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A Note on Estimation of Order Restricted Parameters of Two Uniform Distributions

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Abstract

Uniform distribution is often used in biological and industrial research. Hooda et al. (2007) derived improved estimators of ordered parameters of two uniform distributions with known ordering. In the present paper, the results of Hooda et al. (2007) have been extended for unequal sample sizes. The improved estimators have also been numerically compared in terms of squared error loss with the natural estimators of the ordered parameters with known-ordering through simulation study. The percentage risk improvements of improved estimators over the natural estimators have been worked out for various combinations of parameters and sample sizes.

Keywords: Ordered Parameters; Uniform Distribution; Maximum Likelihood; Equivariant Estimators; Squared Error Loss.

1. Introduction

Estimation of ordered parameters with known or unknown ordering has attracted attention of many researchers. The problem of ordered parameters with known ordering often arises in various agricultural and biological experiments when a researcher estimates the average yield in the presence or absence of a treatment. Estimation of ordered parameters have been studied by Katz (1963), Blumental and Cohen (1968), Cohen and Sachrowitz (1970), Sachrowitz (1982), Kumar and Sharma (1988) and others. Barlow et al. (1972) and Roberston et al. (1988) cite many situations where problems involving ordered parameters are frequently encountered in biological and economic research. Kushary and Cohen (1989, 1991) established that minimum risk estimators of location and scale parameters in the unrestricted case, which uses information only from one population, are inadmissible in the restricted case.

Elfessi and Pal (1992) considered estimation of ordered parameters of two uniform distributions with unknown ordering. Misra and Dhariyal (1995) extended the results of Elfessi

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and Pal to a general case of k (≥ 2) ordered uniform distributions with unknown and known orderings. For this distribution, Fernandez et al. (1997) compared the restricted and unrestricted maximum likelihood estimators using the universal domination and the squared-error loss when linear functions of the parameters are estimated.

Hooda et al. (2007) proposed two new improved estimators based on equal sample sizes and compared these with the natural estimators of the ordered parameters with known ordering. Improved and scale equivariant estimators were also considered and compared with the restricted maximum likelihood estimators in terms of standardized bias and, risk under squared error loss.

In the present paper, we extend the results of Hooda et al. (2007) on estimation of the ordered parameters of two uniform distributions based on independent random samples of sizes n_1 and n_2 drawn from two uniform distributions defined over the intervals (0, θ_1] and (0, θ_2] respectively, where $\theta_1 \le \theta_2$. The proposed estimators have been compared with the usual MLEs in terms of squared-error loss function. It is shown that under certain conditions the proposed estimators dominate the classical MLEs in the unrestricted case. The improved estimators have also been numerically compared in terms of squared-error loss with the natural estimators of the ordered parameters with known-ordering through simulation study. The percentage risk improvements of improved estimators over the natural estimators have been worked out for various combinations of parameters and sample sizes. The continuous uniform distribution is generally used as a probability model for experiments whose outcome is an interval of numbers that are equally likely in the sense that any two intervals of equal lengths have the same probability associated with them. This distribution is also important from the theoretical point of view due to its simplicity and mathematical tractability. Therefore, the present study is very useful both from practical and theoretical considerations where estimation of order restricted parameters of uniform distributions is required.

2. Maximum Likelihood and Best Scale Equivariant Estimators

Let $X_{i1}, X_{i2}, \ldots\ldots.., X_{in_i}$, i = 1, 2 be independent random samples from two uniform populations defined over the intervals (0, θ_1] and (0, θ_2] respectively, where $\theta_1 \le \theta_2$.

The maximum likelihood estimator of θ_i is given by

$$Y_i = \max(X_{i1}, X_{i2}, \ldots\ldots.., X_{in_i}),\ i = 1, 2. \tag{2.1}$$

It is well known that Y_i is a sufficient statistic for θ_i and have probability density function

$$\varphi_i(y_i, \theta_i) = \frac{n_i y_i^{n_i - 1}}{\theta_i^n}, \quad 0 < y_i \le \theta_i,\ i = 1, 2. \tag{2.2}$$

The risk of Y_i under squared-error loss is

$$R_i(Y_i, \theta_i) = E[Y_i - \theta_i]^2 = \frac{2\theta_i^2}{(n_i+1)(n_i+2)},\ i = 1, 2. \tag{2.3}$$

