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A New Unit Root Test for an Autoregressive Model Subject to Measurement Errors

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Abstract

The unit root test – a test of the null hypothesis that a first-order autoregressive model is a random walk model against the alternative hypothesis that the model is a stationary model - has played a significant role in time series literature. The benchmark unit root test is the well-known Dickey-Fuller test widely extended to cover a variety of applications. However, to the best of our knowledge, all available unit root tests assume no measurement errors in the observed data. In this paper, we first investigate the effects of sampling errors, alternatively called as measurement errors, on the biases of the commonly used estimators of autocorrelation coefficient and the Dickey-Fuller test statistics. We then propose alternative estimators for the autocorrelation coefficient and the Dickey-Fuller test statistics to reduce such biases due to sampling errors. In our study, we prove that the adjusted estimators of the autocorrelation coefficient and the test statistics have the same asymptotic distributions as that of the Dickey-Fuller test statistics. Moreover, we conduct Monte Carlo simulation studies to investigate the performance of our proposed test statistics in terms of unbiasedness, the probability of Type-I error, and power of the test. Our simulation results demonstrate that the proposed estimators can reduce bias due to sampling errors. Finally, we apply the proposed test statistics to the Current Population Survey (CPS) data on unemployment of the United States during the period 1990 - 2013.

Key words: Unit root; Autoregressive coefficient; Sampling errors; Measurement errors; Likelihood ratio.

AMS Subject Classifications: 62F10, 62F12 , 62H20, 62M10

1. Introduction

Measurement errors in time series data occur in different applications of ecology, economics, finance, repeated surveys, and other disciplines. In ecological research, Shenk *et al.* (1998) introduced the concept of sampling errors in the form of measurement errors. Specifically, they investigated the effects of sampling variances on the first-order autoregressive

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population models in order to estimate population abundance. The concept was then studied in the context of time series population models such as the ones given in De Valpine and Hastings (2002), Dennis *et al.* (2006), Buonaccorsi and Staudenmayer (2009).

In Economics and Finance, Walters and Ludwig (1981) studied effects of measurement errors on the estimation of stock-recruitment relationships. Moreover, they obtained estimates of measurement errors. Besides the applications in stock markets, the measurement errors in time series data were also considered in other applications such as the U.K. GDP (Smith *et al.*, 1998) and the U.S. GDP (Aruoba *et al.*, 2016).

Time series data with measurement errors also occur in the context of repeated surveys where the actual characteristics of interest are usually not observed but are estimated by survey direct estimates. The problem was first considered in Scott and Smith (1974) where the authors considered an autoregressive time series model with sampling errors. The study was then further pursued by many researchers, such as Scott *et al.* (1977), Bell and Hillmer (1990) Ludwig and Walters (1981), Bell and Hillmer (1990), Staudenmayer and Buonaccorsi (2005), Rossi and Santucci de Magistris (2018).

Beside parameter estimation, one crucial tool for autoregessive time series analysis is the test of unit root. The benchmark unit root test was introduced by Dickey and Fuller (1979), where they obtained the test statistic and derived the asymptotic distribution of their test statistic under the null hypothesis of unit root. The test has been widely extended to higher order time series models and applied in many contexts during the last few decades. However, the test statistic was originally designed for real-time series data without accounting for sampling errors commonly found in repeated survey data. Ignoring sampling errors could cause biases to the test statistic and lead to a wrong conclusion of the unit root test in the presence of sampling errors. Therefore, to avoid such biases, effects of sampling errors to the unit root test deserve investigation and an effective adjustment to the test statistics is required. However, to the best of our knowledge, there is no unit root test for time series data with measurement errors available in literature.

In this paper, we investigate the effect of sampling errors on the unit root test of Dickey and Fuller (1979). Our study suggests that ignoring sampling errors could cause biases in the estimation of autocorrelation coefficient and the Dickey-Fuller unit root test statistics. Thus, we propose a modification of the Dickey-Fuller test that is bias-corrected for sampling errors. We derive its asymptotic properties, and conduct Monte Carlo simulation studies to investigate the performance of our proposed method by considering the unbiasedness, the probability of Type-I error, and the power of the test. Moreover, we apply the proposed test statistics to the Current Population Survey (CPS) data on unemployment of the United States during the period 1990 to 2013. The numerical results demonstrate that the new test can reduce the bias of the original Dickey-Fuller test when there is a present of sampling errors.

The organization of this paper is as follows. In Section 2, we review the Dickey-Fuller unit root test statistic for the first-order autoregressive model. In Section 3, we propose an adjusted estimate of the Dickey-Fuller unit root in the presence of sampling errors. In Section 4, we demonstrate Monte Carlo simulations to study the performance of the proposed test statistic in different aspects such as bias, probability of Type-I error, and power of the test. In Section 5, we apply the proposed test statistic to the Current Population Survey (CPS) data on unemployment of the United States during the period 1990 to 2013. In Section 6, we offer some concluding remarks. Finally the proofs of theoretical properties of the proposed test statistic and important lemmas are provided in Section 7.

2. Unit root test for AR(1) model

Consider the first order autoregressive model for the time series $\{Y_t : t = 1, 2, \ldots, T\}$, defined as

$$Y_t = \rho Y_{t-1} + e_t, \tag{1}$$

where ρ is the regression coefficient and $\{e_t\}$ is a sequence of independent normal random variables with mean zero and unknown variance σ_e^2.
The least squares estimate $\hat{\rho}_Y$ of the autocorrelation coefficient ρ is defined as

$$\hat{\rho}_Y = \frac{S_{Y,T}(1)}{S_{Y,T}(0)}, \tag{2}$$

where $S_{Y,T}(k) = \sum_{t=2}^{T} Y_{t-1}Y_{t+k-1}$.

