1-\alpha)\,100\%$ PIs for σ_{n+h}^2 is given by:

$$\left[0,\ K_{n+h}^*(1-\alpha) \right] \tag{30}$$

where $K^*_{n+h}(1-\alpha)$ is the $(1-\alpha)$ quantile of $\hat{F}^*_{\sigma^{2*}_{n+h}}$.

In the SB method by Tresch (2015), the future volatilities have been calculated by the recursion of $\sigma^{2*}_{n+h} = x^*_{n+h} - \sum_{j=1}^{\hat{p}} \widehat{\varphi}^*_j x^*_{n+h-j}$ for $h = 1, 2, \ldots, s$. This has been changed in RSB and given in (28). It is also noted that the use of $\overline{x}^*$ in the second proposed method has been done which incorporates the advantage of the Bootstrap sampling variability on future volatilities.

A schematic diagram of the method in Section 2.3.1. is given in the Figure 1 below.

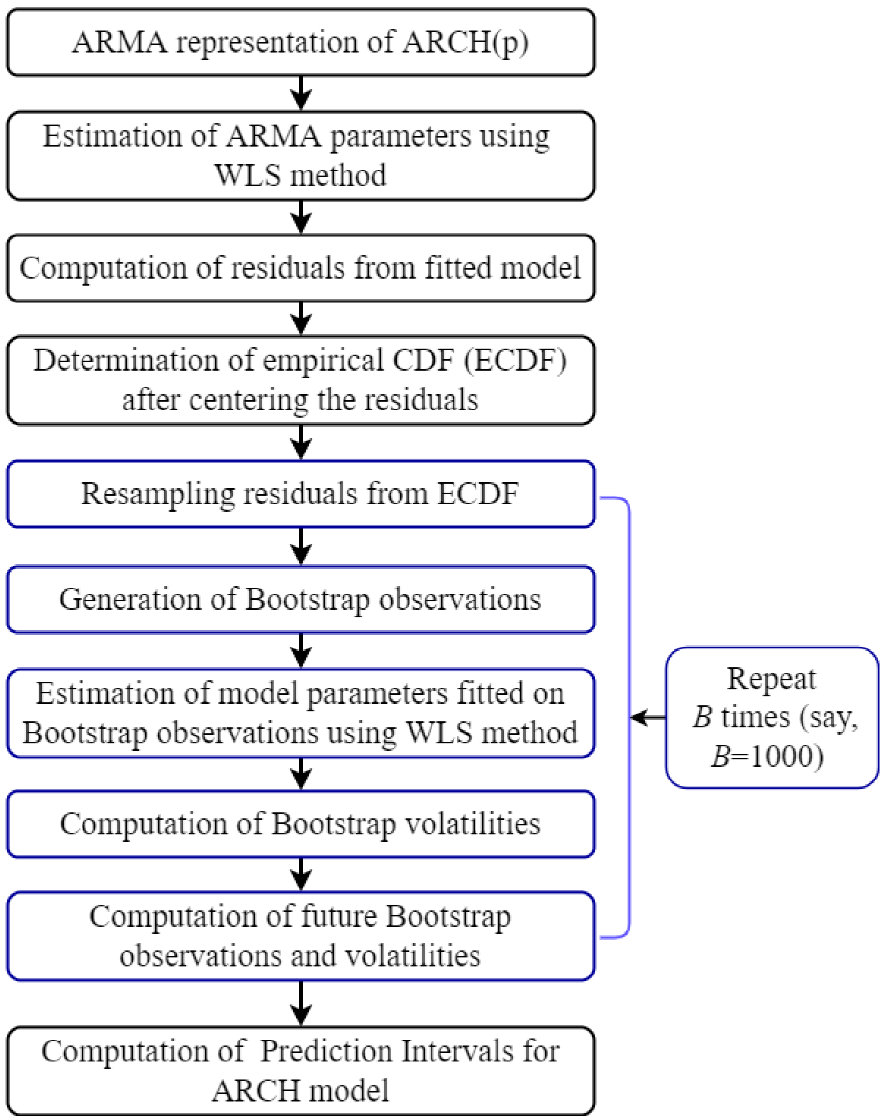

Figure 1: A Schematic diagram of the algorithm in 2.3.1.

3. Simulation results

To compare the finite sample performance of the proposed Bootstrap methods with the existing Bootstrap methods, a Monte-Carlo simulation study has been carried out on an ARCH(2) model for varying sample sizes and with data having no contamination and also with contamination (read innovative outliers). Data were generated using the following ARCH(2) model for heteroscedastic errors:

$$y_t = \sigma_t \varepsilon_t \tag{31}$$

$$\sigma_t^2 = 0.1 + 0.2 y_{t-1}^2 + 0.15 y_{t-2}^2 \tag{32}$$

generated separately considering two different distributions for the innovation process $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ given as (i) $N(0,1)$ and (ii) $(1-\zeta)N(0,1)+\zeta N(0,10)$. Here the level of contamination has been taken as $\zeta = 0.05$. The sample sizes considered were 300 and 1000. For each combination of error distribution and sample size, to start with, the simulated datasets, y_t and σ_t^2, from ARCH(2) process were generated and then $R = 1000$ future values, y_{n+h} and σ_{n+h}^2, for each future lead $h = 1,\ 2, \ldots,\ 20$ were generated from the underlying model using the true values of the parameter coefficients for each simulation. Furthermore, for each Bootstrap procedure (both existing and proposed), $B = 1000$ Bootstrap pseudo-series were generated to obtain Bootstrap PIs for nominal coverages of 95%. These procedures were repeated $N = 1000$ times to calculate the average values of the performance metrics described subsequently.

The empirical or theoretical length of the PIs of y_{t+h} for i^{th} simulation run, $i = 1, 2, \ldots N$, was calculated as $L_{T,y}(i) = \left[y_{n+h}^{(R)}(1-\alpha/2) - y_{n+h}^{(R)}(\alpha/2)\right]$, the difference between $(1-\alpha/2)\,100^{th}$ and $(\alpha/2)\,100^{th}$ percentile point of the empirical distribution of the R future returns. Then mean theoretical length of return is $\overline{L}_{T,y} = N^{-1}\sum_{i=1}^{N} L_{T,y}(i)$. Similarly the mean theoretical length of the PIs of σ_{n+h}^2 is calculated as: $\overline{L}_{T,\sigma^2} = N^{-1}\sum_{i=1}^{N} L_{T,\sigma^2}(i)$, where $L_{T,\sigma^2}(i) = \left[\sigma_{n+h}^{2,(R)}(1-\alpha/2) - \sigma_{n+h}^{2,(R)}(\alpha/2)\right]$, the difference between $(1-\alpha/2)\,100^{th}$ and $(\alpha/2)\,100^{th}$ percentile point of the empirical distribution of the R future volatilities.