Let $\boldsymbol{\theta} = (\theta_1, \theta_2)$ and $\mathbf{Y} = (Y_1, Y_2)$, the probability density function of $\mathbf{Y}$ is given by

$$h(\mathbf{y}, \boldsymbol{\theta}) = \frac{n_1 n_2}{\theta_1^{n_1}\theta_2^{n_2}} y_1^{n_1-1} y_2^{n_2-1}, \quad 0 < y_1 \le \theta_1, \qquad 0 < y_2 \le \theta_2. \tag{2.4}$$

The restricted parameter space is denoted by $\Omega = \{\boldsymbol{\theta};\ \boldsymbol{\theta} = (\theta_1, \theta_2),\ 0 < \theta_1 \le \theta_2 < \propto\}$. For the ordered uniform distributions, the restricted maximum likelihood estimator of θ_1 remains the same, i.e., $\tau_1 = Y_1$, but that of θ_2 becomes

$$\tau_2 = \max(Y_1, Y_2). \tag{2.5}$$

For comparing the order restricted maximum likelihood estimator τ_2 of θ_2 with the natural maximum likelihood estimator Y_2, we compute risk of τ_2

$$\begin{aligned}
R_2(\tau_2, \boldsymbol{\theta}) &= E[\tau_2 - \theta_2]^2 \\
&= E[\max(Y_1, Y_2) - \theta_2]^2 \\
&= \int_0^{\theta_1}\int_{y_1}^{\theta_2} (y_2-\theta_2)^2\, h(\mathbf{y},\boldsymbol{\theta})\,dy_2\,dy_1 + \int_0^{\theta_1}\int_0^{y_1} (y_1-\theta_2)^2\, h(\mathbf{y},\boldsymbol{\theta})\,dy_2\,dy_1 \\
&= \frac{n_1 n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\int_0^{\theta_1}\int_{y_1}^{\theta_2} (y_2-\theta_2)^2\, y_1^{n_1-1} y_2^{n_2-1}\,dy_2\,dy_1 + \int_0^{\theta_1}\int_0^{y_1} (y_1-\theta_2)^2\, y_1^{n_1-1} y_2^{n_2-1}\,dy_2\,dy_1\right] \\
&= \frac{2n_1\theta_1^{n_2+1}}{\theta_2^{n_2}}\left[\frac{\theta_1}{(n_2+2)(n_1+n_2+2)} - \frac{\theta_2}{(n_2+1)(n_1+n_2+1)}\right] + \frac{2\theta_2^2}{(n_2+1)(n_2+2)}.
\end{aligned} \tag{2.6}$$

Subtracting the risk of Y_2 in (2.3) from the risk of τ_2 in (2.6) we get

$$R_2(\tau_2, \boldsymbol{\theta}) - R_2(Y_2, \theta_2) = \frac{2n_1\theta_1^{n_2+1}}{\theta_2^{n_2}}\left[\frac{\theta_1}{(n_2+2)(n_1+n_2+2)} - \frac{\theta_2}{(n_2+1)(n_1+n_2+1)}\right]. \tag{2.7}$$

Thus, for $\theta_1 < \theta_2$, τ_2 dominates the usual maximum likelihood estimator of θ_2.

3. Improved Estimators of θ_1 under Order Restriction

Let Y_1 and Y_2 be the MLEs of two ordered uniform parameters considered in (2.1). An improved estimator $\tau_1(c, d)$ of θ_1 is proposed and it is shown that it improves upon the MLE of θ_1 with respect to the squared error loss.

Define

$$\tau_1(c, d) = c\, Y_1\, I(Y_1 \le Y_2) + d\, Y_1\, I(Y_1 > Y_2). \tag{3.1}$$

When $c = d = 1$, $\tau_1(c, d) = Y_1$ is the usual maximum likelihood estimator of θ_1.