Dickey and Fuller (1979) constructed the unit root test statistic under the null hypothesis that $\rho = 1$ as

$$\hat{\tau} = \frac{(\hat{\rho}_Y - 1)\sqrt{\sum_{t=1}^{T} Y_t^2}}{\sqrt{\hat{\sigma}^2}}, \tag{3}$$

where

$$\hat{\sigma}^2 = \frac{1}{T-2}\sum_{t=2}^{T}(Y_t - \hat{\rho}_Y Y_{t-1})^2.$$

Moreover, they obtained the asymptotic distribution of $\hat{\rho}_Y$ as

$$T(\hat{\rho}_Y - 1) \xrightarrow{d} \frac{\left(\sum_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sum_{i=1}^{\infty} \gamma_i^2 Z_i^2},$$

where $Z_i \overset{iid}{\sim} N(0,1)$ and $\gamma_i = (-1)^{i+1}\dfrac{2}{(2i-1)\pi}$.

Consequently, the asymptotic distribution of the test statistic $\hat{\tau}$ is obtained as

$$\hat{\tau} \xrightarrow{d} \frac{\left(\sum_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sqrt{\sum_{i=1}^{\infty} \gamma_i^2 Z_i^2}}. \tag{4}$$

3. Unit root test for AR(1) with measurement errors

In this section, we consider the model in (1) when the actual time series $\{Y_t : t = 1, 2, \ldots, T\}$ is unobserved but its predicted value from a survey $\{W_t : t = 1, 2, \ldots, T\}$ can be obtained. Specifically, the model considered in this section consists of two sub-models: the autoregressive model for the actual time series defined in (1) and the sampling model assuming that the observed value can be written as a sum of the actual value and a sampling error. In particular, the sampling model is

$$W_t = Y_t + u_t, \tag{5}$$

where $\{W_t : t = 1, 2, \ldots, T\}$ is the sequence of observed variables with $W_0 = 0$ and $\{u_t : t = 1, 2, \ldots, T\}$ is the sequence of sampling errors assumed to be independently normally distributed with mean zero and known variances σ^2_{ut}. The assumption of known sampling variances σ^2_{ut} often follows from the asymptotic variances of transformed direct designed-based estimates such as in Efron and Morris (1975), Carter and Rolph (1974), Lahiri and Suntornchost (2015), and Marhuenda García *et al.* (2016).

To construct an adjustment of the unit root test, we first investigate the effect of ignoring the sampling errors to the estimations of the autocorrelation coefficient and the Dickey- Fuller unit root test statistic. By substituting Y_t with the survey estimate W_t in (2), the naive estimate of the autocorrelation coefficient is

$$\hat{\rho}_W = \frac{S_{W,T}(1)}{S_{W,T}(0)}$$

and the naive test statistic is

$$\hat{\tau}_{naive} = \frac{(\hat{\rho}_W - 1)\sqrt{S_{W,T}(0)}}{\sqrt{\hat{\sigma}^2_{W,e}}}, \tag{6}$$

where

$$\hat{\sigma}^2_{W,e} = \frac{1}{T-2}\sum_{t=2}^{T}(W_t - \hat{\rho}_W W_{t-1})^2.$$

Applying the conditional expectation, we found that

$$\mathbb{E}\left(S_{W,T}(0)|Y_t\right) = \sum_{t=2}^{T} Y^2_{t-1} + \sum_{t=2}^{T} \sigma^2_{u,t-1},$$
$$\mathbb{E}\left(S_{W,T}(1)|Y_t\right) = \sum_{t=2}^{T} Y_t Y_{t-1}.$$

Therefore, by applying the first order Taylor series approximation, we can show that the naive estimator of the autocorrelation coefficient, $\hat{\rho}_W$, is asymptotically biased and then the estimator is not reliable. Hence, following Lahiri and Suntornchost (2015), we propose an adjustment to each component in $\hat{\rho}_W$ by removing the biases of $S_{W,T}(0)$ and $S_{W,T}(1)$. Therefore, the proposed estimate of the autoregressive coefficient ρ is defined as

$$\hat{\rho}_{Adj} = \frac{S_{W,T}(1)}{\widetilde{S}_{W,T}(0)},$$

where $\widetilde{S}_{W,T}(0) = S_{W,T}(0) - S_{\sigma_u}(0)$, and $S_{\sigma_u}(0) = \sum_{t=2}^{T} \sigma^2_{u,t-1}$. Applying the first order Taylor series approximation, we prove in Theorem 1 that

$$\hat{\rho}_{Adj} - \hat{\rho}_Y = o_p(1), \tag{7}$$

under the assumption $\rho = 1$.
Moreover, we show in Theorem 2 that $T(\hat{\rho}_{Adj} - 1)$ has the same asymptotic distribution as $T(\hat{\rho}_Y - 1)$. In particular,

$$T(\hat{\rho}_{Adj} - 1) \xrightarrow{d} \frac{\left(\sum_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sum_{i=1}^{\infty} \gamma_i^2 Z_i^2},$$

where $Z_i \overset{iid}{\sim} N(0,1)$ and $\gamma_i = (-1)^{i+1}\dfrac{2}{(2i-1)\pi}$.