The coverage probability (CP) of returns y_{t+h} for i^{th} simulation run is then calculated as the $C_y(i) = R^{-1}\sum_{r=1}^{R} I_{\left[Q^*(\alpha/2)\le y_{n+h}^{(r)}(i)\le Q^*(1-\alpha/2)\right]}$, where $Q^*(\alpha/2)$ is the $(\alpha/2)^{th}$ quantile of the estimated Bootstrap distribution and $y_{n+h}^{(r)}(i)$ is r^{th} future return value, $r = 1,\ 2, \ldots, R$, generated at i^{th} simulation, $i = 1, 2, \ldots N$. Similarly, the CP of volatility σ_{n+h}^2 for i^{th} simulation $\sigma_{n+h}^{2,(r)}(i)$ is then calculated as the $C_{\sigma^2}(i) = R^{-1}\sum_{r=1}^{R} I_{\left[0\le\sigma_{n+h}^{2,(r)}(i)\le K_{n+h}^*(1-\alpha)\right]}$, where $K^*(\alpha)$ is the α^{th} quantile of the estimated Bootstrap distribution and $\sigma_{n+h}^{2,(r)}(i)$ is r^{th} future volatility value, $r = 1,\ 2, \ldots,\ R$, generated at i^{th} simulation.

The Bootstrap length of returns y_{t+h} and volatility σ_{n+h}^2 for i^{th} simulation run is calculated as $L_{B,y}(i) = [Q^*(1-\alpha/2) - Q^*(\alpha/2)]$ and $L_{B,\sigma^2}(i) = K^*(1-\alpha)$, respectively. Finally, the following performance evaluation measures were calculated:

- Mean Return Coverage (CVR_{ret}): $\overline{C}_y = N^{-1}\sum_{i=1}^{N} C_y(i)$
- Mean Volatility Coverage (CVR_{vol}): $\overline{C}_{\sigma^2} = N^{-1}\sum_{i=1}^{N} C_{\sigma^2}(i)$
- Standard Error of CVR_{ret}: $se\left(\overline{C}_y\right) = \left\{[N(N-1)]^{-1}\sum_{i=1}^{N}\left[C_y(i) - \overline{C}_y\right]^2\right\}^{1/2}$

- Standard Error of CVR_{vol}: $se\left(\overline{C}_{\sigma^2}\right) = \left\{[N(N-1)]^{-1}\sum_{i=1}^{N}\left[C_{\sigma^2}(i) - \overline{C}_{\sigma^2}\right]^2\right\}^{1/2}$
- Mean length of Return (LEN_{ret}): $\overline{L}_{B,y} = N^{-1}\sum_{i=1}^{N} L_{B,y}(i)$
- Mean length of Volatility (LEN_{vol}): $\overline{L}_{B,\sigma^2} = N^{-1}\sum_{i=1}^{N} L_{B,\sigma^2}(i)$
- Standard Error of LEN_{ret}: $se\left(\overline{L}_{B,y}\right) = \left\{[N(N-1)]^{-1}\sum_{i=1}^{N}\left[L_{B,y}(i) - \overline{L}_{B,y}\right]^2\right\}^{1/2}$
- Standard Error of LEN_{vol}: $se\left(\overline{L}_{B,\sigma^2}\right) = \left\{[N(N-1)]^{-1}\sum_{i=1}^{N}\left[L_{B,\sigma^2}(i) - \overline{L}_{B,\sigma^2}\right]^2\right\}^{1/2}$
  $CQ_{ret} = \left|1 - \left(\overline{L}_{B,y}/\overline{L}_{T,y}\right)\right| + \left|1 - \left(CVR_{ret}/CVR_{T,y}\right)\right|$
  $CQ_{vol} = \left|1 - \left(\overline{L}_{B,\sigma^2}/\overline{L}_{T,\sigma^2}\right)\right| + \left|1 - \left(CVR_{vol}/CVR_{T,vol}\right)\right|$

where $CVR_{T,(.)}$ is the $(1-\alpha)\,\%$ nominal coverage. Here, CQ is an index of coverage quality. Therefore the simulation results have been summarized in different tables that contain the mean coverage (CVR), mean length of the intervals (LEN), standard error of mean coverage (SE), and standard error of mean length of the intervals (SE) for different combinations. The performances of the proposed methods were compared to the existing unconditional Sieve Bootstrap (USB) proposed by Chen *et al.* (2011) method and Sieve Bootstrap (SB) by Tresch (2015) for constructing PIs. The proposed approaches are given in Sections 2.3.1 and 2.3.2 respectively.

It is noted that, for the case of $h = 1$, equations (21) and (28) both will have their Bootstrap volatilities as constant and hence the computation of PIs of their one-step-ahead forecast volatilities are not appropriate and hence not given in the following tables.

In Tables 1 through 4, results of the comparisons of PIs for h= 1, 5, 10, 15 and 20 steps ahead of the described methods have been presented for comparison purposes.

Tables 1 and 2 provide the results pertaining to the ARCH(2) model without contaminated innovations. From Tables 1 and 2, it can be seen that all methods have almost similar results in terms of coverage and length of intervals. It can also be seen that the proposed method RSB is performing almost at par with SB when coverage probabilities are compared while lengths of PIs of RUSB are always found to be less than the existing method *i.e.* USB. The same conclusion can be drawn when we compare the proposed method RUSB with the existing method USB. When the lengths of PIs of two proposed methods RSB and RUSB are compared, by and large, RUSB is always better than RSB both for returns and volatilities.

From Tables 3 and 4, a striking feature of the proposed method RUSB which can be seen is that the length of PIs across all forecast horizons for both returns and volatilities have been found to be less as compared to those of the existing methods SB and USB and also of the proposed method RSB when the data is contaminated. The feature of obtaining the order of model by Sieve approximation rather than assumed to be fixed beforehand has yielded better coverage in the case of the proposed method RSB and the existing method SB (in which such a feature is there) as compared to the other two methods *viz.* proposed method RUSB and existing method USB. It can also be seen from Tables 3 and 4 that the proposed methods were able to tackle the inflation of variances and at the same time maintains the length of PIs.