Lemma 3.1: For the estimator $\tau_1(c, d)$ defined in (3.1), the following expectations hold.

i) $$E[\tau_1(c, d)] = n_1\theta_1\left[\frac{c}{n_1+1} + \left(\frac{d-c}{n_1+n_2+1}\right)\left(\frac{\theta_1}{\theta_2}\right)^{n_2}\right] \tag{3.2}$$

ii) $E[\tau_1^2(c,d)] = n_1\theta_1^2\left[\frac{c^2}{n_1+2}+\frac{d^2-c^2}{(n_1+n_2+2)}\left(\frac{\theta_1}{\theta_2}\right)^{n_2}\right]$ (3.3)

Proof: For $\tau_1(c, d) = c\, Y_1\, I(Y_1 \le Y_2) + d\, Y_1\, I(Y_1 > Y_2)$, defined in (3.1), and using the joint probability density function $h(\mathbf{y}, \boldsymbol{\theta})$ of Y_1 and Y_2, we have

i) $$E[\tau_1(c,d)] = \int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_1)h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1+\int_0^{\theta_1}\int_0^{y_1}(dy_1)h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_1)y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1+\int_0^{\theta_1}\int_0^{y_1}(dy_1)y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1\right]$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[c\left(\frac{\theta_1^{n_1+1}\theta_2^{n_2}}{n_2(n_1+1)}-\frac{\theta_1^{n_1+n_2+1}}{n_2(n_1+n_2+1)}\right)+\frac{d\theta_1^{n_1+n_2+1}}{n_2(n_1+n_2+1)}\right]$$

$$= n_1\theta_1\left[\frac{c}{n_1+1}+\left(\frac{d-c}{n_1+n_2+1}\right)\left(\frac{\theta_1}{\theta_2}\right)^{n_2}\right]. \quad (3.4)$$

ii) $$E[\tau_1^2(c,d)] = \int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_1)^2h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1+\int_0^{\theta_1}\int_0^{y_1}(dy_1)^2h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_1)^2y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1+\int_0^{\theta_1}\int_0^{y_1}(dy_1)^2y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1\right]$$

$$= n_1\theta_1^2\left[c^2\left(\frac{1}{n_1+2}-\frac{1}{(n_1+n_2+2)}\frac{\theta_1^{n_2}}{\theta_2^{n_2}}\right)+\frac{d^2}{(n_1+n_2+2)}\frac{\theta_1^{n_2}}{\theta_2^{n_2}}\right]$$

$$= n_1\theta_1^2\left[\frac{c^2}{n_1+2}+\frac{d^2-c^2}{(n_1+n_2+2)}\left(\frac{\theta_1}{\theta_2}\right)^{n_2}\right]. \quad (3.5)$$

Considering the squared-error loss function and utilizing the results of Lemma 3.1, the risk of $\tau_1(c, d)$ is

$$R_1[\tau_1(c,d),\boldsymbol{\theta}] = \frac{[n_1(n_1+1)c^2-2n_1(n_1+2)c+(n_1+1)(n_1+2)]}{(n_1+1)(n_1+2)}\theta_1^2$$

$$-n_1(c-d)\left[\frac{c+d}{n_1+n_2+2}-\frac{2}{n_1+n_2+1}\right]\left(\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right). \quad (3.6)$$

Taking difference of risks in (3.6) and (2.3) and on rearrangement of terms, we get

$$R_1[\boldsymbol{\theta},\tau_1(c,d)] - R_1[\theta_1, Y_1] = \frac{n_1\theta_1^2}{(n_1+1)(n_1+2)}Q_1(c)-n_1\left(\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right)Q_2(c,d). \quad (3.7)$$

where, $Q_1(c) = (n_1+1)c^2 - 2(n_1+2)c + (n_1+3)$ and

$$Q_2(c,d) = (c-d)\left[\frac{c+d}{n_1+n_2+2} - \frac{2}{n_1+n_2+1}\right].$$

The two roots of $Q_1(c) = 0$ are found to be $c = 1$ and $c = \frac{n_1+3}{n_1+1}$. It can be easily shown from (3.7) that $\tau_1(c, d) = c\, Y_1 I(Y_1 \le Y_2) + d\, Y_1 I(Y_1 > Y_2)$ dominates the usual MLE of θ_1 if either of the following conditions is satisfied

i) $c = 1$ and $1 < d < \frac{n_1+n_2+3}{n_1+n_2+1}$ (3.8)

ii) $c = \frac{n_1+3}{n_1+1}$ and $\frac{n_1^2+n_1n_2+2n_1-n_2+1}{(n_1+n_2+1)(n_1+1)} < d < \frac{n_1+3}{n_1+1}$ (3.9)

iii) $\frac{n_1+n_2+2}{n_1+n_2+1} \le c \le \frac{n_1+3}{n_1+1}$ and $\frac{2(n_1+n_2+2)}{(n_1+n_2+1)} - c \le d \le c$ (3.10)

iv) $1 \le c \le \frac{n_1+n_2+2}{n_1+n_2+1}$ and $c \le d \le \frac{2(n_1+n_2+2)}{(n_1+n_2+1)} - c$. (3.11)

