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Variance–sum group–divisible third order slope rotatable designs

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Abstract

In this paper an attempt is made to introduce Variance – Sum Group – Divisible Third Order Slope Rotatable Designs.

Key words: Slope rotatable designs; Third order slope rotatable designs; Group divisible designs; Variance- sum third order slope rotatable designs.

1 Introduction

As an alternative series of second order response surface designs, Das and Dey (1967) introduced Group-Divisible Second Order Rotatable Designs (GDSORD) by modifying the restrictions on the levels of the factors in a second order rotatable design. In these designs the *v*-dimensional space corresponding to *v*-factors is divided into two mutually orthogonal spaces, one of *p*-dimensional and the other of (*v*-*p*) dimension. Given any point *i.e.*, a treatment combination, in the *v*-dimensional space, we can visualize its projection. Let the distances of the projection of the points in each of the subspaces from a suitable origin be d_1^2 and d_2^2 respectively.

$$\sum_{i=1}^{p} x_i^2 = d_1^2$$

$$\sum_{j=p+1}^{v} x_j^2 = d_2^2$$

In rotatable designs, the variance of the estimated response at a point is a function of the distance of that point from origin. In GDSORD the variance of a response estimated at the point $(x_{1,0}, x_{2,0}, ..., x_{v,o})$ is a function of the distances d_1^2 and d_2^2. As the factors get divided into two groups, thus these might be called "Group – Divisible Rotatable Designs" such that for the factors within each group the design is rotatable.

Adikary and Panda (1980, 1984) introduced new methods of constructions of GDSORD and Group-Divisible Third Order Rotatable Designs, when the factors are divided into s $(s \geq \lambda)$ groups. Narasimham *et al*.(1983) gave a new method of construction of GDSORD through BIBDs.

Hader and Park (1978) extended the notion of rotatability to cover the slope for the second order models. They catalogued designs that result in slope rotatability, that is, the variance of the estimated derivatives is constant for all points equidistant from the design center. Anjaneyulu et al (1997) established that slope rotatable designs over all directions have another property known as Variance Sum Second Order Slope Rotatablity. Anjaneyulu et al. (1998) introduced and constructed Group-Divisible Second Order Slope Rotatable Designs over an axial directions. Later, Anjaneyulu et al (2002) introduced and constructed Variance Sum Group Divisible Second Order Slope Rotatable Designs. Anjaneyulu et al (1995, 2000) introduced and constructed third order slope rotatable design over an axial direction. Slope rotatability is potentially useful in cases where rates of reaction (in chemical experiments), rates of deterioration (for example, in a food product), data increase (for example, in cost of manufacture or in deaths when pests are sprayed with a poison) are measured. The U.S. department of Agriculture has used response surface techniques as applied by Baker and Bargman (1985) to plant process simulation models as an aid in the identification of interrelationships among yield and single-valued and functional parameters. Several examples are given about the application of third order response surfaces such as Florida soyabeen model, Texas wheat model to investigate the effectiveness of those higher order surfaces particularly third order response surface models and illustrated how less precise and less costly measurements may be possible in building and using these models.

Anjaneyulu et al (2004) introduced that any Variance-Sum Third Order Slope Rotatable Design is a Third Order Slope Rotatable Design Over All Directions introduced by Park and Lee (1995). Now we introduce Variance – Sum Group- Divisible Third Order Slope Rotatable Designs in this paper.

2 Variance – sum group divisible third order slope rotatable designs

Now we introduce Variance – Sum Group Divisible Third Order Slope Rotatability, that is, the sum of the variance of estimates of the derivative in the direction of any factor axis in each group at any point must be a function of the distances from the design origin.

Let $\mathbf{D} = \left(\left(x_{ik}\right)\right)$ be a set of N design points and $y_1, y_2, ..., y_N$ be the N responses to fit the following third order response surface,

$$Y(X) = b_0 + \sum b_i x_i + \sum b_j x_j + \sum\sum b_{ij} x_i x_j + \sum\sum b_{ii^1} x_i x_{i^1} + \sum\sum b_{jj^1} x_j x_{j^1}$$
$$+ \sum b_{ii} x_i^2 + \sum b_{jj} x_j^2 + \sum b_{iii} x_i^3 + \sum b_{jjj} x_j^3 + \sum\sum b_{ijj} x_i x_j^2$$
$$+ \sum\sum b_{ii^1 i^1} x_i x_{ii^1}^2 + \sum\sum b_{jj^1 j^1} x_j x_{j^1}^2 + \sum\sum\sum b_{ijk} x_i x_j x_k + e, \tag{2.1}$$

where e's are independent random errors with same mean and variance σ^2.