Another inference that can be drawn when the coverage of volatility are considered is that the proposed methods performed well in case of contaminated data. It can also be seen that the length of PIs for the proposed method RSB is always less than those of the existing methods USB and SB. Even though the lengths of PIs of the proposed method RSB are larger than the RUSB, it can be seen that the coverages obtained from RSB are always better than RUSB for both returns and volatilities in the case of contaminated data. When both coverages and lengths of PIs are considered together, as per the combined measure CQ_{ret} and CQ_{vol}, the RUSB has been found far better than others.

4. Case Study

In this section, the performance of the proposed methods RSB and RUSB in comparison with the existing methods USB and SB have been presented with real-time series data. Monthly onion price (Rs/quintal) data at Delhi market has been used for validating the methods. It pertains to the period January 2003 to February 2022, with a total of 230 observations. Data were collected from the secondary source available at National Horticultural Research and Development Foundation, New Delhi, India (NHRDF, 2003-2022). The methods were applied to the return of monthly Onion price at Delhi market data. The returns are more frequently used than the price time series, because returns do not depend on units, making the comparison easier. The return series is obtained as follows:

$$y_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{33}$$

where P_t is the monthly onion price at time t. The price series is shown in Figure 2 and the return series y_t is shown in Figure 3. ADF test has been employed on the return series y_t which revealed that it is stationary. From Figure 3 and Figure 4, the data reveals the presence of outliers. Table 5 presents the summary statistics of the return data series. As the estimated kurtosis is higher than 3, indicates that the return series is leptokurtic.

Now Lagrange-Multiplier (LM) test confirmed the presence of the ARCH effect on this return series. It was found that ARCH(1) is a suitable model for return series y_t. The data set has been partitioned into an in-sample estimation set from January 2003 to December 2020 and an out-sample set from January 2021 to February 2022 for validation. That is out of 230 sample observations 14 observations have been set aside for predictions purpose. From equation (9), by fitting AR(1) model on y_t^2 using LS estimation, the resulting estimated model is

$$y_t^2 = \ 0.0670 + 0.1026 y_{t-1}^2 \tag{34}$$

i.e. $\widehat{\alpha}_0 = 0.0670$ and $\widehat{\alpha}_1 = 0.1026$. It can also be seen that $\widehat{\alpha}_1^2 \leq 1/3$, and hence indicates strictly stationarity (and hence weakly stationarity also) of return series (Tsay, 2010; Box *et al.*, 2015).

Figures 5 and 6 pertain to the PIs for the returns and volatilities from thevarious methods. In case of PIs of returns, lower and upper boundaries of the PIs were obtained,

Table 1: Simulated results of ARCH(2) model for sample size 300 and standard normal innovation and no contamination

h	Method	CVR_{ret} (SE)	LEN_{ret} (SE)	CQ_{ret}	CVR_{vol} (SE)	LEN_{vol} (SE)	CQ_{vol}
1	-	95%	1.501	-	95%	-	-
	USB	0.9481 (0.0014)	1.514 (0.0065)	0.0098	-	-	-
	SB	0.9461 (0.0012)	1.511 (0.0051)	0.0098	-	-	-
	RSB	0.9462 (0.0012)	1.512 (0.0049)	0.0101	-	-	-
	RUSB	0.9480 (0.0014)	1.514 (0.0065)	0.0096	-	-	-
5	-	95%	1.535	-	95%	0.274	-
	USB	0.9465 (0.0009)	1.542 (0.0063)	0.0082	0.9162 (0.0127)	0.273 (0.0040)	0.0418
	SB	0.9468 (0.0006)	1.543 (0.0046)	0.0086	0.9021 (0.0142)	0.273 (0.0032)	0.0563
	RSB	0.9467 (0.0006)	1.541 (0.0043)	0.0073	0.9083 (0.0098)	0.272 (0.0027)	0.0519
	RUSB	0.9462 (0.0009)	1.539 (0.0061)	0.0069	0.9153 (0.0127)	0.270 (0.0036)	0.0529
10	-	95%	1.539	-	95%	0.273	-
	USB	0.9465 (0.0008)	1.542 (0.0063)	0.0057	0.9156 (0.0127)	0.273 (0.0040)	0.0373
	SB	0.9462 (0.0006)	1.542 (0.0047)	0.0058	0.9015 (0.0142)	0.275 (0.0034)	0.0576
	RSB	0.9463 (0.0006)	1.540 (0.0044)	0.0046	0.9078 (0.0098)	0.274 (0.0028)	0.0463
	RUSB	0.9463 (0.0008)	1.540 (0.0061)	0.0044	0.9145 (0.0127)	0.270 (0.0036)	0.0498
15	-	95%	1.535	-	95%	0.274	-
	USB	0.9458 (0.0008)	1.541 (0.0062)	0.0082	0.9157 (0.0127)	0.274 (0.0041)	0.0368
	SB	0.9468 (0.0006)	1.540 (0.0045)	0.0065	0.9023 (0.0142)	0.275 (0.0033)	0.0520
	RSB	0.9468 (0.0006)	1.538 (0.0042)	0.0053	0.9082 (0.0098)	0.273 (0.0021)	0.0473
	RUSB	0.9456 (0.0008)	1.539 (0.0059)	0.0068	0.9147 (0.0127)	0.270 (0.0036)	0.0521
20	-	95%	1.535	-	95%	0.2740	-
	USB	0.9471 (0.0009)	1.543 (0.0062)	0.0089	0.9163 (0.0127)	0.274 (0.0040)	0.0358
	SB	0.9472 (0.0006)	1.546 (0.0047)	0.0106	0.9021 (0.0142)	0.275 (0.0035)	0.0537
	RSB	0.9472 (0.0006)	1.544 (0.0044)	0.0093	0.9080 (0.0098)	0.273 (0.0027)	0.0475
	RUSB	0.9468 (0.0009)	1.541 (0.0059)	0.0075	0.9153 (0.0127)	0.270 (0.0035)	0.0497

but since the volatility is non-negative, only the upper boundary has been obtained and the lower boundary has been assumed to be zero. It can be found that the PIs for returns developed by all methods contained all the future returns. At some points, it is clearly visible that the proposed methods have smaller lengths as compared to existing methods. As volatilities are not directly observable, once the parameters were estimated, volatilities have been estimated using the following equation (Ullao *et al.*, 2014):

$$\sigma_t^2 = \widehat{\alpha}_0 + \widehat{\alpha}_1 y_{t-1}^2 \tag{35}$$

where y_{t-1} corresponds to the observed past return series. It can be clearly seen that the