4. Improved Estimators of θ_2

Estimators improving upon the MLE Y_2 of θ_2 may be defined as

$\tau_2(c, d) = c\, Y_2 I(Y_1 \le Y_2) + d\, Y_1 I(Y_1 > Y_2)$ (4.1)

and $\tau_2^*(c, d) = c\, Y_2 I(Y_1 \le Y_2) + d\, Y_2 I(Y_1 > Y_2)$ (4.2)

where c and d are to be chosen suitably.

Here, it is to be noted that for $c = d = 1$, $\tau_2(c, d) = \tau_2$ defined in (2.6).

We now prove the following lemma for the estimators $\tau_2(c, d)$ and $\tau_2^*(c, d)$.

Lemma 4.1: For the estimators $\tau_2(c, d)$ and $\tau_2^*(c, d)$ defined in (4.1) and (4.2) the following expectations hold.

i) $E[\tau_2(c, d)] = \frac{n_2 c}{n_2+1}\theta_2 - \frac{n_1 n_2}{n_1+n_2+1}\left[\frac{c}{n_2+1} - \frac{d}{n_2}\right]\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right)$ (4.3)

ii) $E[\tau_2^*(c, d)] = \frac{n_1 n_2}{n_2+1}\left[\frac{c\theta_2}{n_1} + \frac{d-c}{n_1+n_2+1}\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right)\right]$ (4.4)

iii) $E[\tau_2^2(c, d)] = n_1 n_2\left[\frac{c^2\theta_2^2}{n_1(n_2+2)} + \frac{1}{n_1+n_2+2}\left(\frac{d^2}{n_2} - \frac{c^2}{n_2+2}\right)\left(\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right)\right]$ (4.5)

iv) $E[\tau_2^{*2}(c, d)] = \frac{n_1 n_2}{n_2+2}\left[\frac{c^2\theta_2^2}{n_1} + \frac{d^2-c^2}{(n_1+n_2+2)}\left(\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right)\right]$ (4.6)

Proof : With $\tau_2(c, d)$ and $\tau_2^*(c, d)$ defined in (4.1) and (4.2) and using the pdf of Y_1 and Y_2 from (2.4) we can prove the followings.

i) $$E[\tau_2(c, d)] = \int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_1)h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_1)y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1\right]$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\frac{c}{n_2+1}\left(\frac{\theta_1^{n_1}\theta_2^{n_2+1}}{n_1} - \frac{\theta_1^{n_1+n_2+1}}{n_1+n_2+1}\right) + \frac{d}{n_2(n_1+n_2+1)}\theta_1^{n_1+n_2+1}\right]$$

$$= n_1n_2\left[\frac{c\theta_2}{n_1(n_2+1)} - \left(\frac{c}{(n_2+1)(n_1+n_2+1)} - \frac{d}{n_2(n_1+n_2+1)}\right)\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right]$$

$$= \frac{n_2c}{n_2+1}\theta_2 - \frac{n_1n_2}{n_1+n_2+1}\left[\frac{c}{n_2+1} - \frac{d}{n_2}\right]\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right).$$

ii) $$E[\tau_2^*(c, d)] = \int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_2)h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_2)y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1\right]$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\frac{c}{n_2+1}\left(\frac{\theta_1^{n_1}\theta_2^{n_2+1}}{n_1} - \frac{\theta_1^{n_1+n_2+1}}{n_1+n_2+1}\right) + \frac{d}{(n_2+1)(n_1+n_2+1)}\theta_1^{n_1+n_2+1}\right]$$

$$= \frac{n_1n_2}{n_2+1}\left[\frac{c\theta_2}{n_1} - \frac{c}{n_1+n_2+1}\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}} + \frac{d}{n_1+n_2+1}\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right]$$

$$= \frac{n_1n_2}{n_2+1}\left[\frac{c\theta_2}{n_1} + \frac{d-c}{n_1+n_2+1}\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right)\right].$$