Furthermore, we construct an adjusted estimate for σ^2 subject to sampling errors, defined as

$$\hat{\sigma}^2_{Adj,e} = |\hat{\sigma}^2_{W,e,1} - \hat{\sigma}^2_{W,e,2}|, \tag{8}$$

where

$$\hat{\sigma}^2_{W,e,1} = \frac{1}{T-2}\sum_{t=2}^{T}(W_t - \hat{\rho}_{Adj}W_{t-1})^2,$$

and

$$\hat{\sigma}^2_{W,e,2} = \frac{1}{T-2}\sum_{t=2}^{T}\left(\sigma^2_{u,t} + \hat{\rho}^2_{Adj}\sigma^2_{u,t-1}\right).$$

Then, we propose an adjusted test statistic for the unit root test of the first order autoregressive model subject to measurement errors defined as

$$\hat{\tau}_{Adj} = \frac{(\hat{\rho}_{Adj} - 1)\sqrt{\widetilde{S}_{W,T}(0)}}{\sqrt{\hat{\sigma}^2_{Adj,e}}}. \tag{9}$$

Moreover, we prove in Theorem 3 that the proposed test statistic has the same asymptotic distribution as the true estimate $\hat{\tau}_Y$. In particular,

$$\hat{\tau}_{Adj} \xrightarrow{d} \frac{\left(\sum_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sqrt{\sum_{i=1}^{\infty} \gamma_i^2 Z_i^2}}, \tag{10}$$

where $\gamma_i = (-1)^{i+1}\frac{2}{(2i-1)\pi}$ and $Z_i \overset{iid}{\sim} N(0,1)$.

4. Monte carlo simulations

In this section, we conduct Monte Carlo simulations to study the performance of the proposed test statistic compared to the naive test that ignores sampling errors. For our simulation experiment, we set the true sampling variances of u_t, σ^2_{ut} in model (5), by using estimated variances of 288 monthly survey-weighted direct estimates of the number of unemployed workers obtained from the U.S. Current Population Survey (CPS) conducted during the period 1990 - 2013. There are 12 simulation settings based on four selected states with different ranges of sampling standard deviations and three different values of the regression standard deviation σ_e of the autoregressive model (1). The values of σ_e are specified by the ratio $k = \dfrac{\bar{\sigma}_u}{\sigma_e}$ where $\bar{\sigma}_u$ is the average of sampling standard deviations defined as $\bar{\sigma}_u = T^{-1} \sum_{t=1}^{T} \sigma_{ut}$. The three values of k considered are $0.75, 1$, and 1.25 representing the cases where the average of standard deviations of sampling errors is smaller than, equal to, and larger than the regression standard deviation, respectively. In addition, we consider four different lengths (T) of time series, $T \in \{25, 50, 100, 250\}$, to study asymptotic behaviours of the test statistics. Each setting is repeated for 20,000 simulation runs. In particular,the steps of simulation are as follows.

1. For each combination of state and k, calculate the regression variance σ^2_e from $\sigma_e = \dfrac{\bar{\sigma}_u}{k}$.
2. For each simulation setting and each $l = 1, 2, \ldots, 20,000$,
   (a) generate the variance components and sampling errors $\{(u_t^{(l)}, e_t^{(l)}) : t = 1, 2, \ldots, 250\}$,
   (b) calculate the time series $\{Y_t^{(l)} : t = 1, 2, \ldots, 250\}$, from model (1) with $\rho = 1$,
   (c) generate $\{W_t^{(l)} : t = 1, 2, \ldots, 250\}$ from model (5),
   (d) calculate $\hat{\tau}^{(l)}_{true}$, $\hat{\tau}^{(l)}_{naive}$, and $\hat{\tau}^{(l)}_{Adj}$ from the fomula in (3), (6), and (9), respectively.

To study the performances of the test statistics, we first consider different percentiles of the estimated test statistics and the estimated values of the probability of Type-I error. The estimates of the test statistics in different percentiles by using data from one selected state, State 3, are presented in Tables 1 - 3, respectively for the cases of $k = 0.75, 1$, and 1.25.

From Tables 1 - 3, we can see that the percentiles of the true test statistics and the proposed test statistic are close together, particularly those values between the 10th and 90th percentiles. In contrast, the naive test statistics are much lower than the true estimates in all cases. These results suggest that the naive estimator of the Dickey-Fuller test statistic underestimates the true test statistic, while the proposed estimator can reduce such underestimation.

Next, we consider the accuracy of the estimated probability of Type-I error, computed as the portion of the number of replications in which the unit root hypothesis is rejected when the actual time series is generated from the true autoregressive model (1) with $\rho = 1$. In particular, the estimated probability of Type-I error is computed as

$$\hat{\alpha} = \frac{1}{L} \sum_{l=1}^{L} \mathbb{1}_{\{\hat{\tau}_{\cdot}^{(l)} \text{ reject } H_0\}},$$

Table 1: The empirical percentiles of the different test statistics for $k = 0.75$

Length (T)	Statistics	Percentiles						
		1	10	25	50	75	90	99
$T = 25$	$\hat{\tau}_{true}$	-2.58	-1.61	-1.06	-0.51	0.21	0.87	2.28
	$\hat{\tau}_{naive}$	-3.88	-2.40	-1.65	-0.98	-0.34	0.25	1.24
	$\hat{\tau}_{Adj}$	-3.41	-1.78	-1.07	-0.46	0.32	1.24	4.16
$T = 50$	$\hat{\tau}_{true}$	-2.60	-1.68	-1.11	-0.53	0.22	0.90	2.08
	$\hat{\tau}_{naive}$	-3.91	-2.58	-1.82	-1.06	-0.36	0.20	1.10
	$\hat{\tau}_{Adj}$	-3.10	-1.78	-1.11	-0.47	0.31	1.17	3.21
$T = 100$	$\hat{\tau}_{true}$	-2.65	-1.61	-1.09	-0.54	0.23	0.86	2.06
	$\hat{\tau}_{naive}$	-3.87	-2.48	-1.76	-1.05	-0.37	0.19	1.13
	$\hat{\tau}_{Adj}$	-2.64	-1.65	-1.08	-0.48	0.27	0.97	2.41
$T = 250$	τ_{true}	-2.69	-1.62	-1.12	-0.55	0.19	0.87	2.16
	$\hat{\tau}_{naive}$	-3.88	-2.46	-1.78	-1.09	-0.38	0.22	1.12
	$\hat{\tau}_{Adj}$	-2.56	-1.64	-1.10	-0.54	0.21	0.91	2.20