Table 2: Simulated results of ARCH(2) model for sample size 1000 and standard normal innovation and no contamination

h	Method	$\boldsymbol{CVR_{ret}}$ **(SE)**	$\boldsymbol{LEN_{ret}}$ (SE)	$\boldsymbol{CQ_{ret}}$	$\boldsymbol{CVR_{vol}}$ (SE)	$\boldsymbol{LEN_{vol}}$ (SE)	$\boldsymbol{CQ_{vol}}$
1	-	95%	1.513	-	95%	-	-
	USB	0.9479 (0.0015)	1.522 (0.0053)	0.0080	-	-	-
	SB	0.9471 (0.0011)	1.519 (0.0038)	0.0068	-	-	-
	RSB	0.9473 (0.0011)	1.519 (0.0038)	0.0070	-	-	-
	RUSB	0.9478 (0.0015)	1.522 (0.0052)	0.0081	-	-	-
5	-	95%	1.538	-	95%	0.275	-
	USB	0.9475 ()0.0006	1.539 (0.0038)	0.0031	0.9381 (0.0144)	0.275 (0.0023)	0.0125
	SB	0.9482 (0.0004)	1.540 (0.0027)	0.0031	0.9374 (0.0136)	0.274 (0.0017)	0.0144
	RSB	0.9481 (0.0004)	1.539 (0.0027)	0.0025	0.9371 (0.0086)	0.272 (0.0015)	0.0216
	RUSB	0.9472 (0.0006)	1.537 (0.0037)	0.0038	0.9369 (0.0144)	0.271 (0.0021)	0.0251
10	-	95%	1.537	-	95%	0.274	-
	USB	0.9471 (0.0006)	1.537 (0.0039)	0.0031	0.9385 (0.0144)	0.275 (0.0023)	0.0183
	SB	0.9483 (0.0004)	1.539 (0.0027)	0.0030	0.9375 (0.0136)	0.275 (0.0017)	0.0183
	RSB	0.9482 (0.0004)	1.538 (0.0026)	0.0024	0.9373 (0.0086)	0.273 (0.0015)	0.0170
	RUSB	0.9469 (0.0006)	1.535 (0.0038)	0.0046	0.9374 (0.0144)	0.272 (0.0020)	0.0184
15	-	95%	1.537	-	95%	0.274	-
	USB	0.9483 (0.0006)	1.538 (0.0038)	0.0024	0.9387 (0.0144)	0.275 (0.0023)	0.0166
	SB	0.9482 (0.0004)	1.540 (0.0026)	0.0040	0.9374 (0.0136)	0.274 (0.0016)	0.0155
	RSB	0.9480 (0.0004)	1.538 (0.0026)	0.0031	0.9370 (0.0086)	0.272 (0.0015)	0.0195
	RUSB	0.9481 (0.0006)	1.536 (0.0037)	0.0027	0.9374 (0.0144)	0.272 (0.0020)	0.0202
20	-	95%	1.539	-	95%	0.274	-
	USB	0.9473 (0.0006)	1.535 (0.0039)	0.0054	0.9384 (0.0144)	0.275 (0.0022)	0.0162
	SB	0.9475 (0.0004)	1.536 (0.0026)	0.0049	0.9369 (0.0136)	0.274 (0.0016)	0.0167
	RSB	0.9474 (0.0004)	1.535 (0.0025)	0.0054	0.9367 (0.0086)	0.272 (0.0015)	0.0191
	RUSB	0.9471 (0.0006)	1.534 (0.0038)	0.0068	0.9372 (0.0144)	0.272 (0.0020)	0.0190

proposed method RSB and SB are almost close to each other and cover all future volatilities. RUSB has a very small length for PIs.

5. Conclusion

In this study, two new robust Sieve Bootstrap approaches based on weighted least squares estimation have been proposed to deal with the presence of outliers for developing PIs for both returns and volatilities in the ARCH model setup. The performances of the proposed methods *viz.*, Robust Unconditional Sieve Bootstrap (RUSB) and Robust Sieve Bootstrap (RSB) for constructing PIs using both simulated as well as real data sets have been found

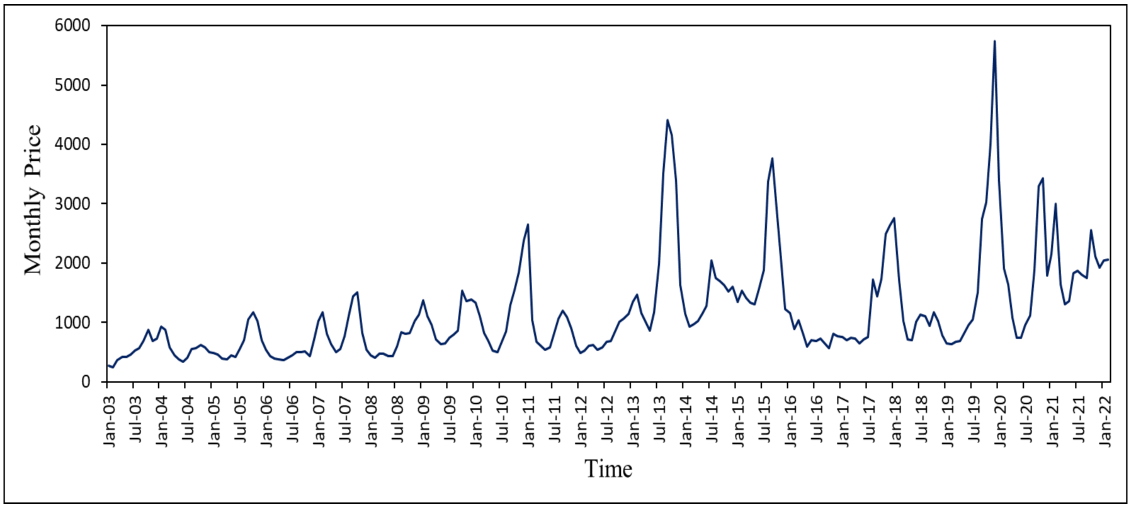

Figure 2: Monthly Onion price at Delhi market data from January 2003 to February 2022, with a total of 230 observations