iii) $$E[\tau_2^2(c, d)] = \int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)^2h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_1)^2h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)^2y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_1)^2y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1\right]$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\frac{c^2\theta_1^{n_1}}{n_2+2}\left(\frac{\theta_2^{n_2+2}}{n_1} - \frac{\theta_1^{n_2+2}}{(n_1+n_2+2)}\right) + \frac{d^2}{n_2(n_1+n_2+2)}\theta_1^{n_1+n_2+2}\right]$$

$$= n_1n_2\theta_2^2\left[\frac{c^2}{n_1(n_2+2)} + \left(\frac{d^2}{n_2} - \frac{c^2}{n_2+2}\right)\frac{1}{(n_1+n_2+2)}\left(\frac{\theta_1}{\theta_2}\right)^{n_2+2}\right]$$

$$= n_1n_2\left[\frac{c^2\theta_2^2}{n_1(n_2+2)} + \frac{1}{n_1+n_2+2}\left(\frac{d^2}{n_2} - \frac{c^2}{n_2+2}\right)\left(\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right)\right].$$

iv) $$E[\tau^{*2}_2(c, d)] = \int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)^2 h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_2)^2 h(\mathbf{y},\boldsymbol{\theta})dy_2dy_1$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\int_0^{\theta_1}\int_{y_1}^{\theta_2}(cy_2)^2 y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1 + \int_0^{\theta_1}\int_0^{y_1}(dy_2)^2 y_1^{n_1-1}y_2^{n_2-1}dy_2dy_1\right]$$

$$= \frac{n_1n_2}{\theta_1^{n_1}\theta_2^{n_2}}\left[\frac{c^2}{n_2+2}\left(\frac{\theta_1^{n_1}\theta_2^{n_2+2}}{n_1} - \frac{\theta_1^{n_1+n_2+2}}{(n_1+n_2+2)}\right) + \frac{d^2}{(n_1+n_2+2)(n_2+2)}\theta_1^{n_1+n_2+2}\right]$$

$$= \frac{n_1n_2}{(n_2+2)}\left[\frac{c^2}{n_1}\theta_2^2 - \frac{c^2}{(n_1+n_2+2)}\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}} + \frac{d^2}{(n_1+n_2+2)}\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right]$$

$$= \frac{n_1n_2}{n_2+2}\left[\frac{c^2\theta_2^2}{n_1} + \frac{d^2-c^2}{(n_1+n_2+2)}\left(\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right)\right].$$

Using the Lemma 4.1, it can be easily shown that risk of the estimator $\tau_2(c, d) = c\, Y_2\, I(Y_1 \le Y_2) + d\, Y_1\, I(Y_1 > Y_2)$ under squared error loss is

$$R_2[\tau_2(c, d), \boldsymbol{\theta}] = \theta_2^2\left(\frac{n_2c^2}{n_2+2} - \frac{2n_2c}{n_2+1} + 1\right) - \frac{n_1}{(n_1+n_2+2)}\left(\frac{\theta_1^{n_2+2}}{\theta_2^{n_2}}\right)\left[\frac{n_2c^2}{n_2+2} - d^2\right]$$
$$- \frac{2n_1}{(n_1+n_2+1)}\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2-1}}\right)\left[d - \frac{n_2c}{n_2+1}\right]. \qquad (4.7)$$

Taking difference of risks in (2.3) and (4.7) we can easily show that $\tau_2(c, d)$ dominates upon the usual MLE Y_2 of θ_2 when

$$c \in \left[1, \frac{n_2+3}{n_2+1}\right] \text{ and } d \in \left[\frac{n_2c}{n_2+1}, c\sqrt{\frac{n_2}{n_2+2}}\right]. \qquad (4.8)$$

Also, using the results of Lemma 4.1, risk of $\tau_2^*(c, d)$ under the squared error loss is

$$R_2[\tau_2^*(c, d), \boldsymbol{\theta}] = E[\tau_2^*(c, d) - \theta_2]^2$$
$$= \frac{n_2(n_2+1)c^2 - 2n_2(n_2+2)c + (n_2+1)(n_2+2)}{(n_2+1)(n_2+2)}\theta_2^2$$
$$+ n_1n_2(d-c)\left[\frac{(c+d)\theta_1}{(n_1+n_2+2)(n_2+2)} - \frac{2\theta_2}{(n_1+n_2+1)(n_2+1)}\right]\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right). \qquad (4.9)$$