Then we must have,

$$\sum_{i=1}^{v} V\left(\frac{\partial \hat{y}}{\partial x_i}\right) = f\left(d_1^2, d_2^2\right),$$

where $d_1^2 = \sum_{i=1}^{p} x_i^2$, $d_2^2 = \sum_{j=p+1}^{v} x_j^2$,

d_1^2 and d_2^2 are the distances of the projections of the points in p dimensional and $(v-p)$ dimensional spaces from a suitable origin.

Let us consider the following symmetry conditions. (c.f. Anjaneyulu *et al*. (2004))

A: All sums of products in which at least one of the *x*'s with an odd power are zero.

B: (i) $\sum x_i^2$ = constant for $i = 1, 2, 3, \ldots, p$

$\sum x_j^2$ = constant $j = p+1, p+2, \ldots, v$

(ii) $\sum x_i^4$ = constant for $i = 1, 2, 3, \ldots, p$

$\sum x_j^4$ = constant $j = p+1, p+2, \ldots, v$

(iii) $\sum x_i^6$ = constant for $i = 1, 2, 3, \ldots, p$

$\sum x_j^6$ = constant $j = p+1, p+2, \ldots, v$

C: (1) (i) $\sum x_1^2 x_j^2$ = constant , for $i \neq j$ for $i = 1, 2, 3, \ldots, p$
$j = p+1, p+2, \ldots v$

(ii) $\sum_{j \neq j^1} x_j^2 x_{j^1}^2$ = constant for $j, j^1 = p+1, p+2, \ldots, v$

(iii) $\sum_{i \neq i^1} x_i^2 x_{i^1}^2$ = constant $i, i^1 = 1, 2, 3, \ldots, p$

(2) (i) $\sum x_1^2 x_j^4$ = constant , for i≠j for $i = 1, 2, 3, \ldots, p$
j = $p+1, p+2, \ldots, v$

(ii) $\sum x_j^2 x_{j^1}^4$ = constant for $i, i^1 =$ p+1, p+2, …, v

(iii) $\sum x_i^2 x_{i^1}^4$ = constant for i, i^1 for $i = 1, 2, 3, \ldots, p$

$$\text{D:} \quad \sum\sum\sum x_i^2 x_j^2 x_k^2 = \text{Constant} \quad \text{for } i \neq j \neq k \tag{2.2}$$

The above summations are taken over all design points. By differentiating (2.1) with respect to x_i and x_j, the variances $V\left(\frac{\partial y}{\partial x_i}\right)$ and $V\left(\frac{\partial y}{\partial x_i}\right)$ are functions of distance of the point from the design origin. Hence the considered response surface is a Variance - Sum - Group Divisible Third Order Slope Rotatable Designs.

3 Construction of variance – sum group- divisible TOSRD using Central Composite type design points

So far no method of construction of Variance Sum Group Divisible Third Order Slope Rotatable Design is available. Hence we make an attempt to evolve a method of construction of Variance – Sum Group Divisible Third Order Slope Rotatable Design using Central Composite Type Design Points.

Consider the following VSTOSRD with Central Composite Type of design points in v dimensions,

$$\begin{bmatrix} \pm\alpha & \pm\alpha & \pm\alpha & \pm\alpha \ldots\ldots\ldots\ldots \pm\alpha \\ \pm\beta & \pm\beta & \pm\beta & \pm\beta \ldots\ldots\ldots\ldots \pm\beta \\ \pm\gamma & \pm\gamma & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ \pm\delta & 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 \\ \pm\varepsilon & 0 & 0 & 0 \ldots\ldots\ldots\ldots 0 \end{bmatrix} \tag{3.1}$$

The total number of design points is $N = 2^v + 2^v + 2v(v-1) + 4v$

Let the design given in (3.1) be divided in to the two Groups as Group one contains p – factors and Group two contains (v-p) factors.