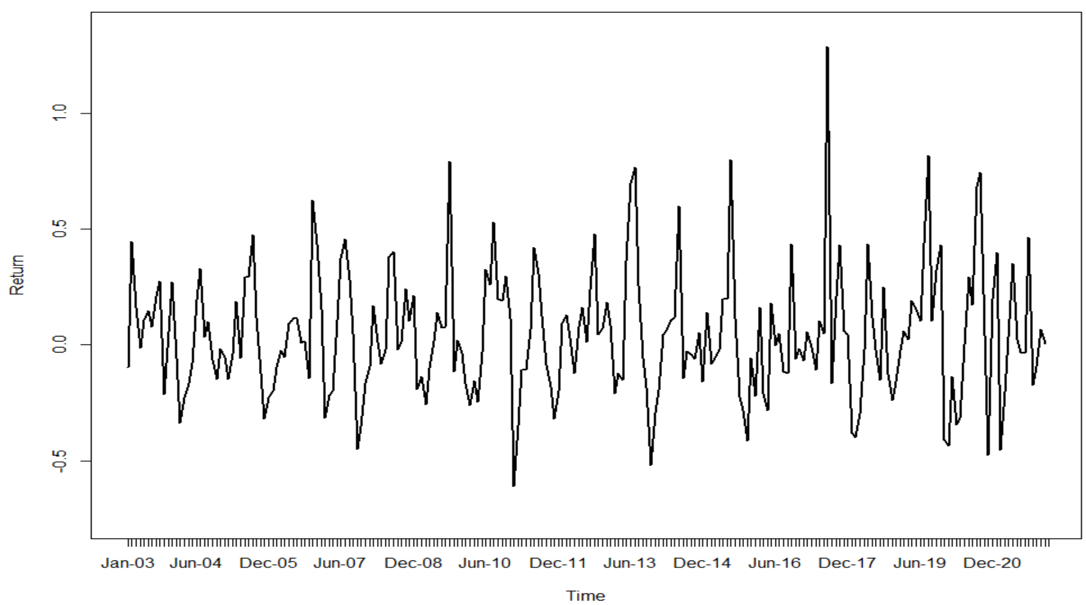

Figure 3: Time plot of returns of monthly onion price data of Delhi market

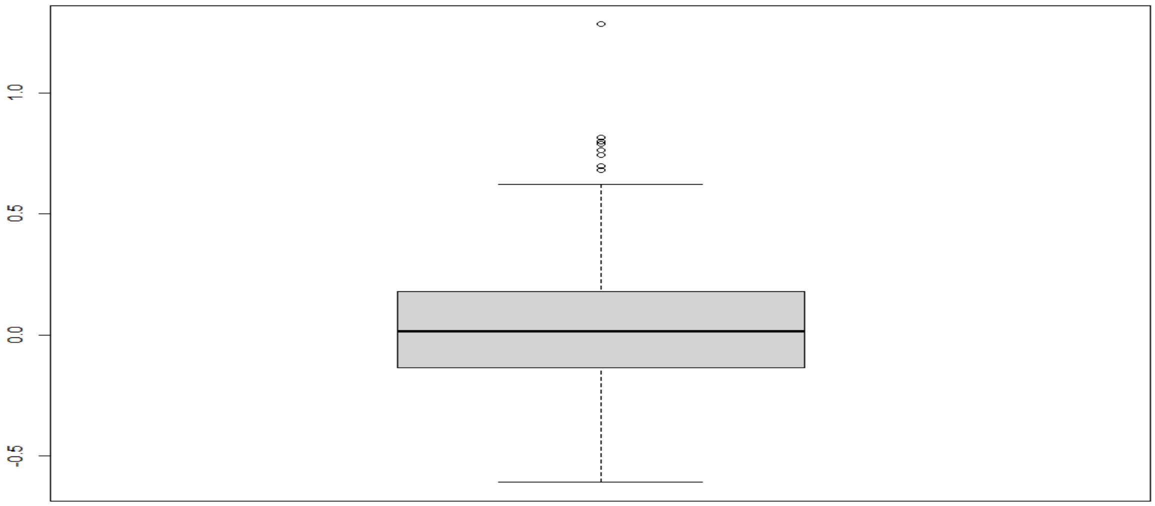

Figure 4: Box plot of return series of monthly onion price of Delhi market

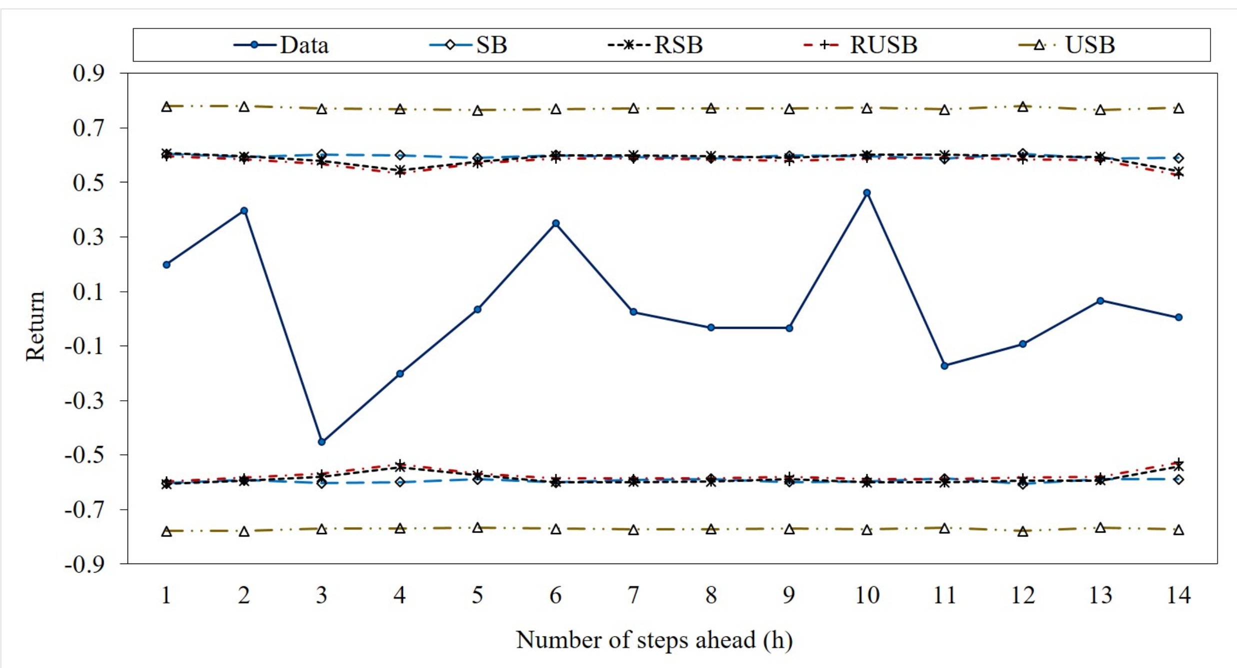

Figure 5: Prediction intervals for returns of monthly onion price of Delhi market for forecast horizons $h = 1, 2, \cdots, 14$