Subtracting the risk of the MLE Y_2 of θ_2 in (2.3) from the risk of $\tau_2^*(c, d)$ in (4.9) and after simplification, we get

$$R_2[\tau_2^*(c, d), \theta_2] - R_2(Y_2, \theta_2) = \frac{n_2\theta_2^2}{(n_2+1)(n_2+2)}\left[(n_2+1)c^2 - 2(n_2+2)c + (n_2+3)\right]$$

$$+n_1 n_2 (d-c)\left[\frac{(c+d)\theta_1}{(n_1+n_2+2)(n_2+2)}-\frac{2\theta_2}{(n_1+n_2+1)(n_2+1)}\right]\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right)$$

$$=\frac{n_2\theta_2^2}{(n_2+1)(n_2+2)}Q_1(c)+n_1 n_2\left(\frac{\theta_1^{n_2+1}}{\theta_2^{n_2}}\right)Q_2(c,d)\,. \tag{4.10}$$

where, $Q_1(c)=(n_2+1)c^2-2(n_2+2)c+(n_2+3)$ and

$$Q_2(c,d)=(d-c)\left[\frac{(c+d)\theta_1}{(n_1+n_2+2)(n_2+2)}-\frac{2\theta_2}{(n_1+n_2+1)(n_2+1)}\right].$$

$Q_1(c) = 0$ has roots c = 1 and c = $\frac{n_2+3}{n_2+1}$. Also, first term in (4.10) is non-positive for $c\in\left[1,\ \frac{n_2+3}{n_2+1}\right]$. The second term in (4.10) is also non-positive when $c\le d\le\frac{2(n_1+n_2+2)(n_2+2)}{(n_1+n_2+1)(n_2+1)}-c$ for all $0<\theta_1\le\theta_2<\infty$.

Thus, $\tau_2^*(c, d)$ improves upon the MLE of θ_2 when

$$c\in\left[1,\ \frac{(n_1+n_2+2)(n_2+2)}{(n_1+n_2+1)(n_2+1)}\right]\text{and } d\in\left[c,\frac{2(n_1+n_2+2)(n_2+2)}{(n_1+n_2+1)(n_2+1)}-c\right] \text{ for all } 0<\theta_1\le\theta_2. \tag{4.11}$$

5. Comparison of Improved and Natural Estimators

The improved and natural estimators were empirically compared by generating observations from suitable uniform distributions. Point estimators and their risks were computed for various combinations [(2, 5), (5, 10) and (10, 20)] of parameters θ_1& θ_2 and sample sizes n_1 and n_2. Simulation study was conducted for c and d values satisfying conditions in equations (3.10), (4.8) and (4.11) and the results have been presented in tables 5.1, 5.2 and 5.3 respectively. The values of c and marked by * have been taken where above conditions are not satisfied. The procedure was then repeated 10,000 times to approximate the risk by the average of 10,000 values. The risk improvement (*RI*) of the improved estimator over a natural estimator was obtained by the following formula suggested by Jin and Pal (1991).

$$RI(\%)=\left[\frac{Risk(MLE)-Risk(improved)}{Risk(MLE)}\times 100\right] \tag{5.1}$$

The simulation results in tables 5.1 through table 5.3 indicate that the higher risk improvements are obtained for large samples where θ_1 and θ_2 are estimated by the proposed improved estimators. The risk improvements of τ_1 and τ_2^* over the natural estimator of θ_2 for appropriate choices of c and d are almost same. Further, negative values of RI(%) in table 5.1 and table 5.2 indicate no improvement in the estimators for values of c and d not satisfying the conditions (3.10) and (4.8).