From the central composite type design points, we have,

1) All odd order moments are equal to Zero in both the groups

i,e., $\sum x_i, \sum x_j, \sum x_i x_j^2, \sum x_i x_{i^1}^2, \sum x_i^3, \sum x_j^3, \sum x_i x_j x_k, \sum x_i x_{i^1} x_j$......etc are Zero

2) (i) $\sum x_i^2$ = constant for $i = 1, 2, 3, \ldots, p$

(ii) $\sum x_j^2$ = constant for $j = p+1, p+2, \ldots, v$

(iii) $\sum x_i^4$ = constant for $i = 1, 2, 3, \ldots, p$

(iv) $\sum x_j^4$ = constant for $j = p+1, p+2, \ldots, v$

(v) $\sum x_i^6$ = constant for $i = 1, 2, 3, \ldots, p$

(vi) $\sum x_j^6$ = constant for $j = p+1, p+2, \ldots, v$

(vii) $\sum x_i^2 x_j^2$ = constant , for $i \neq j$ for $i = 1, 2, 3, \ldots, p$; $j = p+1, p+2, \ldots, v$

(viii) $\sum x_i^2 x_{i1}^2$ = constant for $i, i^1 = 1, 2, 3, \ldots, p$

(ix) $\sum_{j \neq j^1} x_j^2 x_{j^1}^2$ = constant for $j, j^1 = p+1, p+2, \ldots, v$

(x) $\sum_{i \neq j} x_i^4 x_j^2$ = constant for $i = 1, 2, 3, \ldots, p$; $j = p+1, p+2, \ldots, v$

(xi) $\sum x_i^4 x_{i^1}^2$ = constant for $i, i^1 = 1, 2, 3, \ldots, p$

(xii) $\sum x_j^4 x_{j^1}^2$ = constant for $j, j^1 = p+1, p+2, \ldots, v$

(xiii) $\sum x_i^2 x_j^2 x_k^2$ = Constant for $i \neq j \neq k$ for $i = 1, 2, 3, \ldots, p$; $j = p+1, p+2, \ldots, v$; $k = 1, 2, 3, \ldots, v$

Conditions given in (2.2) obviously satisfied by the design points given in (3.1). For these design points, we have the following.

$$\sum x_i^2 = 2^{t(v)} \alpha^2 + 2^{t(v)} \beta^2 + 4(v-1)\gamma^2 + 2\delta^2 + 2\varepsilon^2 = \text{constant}$$
$$\sum x_j^2 = 2^{t(v)} \alpha^2 + 2^{t(v)} \beta^2 + 4(v-1)\gamma^2 + 2\delta^2 + 2\varepsilon^2 = \text{constant}$$

$$\sum x_i^4 = 2^{t(v)} \alpha^4 + 2^{t(v)} \beta^4 + 4(v-1)\gamma^4 + 2\delta^4 + 2\varepsilon^4 = \text{constant}$$
$$\sum x_j^4 = 2^{t(v)} \alpha^4 + 2^{t(v)} \beta^4 + 4(v-1)\gamma^4 + 2\delta^4 + 2\varepsilon^4 = \text{constant}$$

$$\sum x_i^6 = 2^{t(v)} \alpha^6 + 2^{t(v)} \beta^6 + 4(v-1)\gamma^6 + 2\delta^6 + 2\varepsilon^6 = \text{constant}$$
$$\sum x_j^6 = 2^{t(v)} \alpha^6 + 2^{t(v)} \beta^6 + 4(v-1)\gamma^6 + 2\delta^6 + 2\varepsilon^6 = \text{constant}$$

$$\sum x_i^{\,2} x_j^{\,2} = 2^{t(v)}\alpha^4 + 2^{t(v)}\beta^4 + 4\gamma^4 \qquad = \text{constant}$$

$$\sum x_i^{\,2} x_{i1}^{\,2} = 2^{t(v)}\alpha^4 + 2^{t(v)}\beta^4 + 4\gamma^4 \qquad = \text{constant}$$

$$\sum x_j^{\,2} x_{j1}^{\,2} = 2^{t(v)}\alpha^4 + 2^{t(v)}\beta^4 + 4\gamma^4 \qquad = \text{constant}$$