Table 2: The empirical percentiles of the different test statistics for $k = 1$

Length (T)	Statistics	Percentiles						
		1	10	25	50	75	90	99
$T = 25$	$\hat{\tau}_{true}$	-2.68	-1.65	-1.09	-0.54	0.16	0.92	2.16
	$\hat{\tau}_{naive}$	-4.25	-2.81	-2.04	-1.29	-0.59	0.03	0.94
	$\hat{\tau}_{Adj}$	-3.87	-2.04	-1.25	-0.54	0.27	1.26	5.48
$T = 50$	$\hat{\tau}_{true}$	-2.63	-1.70	-1.15	-0.56	0.18	0.84	2.22
	$\hat{\tau}_{naive}$	-4.59	-3.05	-2.25	-1.41	-0.68	-0.09	0.77
	$\hat{\tau}_{Adj}$	-3.50	-1.85	-1.16	-0.51	0.29	1.20	4.94
$T = 100$	$\hat{\tau}_{true}$	-2.52	-1.67	-1.13	-0.57	0.15	0.84	1.88
	$\hat{\tau}_{naive}$	-4.49	-3.02	-2.19	-1.40	-0.67	-0.11	0.67
	$\hat{\tau}_{Adj}$	-2.95	-1.71	-1.10	-0.50	0.23	1.02	3.23
$T = 250$	τ_{true}	-2.58	-1.62	-1.11	-0.51	0.21	0.86	2.04
	$\hat{\tau}_{naive}$	-4.46	-2.95	-2.15	-1.34	-0.64	-0.07	0.86
	$\hat{\tau}_{Adj}$	-2.75	-1.65	-1.09	-0.49	0.24	0.97	2.68

where $\mathbb{1}_{\{\hat{\tau}_{\cdot}^{(l)} \text{ reject } H_0\}}$ is equal to 1 if the specific test statistic $\hat{\tau}_{\cdot}^{(l)} \in \{\hat{\tau}_{true}, \hat{\tau}_{Adj}, \hat{\tau}_{naive}\}$ rejects $\rho = 1$, and is equal to 0 for otherwise. The results for the tests with significance level 0.05 are presented in Table 4 as follows. From Table 4, we can see that the estimated probabilities of Type-I error of the true test statistic $\hat{\tau}_{true}$ and the proposed test statistic $\hat{\tau}_{adj}$ are approximately 0.05 in all cases. In contrast, the naive test statistic $\hat{\tau}_{naive}$ produces estimated probabilities of Type-I error different from 0.05 for all cases. Specifically, the values are approximately 0.2, 0.3, and 0.4 for the cases corresponding to $k = 0.75$, 1, and 1.25, respectively. This result suggests that the bias of the estimated probability of Type-I error obtained from the naive test statistic is higher when the sampling variance is higher. Moreover, the naive test statistic gives different conclusions from the actual test statistic. In contrast, our proposed test provides the same conclusion as the true test even with the large values of sampling variances.

Finally, we investigate the performance of the proposed test regarding the estimation

Table 3: The empirical percentiles of the different test statistics for $k = 1.25$

Length (T)	Statistics	Percentiles						
		1	10	25	50	75	90	99
$T = 25$	$\hat{\tau}_{true}$	-2.58	-1.62	-1.06	-0.47	0.23	0.86	2.17
	$\hat{\tau}_{naive}$	-4.09	-2.45	-1.71	-1.00	-0.35	0.25	1.24
	$\hat{\tau}_{Adj}$	-3.58	-1.76	-1.05	-0.44	0.34	1.33	4.53
$T = 50$	$\hat{\tau}_{true}$	-2.64	-1.61	-1.10	-0.49	0.18	0.87	1.89
	$\hat{\tau}_{naive}$	-4.05	-2.50	-1.81	-1.08	-0.44	0.10	0.93
	$\hat{\tau}_{Adj}$	-3.14	-1.68	-1.06	-0.44	0.31	1.14	3.16
$T = 100$	$\hat{\tau}_{true}$	-2.59	-1.59	-1.08	-0.50	0.20	0.87	2.00
	$\hat{\tau}_{naive}$	-3.84	-2.51	-1.76	-1.06	-0.42	0.16	0.91
	$\hat{\tau}_{Adj}$	-2.65	-1.59	-1.07	-0.46	0.20	1.03	2.34
$T = 250$	$\hat{\tau}_{true}$	-2.62	-1.59	-1.10	-0.50	0.22	0.90	1.98
	$\hat{\tau}_{naive}$	-3.81	-2.50	-1.80	-1.07	-0.37	0.19	0.99
	$\hat{\tau}_{Adj}$	-2.59	-1.61	-1.09	-0.49	0.22	0.96	2.21

Table 4: The empirical estimates of Type-I error

	Values of the ratio k								
	$k = 0.75$			$k = 1$			$k = 1.25$		
	$\hat{\tau}_{true}$	$\hat{\tau}_{naive}$	$\hat{\tau}_{Adj}$	$\hat{\tau}_{true}$	$\hat{\tau}_{naive}$	$\hat{\tau}_{Adj}$	$\hat{\tau}_{true}$	$\hat{\tau}_{naive}$	$\hat{\tau}_{Adj}$
State 1	0.0490	0.2090	0.0495	0.0450	0.3092	0.0485	0.0422	0.4078	0.0492
State 2	0.0450	0.1955	0.0470	0.0445	0.2895	0.0410	0.0511	0.4099	0.0656
State 3	0.0480	0.2010	0.0465	0.0485	0.3210	0.0535	0.0532	0.4104	0.0572
State 4	0.0475	0.1955	0.0455	0.0550	0.2915	0.0565	0.0473	0.4031	0.0488

of the power of the test for different values of the autocorrelation coefficient ρ, varying in the set $\{0.85, 0.9, 0.95, 0.975, 0.99, 0.995\}$. The simulation setting in this post is the same as previous algorithm except in the step 2(b), instead of using the data with a unit root, the time series $\{Y_t^{(l)} : t = 1, 2, \ldots, 250\}$, is generated from model (1) with specific $\rho = \rho_0$, where $\rho_0 \in \{0.85, 0.9, 0.95, 0.975, 0.99, 0.995\}$. The numerical results of the estimated power functions of the true test statistic $\hat{\tau}_{true}$ and the proposed test statistic $\hat{\tau}_{Adj}$ for $k = 0.75, 1, 1.25$ are presented in Figures 1-3, respectively.