Table 3: Simulated results of ARCH(2) model for sample size 300 with 5% contaminated normal innovation

h	Method	CVR_{ret} **(SE)**	LEN_{ret} (SE)	CQ_{ret}	CVR_{vol} (SE)	LEN_{vol} (SE)	CQ_{vol}
1	-	95%	1.820	-	95%	-	-
	USB	0.9607 (0.0017)	2.151 (0.0198)	0.1935	-	-	-
	SB	0.9480 (0.0015)	1.906 (0.0102)	0.0493	-	-	-
	RSB	0.9491 (0.0016)	1.941 (0.0135)	0.0675	-	-	-
	RUSB	0.9423 (0.0022)	1.839 (0.0126)	0.0186	-	-	-
5	-	95%	1.955	-	95%	0.433	-
	USB	0.9626 (0.0009)	2.446 (0.0346)	0.2646	0.9749 (0.0030)	1.138 (0.0641)	1.6523
	SB	0.9518 (0.0007)	2.117 (0.0151)	0.0846	0.9441 (0.0069)	0.581 (0.0134)	0.3459
	RSB	0.9500 (0.0007)	2.072 (0.0162)	0.0597	0.9465 (0.0056)	0.576 (0.0291)	0.3335
	RUSB	0.9440 (0.0009)	1.949 (0.0132)	0.0096	0.9363 (0.0078)	0.436 (0.0084)	0.0200
10	-	95%	1.960	-	95%	0.429	-
	USB	0.9636 (0.0009)	2.490 (0.0380)	0.2849	0.9741 (0.0030)	1.216 (0.0788)	1.8600
	SB	0.9514 (0.0007)	2.126 (0.0163)	0.0864	0.9433 (0.0069)	0.601 (0.0162)	0.4077
	RSB	0.9496 (0.0006)	2.081 (0.0188)	0.0622	0.9455 (0.0056)	0.601 (0.0399)	0.4066
	RUSB	0.9447 (0.0009)	1.970 (0.0150)	0.0106	0.9348 (0.0078)	0.443 (0.0094)	0.0479
15	-	95%	1.965	-	95%	0.432	-
	USB	0.9626 (0.0009)	2.484 (0.0392)	0.2776	0.9746 (0.0030)	1.234 (0.0861)	1.8824
	SB	0.9512 (0.0007)	2.126 (0.0159)	0.0832	0.9430 (0.0069)	0.605 (0.0164)	0.4078
	RSB	0.9491 (0.0007)	2.083 (0.0210)	0.0614	0.9451 (0.0056)	0.605 (0.0496)	0.4045
	RUSB	0.9438 (0.0009)	1.953 (0.0140)	0.0125	0.9354 (0.0078)	0.440 (0.0089)	0.0341
20	-	95%	1.966	-	95%	0.432	-
	USB	0.9628 (0.0009)	2.488 (0.0395)	0.2793	0.9738 (0.0030)	1.241 (0.0885)	1.9020
	SB	0.9511 (0.0007)	2.132 (0.0164)	0.0859	0.9431 (0.0069)	0.609 (0.0179)	0.4193
	RSB	0.9491 (0.0007)	2.090 (0.0216)	0.0643	0.9450 (0.0056)	0.638 (0.0579)	0.4838
	RUSB	0.9444 (0.0009)	1.957 (0.0141)	0.0104	0.9349 (0.0078)	0.438 (0.0087)	0.0307

to be better when compared with their existing counterparts. The results revealed that the proposed method RSB is performing almost at par with SB when coverage probabilities are compared while lengths of PIs of RUSB are always found to be less than the existing method *i.e.* USB. When the lengths of PIs of two proposed methods RSB and RUSB are compared, by and large, RUSB is always better than RSB both for returns and volatilities. For the proposed method RUSB, the length of PIs across all forecast horizons for both returns and volatilities have been found to be less as compared to those of the existing methods SB and USB and also of the proposed method RSB when the data is contaminated. The proposed methods were able to tackle the inflation of variances and at the same time maintain the

Table 4: Simulated results of ARCH(2) model for sample size 1000 with 5% contaminated normal innovation

h	Method	$\boldsymbol{CVR_{ret}}$ **(SE)**	$\boldsymbol{LEN_{ret}}$ (SE)	$\boldsymbol{CQ_{ret}}$	$\boldsymbol{CVR_{vol}}$ (SE)	$\boldsymbol{LEN_{vol}}$ (SE)	$\boldsymbol{CQ_{vol}}$
1	-	95%	1.768	-	95%	-	-
	USB	0.9644 (0.0016)	2.155 (0.0164)	0.2072	-	-	-
	SB	0.9535 (0.0011)	1.875 (0.0078)	0.0644	-	-	-
	RSB	0.9517 (0.0012)	1.836 (0.0068)	0.0403	-	-	-
	RUSB	0.9444 (0.0022)	1.806 (0.0096)	0.0278	-	-	-
5	-	95%	1.948	-	95%	0.422	-
	USB	0.9675 (0.0007)	2.491 (0.0294)	0.2922	0.9806 (0.0035)	1.141 (0.0592)	1.7386
	SB	0.9557 (0.0005)	2.114 (0.0095)	0.0914	0.9620 (0.0045)	0.594 (0.0083)	0.4220
	RSB	0.9511 (0.0005)	2.013 (0.0071)	0.0344	0.9504 (0.0035)	0.474 (0.0057)	0.1242
	RUSB	0.9454 (0.0007)	1.928 (0.0091)	0.0154	0.9350 (0.0073)	0.397 (0.0055)	0.0732
10	-	95%	1.959	-	95%	0.428	-
	USB	0.9678 (0.0006)	2.519 (0.0333)	0.3103	0.9807 (0.0035)	1.220 (0.0717)	1.8841
	SB	0.9555 (0.0005)	2.134 (0.0107)	0.0950	0.9610 (0.0045)	0.616 (0.0101)	0.4520
	RSB	0.9506 (0.0004)	2.020 (0.0074)	0.0317	0.9488 (0.0035)	0.476 (0.0058)	0.1128
	RUSB	0.9451 (0.0007)	1.921 (0.0088)	0.0249	0.9332 (0.0074)	0.396 (0.0058)	0.0913
15	-	95%	1.959	-	95%	0.431	-
	USB	0.9673 (0.0006)	2.528 (0.0342)	0.3037	0.9807 (0.0035)	1.242 (0.0764)	1.9108
	SB	0.9552 (0.0005)	2.131 (0.0109)	0.0929	0.9606 (0.0045)	0.626 (0.0111)	0.4626
	RSB	0.9503 (0.0005)	2.015 (0.0075)	0.0285	0.9484 (0.0035)	0.478 (0.0059)	0.1090
	RUSB	0.9443 (0.0007)	1.924 (0.0089)	0.0240	0.9340 (0.0073)	0.399 (0.0060)	0.0922
20	-	95%	1.955	-	95%	0.433	-
	USB	0.9676 (0.0007)	2.528 (0.0348)	0.3120	0.9801 (0.0035)	1.245 (0.0780)	1.9076
	SB	0.9555 (0.0005)	2.131 (0.0108)	0.0958	0.9603 (0.0045)	0.626 (0.0116)	0.4562
	RSB	0.9506 (0.0004)	2.014 (0.0071)	0.0309	0.9481 (0.0035)	0.476 (0.0060)	0.1016
	RUSB	0.9452 (0.0007)	1.931 (0.0095)	0.0171	0.9333 (0.0074)	0.399 (0.0064)	0.0961