$$\sum x_i^{\,4} x_j^{\,2} = 2^{t(v)}\alpha^6 + 2^{t(v)}\beta^6 + 4\gamma^6 \qquad = \text{constant}$$

$$\sum x_i^{\,4} x_{i1}^{\,2} = 2^{t(v)}\alpha^6 + 2^{t(v)}\beta^6 + 4\gamma^6 \qquad = \text{constant}$$

$$\sum x_j^{\,2} x_{j1}^{\,4} = 2^{t(v)}\alpha^6 + 2^{t(v)}\beta^6 + 4\gamma^6 \qquad = \text{constant}$$

$$\sum x_i^{\,2} x_j^{\,2} x_k^{\,2} = 2^{t(v)}\alpha^6 + 2^{t(v)}\beta^6 \qquad = \text{constant}$$

$$\sum x_i^{\,2} x_{i1}^{\,2} x_{i11}^{\,2} = 2^{t(v)}\alpha^6 + 2^{t(v)}\beta^6 \qquad = \text{constant}$$

$$\sum x_j^{\,2} x_{j1}^{\,2} x_{j11}^{\,2} = 2^{t(v)}\alpha^6 + 2^{t(v)}\beta^6 \qquad = \text{constant}$$

Example 3.1: Construction of Variance - Sum GDTOSRD in four dimension using the central composite type design points.

Consider the following design points

$(\pm\alpha, \pm\alpha, \pm\alpha, \pm\alpha.)$ ....16 points

$(\pm\beta, \pm\beta, 0, 0)$..................12 points

$(\pm\gamma, 0, 0, 0)$ ....08 points

$(\pm\delta, 0, 0, 0)$ ....08 points

Here $N = 44$ points.

For the above design points, conditions in (2.2) are obviously satisfied. For the above design points, we have ,

$$\sum x_i^{\,2} = 16\,\alpha^2 + 12\,\beta^2 + 2\gamma^2 + 2\delta^2 = \text{constant}$$

$$\sum x_j^{\,2} = 16\,\alpha^2 + 12\,\beta^2 + 2\,\gamma^2 + 2\delta^2 = \text{constant}$$

$$\sum x_i^{\,4} = 16\,\alpha^4 + 12\,\beta^4 + 2\,\gamma^4 + 2\delta^4 = \text{constant}$$

$$\sum x_j^{\,4} = 16\,\alpha^4 + 12\,\beta^4 + 2\,\gamma^4 + 2\delta^4 = \text{constant}$$

$$\sum x_i^6 = 16\,\alpha^6 + 12\,\beta^6 + 2\,\gamma^6 + 2\delta^6 = \text{constant}$$
$$\sum x_j^6 = 16\,\alpha^6 + 12\,\beta^6 + 2\,\gamma^6 + 2\delta^6 = \text{constant}$$

$$\sum x_i^2 x_j^4 = 16\,\alpha^6 + 4\,\beta^6 = \text{constant}$$
$$\sum x_i^2 x_{i1}^4 = 16\,\alpha^6 + 4\,\beta^6 = \text{constant}$$
$$\sum x_j^2 x_{j1}^4 = 16\,\alpha^6 + 4\,\beta^6 = \text{constant}$$

$$\sum x_i^2 x_j^4 = 16\,\alpha^6 + 4\,\beta^6 = \text{constant}$$
$$\sum x_i^2 x_{i1}^4 = 16\,\alpha^6 + 4\,\beta^6 = \text{constant}$$
$$\sum x_j^2 x_{j1}^4 = 16\,\alpha^6 + 4\,\beta^6 = \text{constant}$$

$$\sum x_i^2 x_j^2 x_k^2 = 16\alpha^6 = \text{constant} \quad ,$$

where *i*, *j*, *k* = 1, 2, 3, 4.

Hence, conditions in (2.2) are satisfied.

The above design points gives a Variance – Sum Group Divisible Third Order Slope Rotatable Design for 4 factors for any values of $\alpha, \beta, \gamma, \delta$.

Constructing the Variance -Sum TOSRD using CCD type design points mentioned in this paper is first time. The Variance-sum TOSRD can also be constructed using Doubly balanced incomplete block design points. Usually, CCD construction gives us less number of points.

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