From Figures 1-3, we can see that the estimated powers of the two tests are lower when the true value of ρ gets closer to one. The powers of the proposed test are close to the powers of the true test. These results suggest that the proposed test performs well in terms of the power of the test.

5. Applications

In this section, we apply the proposed test statistic to the CPS survey data of the four selected states, comparing with the naive test statistic ignoring sampling errors. Numerical results including the test statistics with their associated probabilities of Type-I errors are presented in Table 5.

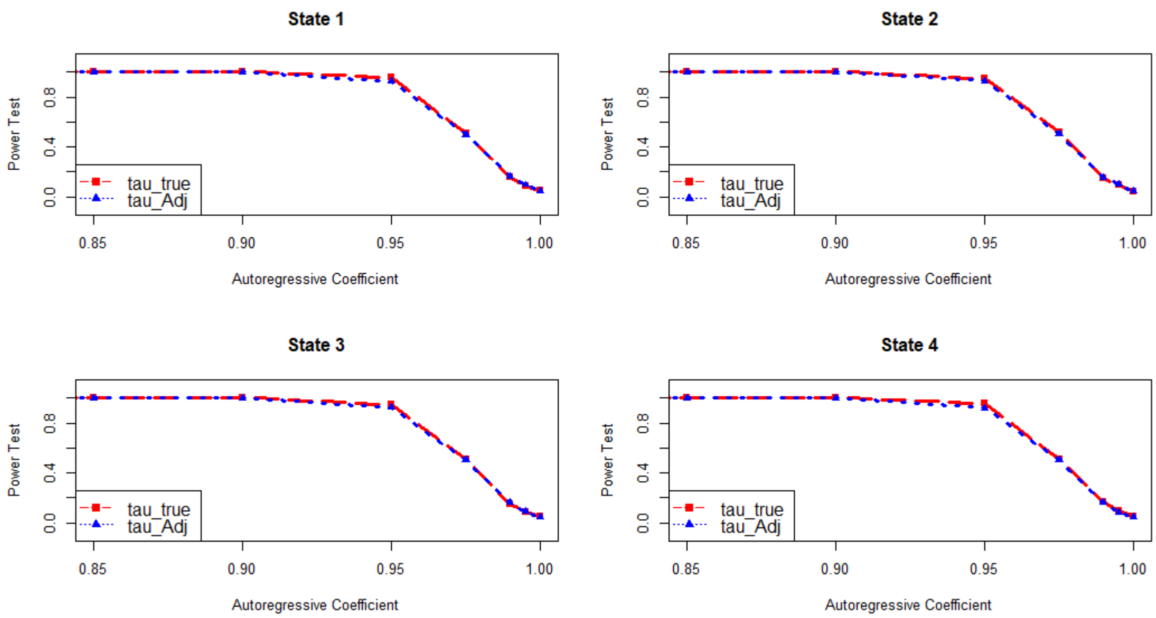

Figure 1: Empirical estimates of the power for $k = 0.75$

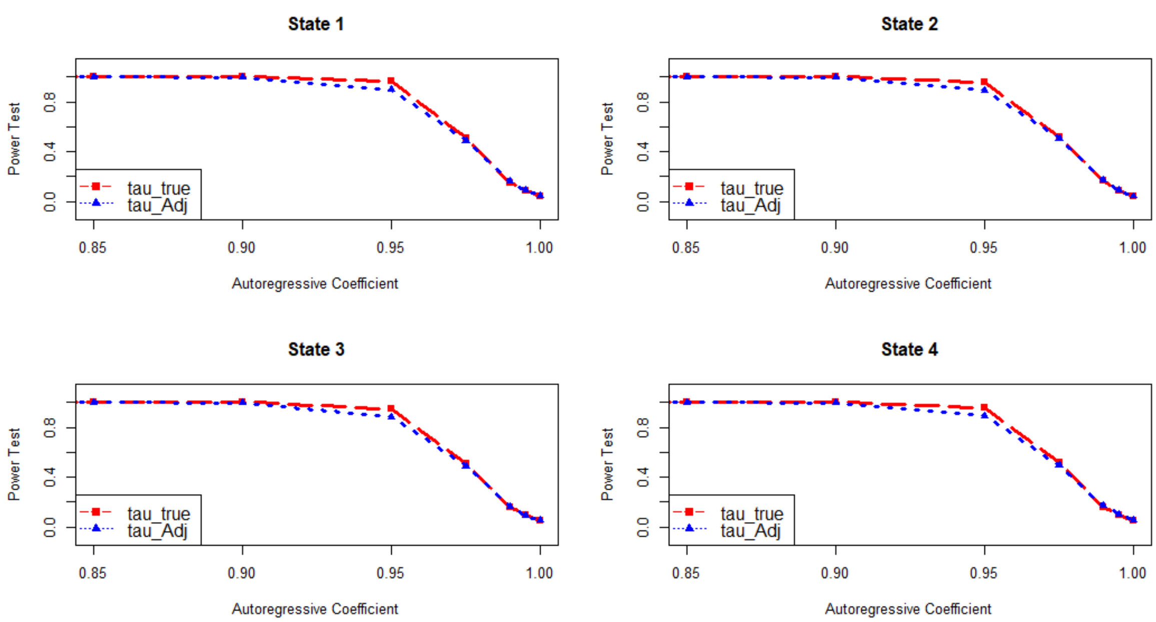

Figure 2: Empirical estimates of the power for $k = 1$

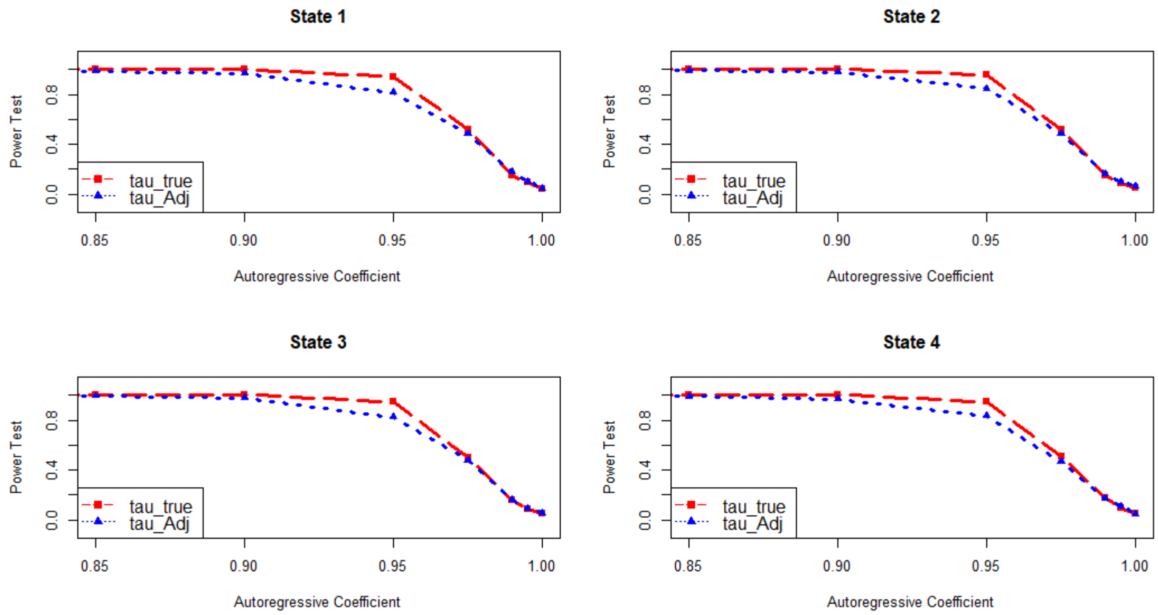

Figure 3: Empirical estimates of the power for $k = 1.25$

Table 5: The estimated test statistics and the corresponding p-values for four selected states

	$\hat{\tau}_{naive}$		$\hat{\tau}_{Adj}$	
	Calculated test Statistic	p-value	Calculated test Statistic	p-value
State 1	-6.59	$< 1 \times 10^{-4}$	-1.32	0.17
State 2	-4.89	$< 1 \times 10^{-4}$	-0.86	0.35
State 3	-7.90	$< 1 \times 10^{-4}$	-1.51	0.12
State 4	-4.18	$< 1 \times 10^{-4}$	-0.76	0.39

From Table 5, we observe the same behavior of the two estimates as the simulation results presented in Tables 1 – 3. In particular, the naive test provides much lower values of the test statistic than the proposed test statistics. The naive test statistics for the four states reject the null hypothesis and conclude that the time series are stationary. In contrast, the proposed test provides larger values of the p-values than 0.01 in all cases. Therefore, the proposed test suggests that the actual time series have a unit root at the significant level 0.01.

6. Conclusions and discussions