Table 5: Summary statistics of return series $\boldsymbol{y_t}$

Mean	Median	SD	Skewness	Kurtosis	Maximum	Minimum
0.0437	0.0158	0.2730	0.8846	5.0058	1.2840	-0.6090

length of PIs. Using the real data set on the monthly onion price of Delhi market, it has been shown that the PIs for returns developed by all methods contained all the future returns and that the proposed methods have smaller lengths as compared to existing methods. Hence the proposed methods can be used as a viable alternative for computing PIs for non-linear

TS models.

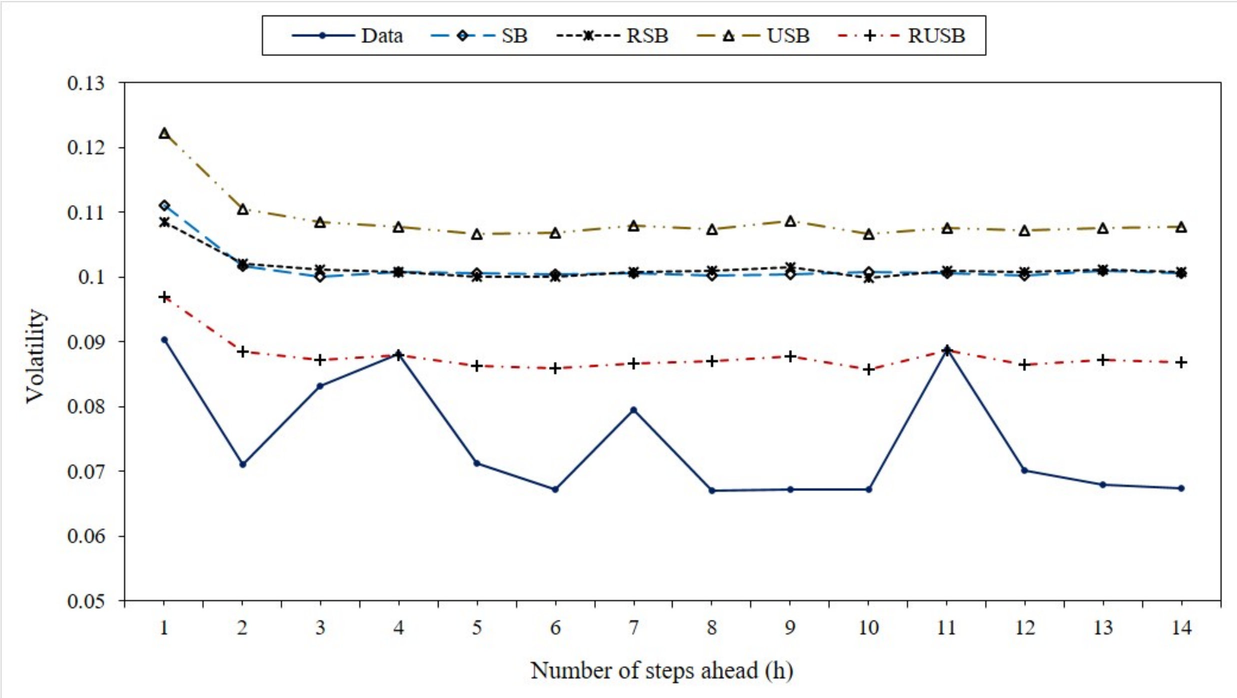

Figure 6: Prediction intervals for volatilities of monthly onion price of Delhi market for forecast horizons $h = 1, 2, \cdots, 14$

Acknowledgements

The facilities provided by ICAR-Indian Agricultural Statistics Research Institute (IASRI), New Delhi and the funding granted to the first author by Indian Council of Agricultural Research in the form of IARI-SRF fellowship is duly acknowledged for carrying out this study, which is part of his doctoral research work being pursued at IASRI. Thanks are also due to the reviewer whose comments have improved the paper to a great extent.

References

Allende, H., Ulloa, G., and Allende-Cid, H. (2015). Prediction intervals in linear and non-linear time series with sieve bootstrap methodology. In: Beran, J., Feng, Y. and Hebbel, H. (Eds.) *Empirical Economic and Financial Research. Advanced Studies in Theoretical and Applied Econometrics*, **48**, Springer, Cham, 255-273. $https://doi.org/10.1007/978-3-319-03122-4\_16$.

Alonso, A. M., Peña, D., and Romo, J. (2002). Forecasting time series with sieve bootstrap. *Journal of Statistical Planning and Inference*, **100**, 1-11.

Alonso, A. M., Peña, D., and Romo, J. (2003). On sieve bootstrap prediction intervals. *Statistics and Probability Letters*, **65**, 13-20.

Alonso, A. M., Peña, D., and Romo, J. (2004). Introducing model uncertainty in time series bootstrap. *Statistica Sinica*, **14**, 155-174.

Andersen, T. G. and Bollerslev, T. (1998). Answering the skeptics: Yes, standard volatility models do provide accurate forecasts. *International Economic Review*, **39**, 885-905.