Table-5.1 Risk improvements of τ_1 under the squared error loss

θ_1	θ_2	n_1	n_2	Y_1	c	d	τ_1	RI (%)
2	5	5	10	1.66	1.198	1.0625	1.988	15.16
2	5	10	10	1.82	1.115	1.0476	2.0293	28.04
2	5	10	20	1.82	1.107	1.0322	2.0148	31.89
2	5	20	20	1.9	1.059	1.024	2.014	37.8
2	5	20	50	1.9	1.055	1.014	2.0038	40.69
2	5	50	50	1.96	1.025	1.0099	2.008	43.67
2	5	100	100	1.98	1.012	1.005	2.0045	48.49
5	10	10	10	4.54	1.115	1.0476	5.061	28.23
5	10	10	20	4.54	1.107	1.0322	5.0259	32.06
5	10	20	20	4.76	1.060	1.024	5.0447	37.61
5	10	20	50	4.76	1.055	1.014	5.0201	40.5
5	10	50	50	4.9	1.025	1.0099	5.0203	42.59
5	10	100	100	4.95	1.012	1.005	5.0113	48.51
10	20	10	10	9.1	1.115	1.0476	10.1439	28.24
10	20	10	20	9.1	1.107	1.0322	10.074	32.07
10	20	20	20	9.53	1.060	1.024	10.100	37.55
10	20	20	50	9.53	1.055	1.014	10.0509	40.45
10	20	50	50	9.8	1.025	1.0099	10.041	42.58
10	20	100	100	9.9	1.012	1.005	10.023	48.52
2	5	5	10	1.66	0.800*	1.200*	1.328	-66.04
2	5	10	20	1.82	0.800*	1.500*	1.456	-233.64

Table-5.2 Risk improvements of τ_2 under the squared error loss

θ_1	θ_2	n_1	n_2	Y_2	c	d	τ_1	θ_2
2	5	5	10	4.54	1.091	0.9968	4.9527	35.19
2	5	10	10	4.54	1.091	0.9938	4.9527	35.19
2	5	10	20	4.76	1.048	0.9983	4.987	42.19
2	5	20	20	4.76	1.048	0.9983	4.987	42.19
2	5	20	50	4.9	1.02	0.9997	4.996	46.98
2	5	50	50	4.9	1.02	0.9997	4.996	46.98
2	5	100	100	4.95	1.01	0.9999	4.999	48.51
5	10	10	10	9.1	1.091	0.9938	9.927	35.21
5	10	10	20	9.53	1.048	0.9983	9.984	42.14
5	10	20	20	9.53	1.048	0.9983	9.984	42.14
5	10	20	50	9.8	1.02	0.9997	9.992	46.97
5	10	50	50	9.8	1.02	0.9997	9.992	46.97
5	10	100	100	9.9	1.01	0.9999	9.998	48.52
10	20	10	10	18.16	1.091	0.9938	19.811	35.21
10	20	10	20	19.05	1.048	0.9983	19.957	42.12
10	20	20	20	19.05	1.048	0.9983	19.957	42.12
10	20	20	50	19.61	1.02	0.9997	19.994	46.93

10	20	50	50	19.61	1.02	0.9997	19.994	46.93
10	20	100	100	19.8	1.01	0.9999	19.996	48.47
2	5	5	10	4.54	0.800*	0.730*	3.632	-25.41
2	5	10	20	4.76	0.800*	0.760*	3.808	-169.36

Table-5.3 Risk improvements of τ_2^* under the squared error loss

θ_1	θ_2	$\mathbf{n_1}$	$\mathbf{n_2}$	$\mathbf{Y_2}$	**c**	**d**	τ_2^*	**RI (%)**
2	5	5	10	4.54	1.09	1.16	4.9527	35.19
2	5	10	10	4.54	1.09	1.15	4.9527	35.2
2	5	10	20	4.76	1.05	1.08	4.987	42.19
2	5	20	20	4.76	1.05	1.07	4.987	42.19
2	5	20	50	4.9	1.02	1.03	4.996	46.98
2	5	50	50	4.9	1.02	1.03	4.996	46.98
2	5	100	100	4.95	1.01	1.015	4.999	48.51
5	10	10	10	9.1	1.09	1.14	9.927	35.29
5	10	10	20	9.53	1.05	1.08	9.984	42.14
5	10	20	20	9.53	1.05	1.07	9.984	42.14
5	10	20	50	9.8	1.02	1.03	9.992	46.97
5	10	50	50	9.8	1.02	1.03	9.992	46.97
5	10	100	100	9.9	1.01	1.015	9.998	48.52
10	20	10	10	18.16	1.09	1.14	19.811	35.29
10	20	10	20	19.05	1.05	1.08	19.957	42.12
10	20	20	20	19.05	1.05	1.07	19.957	42.12
10	20	20	50	19.61	1.02	1.03	19.994	46.93
10	20	50	50	19.61	1.02	1.03	19.994	46.93
10	20	100	100	19.8	1.01	1.015	19.996	48.47

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