In this paper, we investigated the effects of sampling errors on the commonly used autocorrelation coefficient estimator and the well-known Dickey-Fuller unit root test statistic. We found that ignoring sampling errors could cause biases in the estimations of the correlation coefficient and the test statistic. This will lead to a wrong conclusion of the unit root test. Therefore, in our study, we introduced a new autocorrelation coefficient estimator and a unit root test statistic in order to reduce biases caused by sampling errors. Moreover, we obtained asymptotic distributions of our proposed estimator $\hat{\rho}_{Adj}$ and the proposed test statistic $\hat{\tau}_{Adj}$ and showed that the two estimators have the same asymptotic distributions as of the estimators without measurement errors. Furthermore, we conducted simulation studies and applied the proposed method to real data. Numerical results suggested that our proposed method have good performances in terms of bias reduction, the accuracies of the estimated probability of Type-I error and the estimated power of the unit root test.

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APPENDIX

A. Appendix: theoretical properties

In this section, we prove asymptotic properties of the adjusted estimators of the correlation coefficient and the unit root test statistic discussed in Section 3. We first obtain some important moment properties in Lemma 1 and then prove the three main results respectively in Theorem 1, Theorem 2, and Theorem 3.

Lemma 1: Under the assumption that $\rho = 1$, we have

1. $\mathbb{E}\left(S_{Y,T}(0)\right) = \dfrac{1}{2}T(T-1)\sigma_e^2$;
2. $\mathbb{E}\left(S_{Y,T}(1)\right) = \dfrac{1}{2}T(T-1)\sigma_e^2$;
3. $\mathrm{Var}\left(S_{Y,T}(0)\right) = \dfrac{1}{3}T(T-1)(T^2-T+1)\sigma_e^4$;
4. $\mathrm{Var}\left(S_{Y,T}(1)\right) = \dfrac{1}{3}T(T-1)(T^2-T+1)\sigma_e^4$;
5. for any positive integer k, $\mathbb{E}(S_{Y,T}^{-k}(0)) = O(T^{-2k})$; and
6. for any positive integers l and k, $\mathbb{E}(S_{Y,T}^{-k}(0)S_{Y,T}^{l}(1)) = O(T^{2(l-k)})$.