Andersen, T. G., Bollerslev, T., Diebold, F. X., and Labys, P. (2001). The distribution of realized exchange rate volatility. *Journal of the American Statistical Association*, **96**, 42-55.

Baillie, R. T. and Bollerslev, T. (1992). Prediction in dynamic models with time-dependent conditional variances. *Journal of Econometrics*, **52**, 91-113.

Bentes, S. R. (2015). A comparative analysis of the predictive power of implied volatility indices and GARCH forecasted volatility. *Physica A: Statistical Mechanics and its Applications*, **424**, 105-112.

Beyaztas, U. and Shang, H. L. (2020). Robust bootstrap prediction intervals for univariate and multivariate autoregressive time series models. *Journal of Applied Statistics*, 1-24. DOI: 10.1080/02664763.2020.1856351.

Beyaztas, U. and Shang, H. L. (2021). A robust partial least squares approach for function-on-function regression. *arXiv preprint arXiv:2111.01238*.

Bhansali, R. J. (1983). A simulation study of autoregressive and window estimators of the inverse correlation function. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, **32**, 141-149.

Bhardwaj, S. P., Paul, R. K., Singh, D. R., and Singh, K. N. (2014). An empirical investigation of ARIMA and GARCH models in agricultural price forecasting. *Economic Affairs*, **59**, 415-428.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, **31**, 307-327.

Bose, A. and Mukherjee, K. (2009). Bootstrapping a weighted linear estimator of the arch parameters. *Journal of Time Series Analysis*, **30**, 315-331.

Bougerol, P. and Picard, N. (1992). Strict stationarity of generalized autoregressive processes. *The Annals of Probability*, **20**, 1714-1730.

Box, G. E., Jenkins, G. M., Reinsel, G. C., and Ljung, G. M. (2015). *Time Series Analysis: Forecasting and Control*, John Wiley & Sons.

Bühlmann, P. (1997). Sieve bootstrap for time series. *Bernoulli*, **3**, 123-148.

Bühlmann, P. (2002). Bootstraps for time series. *Statistical Science*, **17**, 52-72.

Chatfield, C. (2000). *Time Series Forecasting*, Chapman and Hall, USA.

Chen, B., Gel, Y. R., Balakrishna, N., and Abraham, B. (2011). Computationally efficient bootstrap prediction intervals for returns and volatilities in ARCH and GARCH processes. *Journal of Forecasting*, **30**, 51-71.

Dyhrberg, A. H. (2016). Bitcoin, gold and the dollar–A GARCH volatility analysis. *Finance Research Letters*, **16**, 85-92.

Engle R. F. (1982). Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, **50**, 987–1007.

Hannan, E. J. and Rissanen, J. (1982). Recursive estimation of mixed autoregressive-moving average order. *Biometrika*, **69**, 81-94.

Iqbal, F. (2011). A weighted linear estimator of multivariate ARCH parameters. *Communications in Statistics-Simulation and Computation*, **40**, 544-560.

Iqbal, F. and Chand, S. (2013). Efficient bootstrap forecast intervals for return and volatility using the linear estimator of ARCH models. *Middle-East Journal of Scientific Research*, **14**, 1502-1507.

Kreiss, J. P. and Franke, J. (1992). Bootstrapping stationary autoregressive moving-average models. *Journal of Time Series Analysis*, **13**, 297-317.

Lahiri, S. N. (2003). *Resampling Methods for Dependent Data*, Springer, New York.

Lama, A., Jha, G. K., and Paul, R. K. (2015). Modelling and forecasting of price volatility: An application of GARCH and EGARCH models. *Agricultural Economics Research Review*, **28**, 365-382.

Lindsay, B. G. (1994). Efficiency versus robustness: the case for minimum Hellinger distance and related methods. *The Annals of Statistics*, **22**, 1081-1114.

Markatou, M., Basu, A., and Lindsay, B. G. (1998). Weighted likelihood equations with bootstrap root search. *Journal of the American Statistical Association*, **93**, 740-750.

Miguel, J. A. and Olave, P. (1999). Bootstrapping forecast intervals in ARCH models. *Test*, **8**, 345-364.

Mukhopadhyay, P. and Samaranayake, V. A. (2010). Prediction intervals for time series: a modified sieve bootstrap approach. *Communications in Statistics-Simulation and Computation*, **39**, 517-538.

NHRDF (2003-2022). National Horticultural Research and Development Foundation, *https://nhrdf.org/en-us/MonthWiseMarketArrivals*, accessed on March 20, 2022.

Pan, L. and Politis, D. N. (2016). Bootstrap prediction intervals for linear, nonlinear and nonparametric autoregressions. *Journal of Statistical Planning and Inference*, **177**, 1-27.

Pascual, L., Romo, J., and Ruiz, E. (2006). Bootstrap prediction for returns and volatilities in GARCH models. *Computational Statistics and Data Analysis*, **50**, 2293-2312.

Politis, D. N., Romano, J. P., and Wolf, M. (1999). *Subsampling.* Springer, New York.

Politis, D. N. (2003). The impact of bootstrap methods on time series analysis. *Statistical Science*, **28**, 219-230.

Poon S. H. (2005). *A Practical Guide to Forecasting Financial Market Volatility.* Wiley, Chichester.

Reeves, J. J. (2005). Bootstrap prediction intervals for ARCH models. *International Journal of Forecasting*, **21**, 237-248.

Thombs, L. A. and Schucany, W. R. (1990). Bootstrap prediction intervals for autoregression. *Journal of the American Statistical Association*, **85**, 486-492.

Tresch, G. D. (2015). *Sieve Bootstrap-Based Prediction Intervals for GARCH Processes*, Ph.D. Thesis, Ashland University, Ohio, USA.

Trucíos, C., Hotta, L. K., and Ruiz, E. (2017). Robust bootstrap forecast densities for GARCH returns and volatilities. *Journal of Statistical Computation and Simulation*, **87**, 3152-3174.

Tsay, R. S. (1988). Outliers, level shifts, and variance changes in time series. *Journal of Forecasting*, **7**, 1-20.

Tsay R. S. (2010). *Analysis of Financial Time Series.* Wiley-Interscience, New York.

Ulloa, G., Allende-Cid, H., and Allende, H. (2014). Robust sieve bootstrap prediction intervals for contaminated time series. *International Journal of Pattern Recognition and Artificial Intelligence*, **28**, 1460012.