Proof:

1. Given that $Y_0 = 0$,

$$S_{Y,T}(0) = \sum_{t=1}^{T-1} \left(\sum_{j=1}^{t} e_j \right)^2$$
$$= \sum_{i=1}^{T-1} (T-i) e_i^2 + \sum_{i=2}^{T-1} \sum_{j=1}^{i-1} (T-i) e_i e_j. \quad (11)$$

By the property that $\{e_i\}_{i \geq 1}$ is a sequence of independent random variables with zero mean and variance σ_e^2,

$$\mathbb{E}\left(S_{Y,T}(0)\right) = \sum_{i=1}^{T-1} (T-i)\sigma_e^2 = \frac{1}{2}T(T-1)\sigma_e^2.$$

2. Note that

$$S_{Y,T}(1) = S_{Y,T}(0) + \sum_{t=2}^{T} e_t Y_{t-1}.$$

Since $\mathbb{E}(e_i) = 0$ and e_i and Y_{i-1} are independent, $\mathbb{E}\left(S_{Y,T}(1)\right) = \mathbb{E}\left(S_{Y,T}(0)\right)$.

3. Since $\{e_i\}_{i \geq 1}$ is a sequence of independent random variables with zero mean and variance σ_e^2, $\{e_i^2\}$ and $\{e_i e_j\}$ are uncorrelated sequences of uncorrelated random variables such that $\mathrm{Var}(e_i^2) = 2\sigma_e^4$ and $\mathrm{Var}(e_i e_j) = \sigma_e^4$ for $i \neq j$. From (11),

$$\mathrm{Var}\left(S_{Y,T}(0)\right) = \sum_{i=1}^{T-1} (T-i)^2 \,\mathrm{Var}\left(e_i^2\right) + \sum_{i=2}^{T-1} \sum_{j=1}^{i-1} (T-i)^2 \,\mathrm{Var}\left(e_i e_j\right)$$
$$= T(T-1)(T^2 - T + 1)\sigma_e^4.$$

4. Note that

$$\mathrm{Var}\left(\sum_{t=2}^{T} e_t Y_{t-1} \right) = \sum_{t=2}^{T} \mathrm{Var}(e_t Y_{t-1}) + 2 \sum_{2 \leq i < j \leq T} \mathrm{Cov}(e_i Y_{i-1}, e_j Y_{j-1})$$
$$= \frac{1}{2} T(T-1)\sigma_e^4,$$

and

$$\mathrm{Cov}\left(S_{Y,T}(0), \sum_{t=2}^{T} e_t Y_{t-1} \right) = \mathrm{Cov}\left(\sum_{t=2}^{T} Y_{t-1}^2, \sum_{t=2}^{T} e_t Y_{t-1} \right)$$
$$= \sum_{t=2}^{T} \mathrm{Cov}\left(Y_{t-1}^2, e_t Y_{t-1}\right) + \sum_{t=2}^{T} \sum_{s=2}^{t-1} \mathrm{Cov}\left(Y_{t-1}^2, e_s Y_{s-1}\right)$$
$$+ \sum_{t=2}^{T} \sum_{s=t+1}^{T} \mathrm{Cov}\left(Y_{t-1}^2, e_s Y_{s-1}\right)$$

$$= \frac{1}{3}T(T-1)(T-2)\sigma_e^4.$$

Then,

$$\begin{aligned}
\mathrm{Var}\left(S_{Y,T}(1)\right) &= \mathrm{Var}\left(S_{Y,T}(0)\right) + \mathrm{Var}\left(\sum_{t=2}^{T} e_t Y_{t-1}\right) + 2\,\mathrm{Cov}\left(S_{Y,T}(0), \sum_{t=2}^{T} e_t Y_{t-1}\right) \\
&= \frac{1}{3}T(T-1)(T^2-T+1)\sigma_e^4 + \frac{1}{2}T(T-1)\sigma_e^4 + \frac{2}{3}T(T-1)(T-2)\sigma_e^4 \\
&= \frac{1}{6}T(T-1)(2T^2+2T-3)\sigma_e^4.
\end{aligned}$$

5. To find the order of $\mathbb{E}(S_{Y,T}(0)^{-k})$, we apply the second order Taylor approximation to the function $f(x) = x^{-k}$ about $\mu = \mathbb{E}(S_{Y,T}(0))$ as follows.

$$\begin{aligned}
\mathbb{E}(S_{Y,T}(0)^{-k}) &= \frac{1}{\mathbb{E}^k(S_{Y,T}(0))} + \frac{k(k+1)}{2}\frac{\mathrm{Var}(S_{Y,T}(0))}{\mathbb{E}^{k+2}(S_{Y,T}(0))} + O(T^{-2k}) \\
&= O(T^{-2k}) + O(T^{-2(k+2)})O(T^4) + O(T^{-2k}) \\
&= O(T^{-2k}).
\end{aligned}$$

6. Similarly, we apply the second order Taylor approximation to the function $f(x,y) = y^{-k}x^l$ about $\mu = (\mathbb{E}(S_{Y,T}(1)), \mathbb{E}(S_{Y,T}(0)))$ to find the order of $\mathbb{E}(S_{Y,T}(0)^{-k}S_{Y,T}(1)^l)$ as follows.

$$\begin{aligned}
\left|\mathbb{E}\left(\frac{S_{Y,T}(1)^l}{S_{Y,T}(0)^k}\right)\right| &\le \left|\frac{\mathbb{E}^l(S_{Y,T}(1))}{\mathbb{E}^k(S_{Y,T}(0))}\right| + \left|\frac{l(l-1)}{2}\frac{\mathbb{E}^{l-2}(S_{Y,T}(1))}{\mathbb{E}^k(S_{Y,T}(0))}\mathrm{Var}(S_{Y,T}(1))\right| \\
&\quad + \left|\frac{k(k+1)}{2}\frac{\mathbb{E}^l(S_{Y,T}(1))}{\mathbb{E}^{k+2}(S_{Y,T}(0))}\mathrm{Var}(S_{Y,T}(0))\right| \\
&\quad + \left|2kl\frac{\mathbb{E}^{l-1}(S_{Y,T}(1))}{\mathbb{E}^{k+1}(S_{Y,T}(0))}\mathrm{Cov}(S_{Y,T}(1), S_{Y,T}(0))\right| + O(T^{-2(l-k)}) \\
&\le O(T^{2(l-k)}) + O(T^{2(l-k)}) + O(T^{2(l-k)}) + O(T^{2(l-k)}) + O(T^{-2(l-k)}) \\
&= O(T^{2(l-k)}).
\end{aligned}$$

□

Theorem 1: Under the assumption that $\rho = 1$,

$$\hat{\rho}_{Adj} - \hat{\rho}_Y = o_p(1) \qquad \text{as } T \text{ goes to infinity.}$$

Moreover,

$$\hat{\rho}_{Adj} - \rho = o_p(1) \qquad \text{as } T \text{ goes to infinity.}$$

Proof: To prove the theorem, we will show that $\mathbb{E}(\hat{\rho}_{Adj} - \hat{\rho}_Y)^2 = O(T^{-2})$ by proving the following statements:

(1) $\mathbb{E}(\hat{\rho}_{Adj} - \hat{\rho}_Y) = O(T^{-2})$,

(2) $\text{Var}(\hat{\rho}_{Adj} - \hat{\rho}_Y) = O(T^{-2})$.

To prove (1), apply the second order Taylor series expansion to the function $f(x,y) = \dfrac{x}{y}$ around $(S_{Y,T}(1), S_{Y,T}(0))$ as follows.

$$\begin{aligned}\hat{\rho}_{Adj} - \hat{\rho}_Y =& \frac{1}{S_{Y,T}(0)}(S_{W,T}(1) - S_{Y,T}(1)) - \frac{S_{Y,T}(1)}{S^2_{Y,T}(0)}(\widetilde{S}_{W,T}(0) - S_{Y,T}(0)) \\ &+ \frac{S_{Y,T}(1)}{S^3_{Y,T}(0)}(\widetilde{S}_{W,T}(0) - S_{Y,T}(0))^2 \\ &- \frac{1}{S^2_{Y,T}(0)}(S_{W,T}(1) - S_{Y,T}(1))(\widetilde{S}_{W,T}(0) - S_{Y,T}(0)) + O_p(T^{-2}).\end{aligned}$$

Then, apply the conditional expectation given $\boldsymbol{Y}$, we have

$$\begin{aligned}\mathbb{E}\left(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y}\right) &= \frac{S_{Y,T}(1)}{S^3_{Y,T}(0)}\text{Var}\left(\widetilde{S}_{W,T}(0)\middle|\boldsymbol{Y}\right) - \frac{1}{S^2_{Y,T}(0)}\text{Cov}\left(S_{W,T}(1), \widetilde{S}_{W,T}(0)\middle|\boldsymbol{Y}\right) \\ &= \frac{S_{Y,T}(1)}{S^3_{Y,T}(0)}\left(2\sum_{t=2}^{T}\sigma^4_{u,t-1} + 4\sum_{t=2}^{T}Y^2_{t-1}\sigma^2_{u,t-1}\right) \\ &\quad - \frac{2}{S^2_{Y,T}(0)}\sum_{t=2}^{T}\left(Y_tY_{t-1} + Y_{t-1}Y_{t-2}\right)\sigma^2_{u,t-1} + O_p(T^{-2}).\end{aligned}$$

Let $\sigma^2_u = \max\limits_{1\le t\le T}\sigma^2_{u,t}$. We can show that

$$|\mathbb{E}\left(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y}\right)| \le \frac{2|S_{Y,T}(1)|}{S^3_{Y,T}(0)}T\sigma^4_u + \frac{4|S_{Y,T}(1)|}{S^2_{Y,T}(0)}\sigma^2_u + \frac{5}{S_{Y,T}(0)}\sigma^2_u + O_p(T^{-2}). \tag{12}$$

From Lemma 1, we can show that

$$\begin{aligned}\mathbb{E}\left(\frac{|S_{Y,T}(1)|}{S^3_{Y,T}(0)}\right) &= O(T^{-4}), \\ \mathbb{E}\left(\frac{|S_{Y,T}(1)|}{S^2_{Y,T}(0)}\right) &= O(T^{-2}), \\ \mathbb{E}\left(\frac{1}{S_{Y,T}(0)}\right) &= O(T^{-2}).\end{aligned}$$

Therefore, $|\mathbb{E}(\hat{\rho}_{Adj} - \hat{\rho}_Y)| = O(T^{-2})$.

To prove (2), we note that

$$\begin{aligned}\text{Var}(\hat{\rho}_{Adj} - \hat{\rho}_Y) &= \mathbb{E}\left(\text{Var}(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y})\right) + \text{Var}\left(\mathbb{E}(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y})\right) \\ &\le \mathbb{E}\left(\text{Var}(\hat{\rho}_{Adj}|\boldsymbol{Y})\right) + \mathbb{E}\left(\mathbb{E}^2(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y})\right).\end{aligned} \tag{13}$$

To bound the first term of (13), we apply the first order Taylor approximation to the function $f(x,y) = \dfrac{x}{y}$ around the point $(S_{Y,T}(1), S_{Y,T}(0))$ as follows.

$$\frac{S_{W,T}(1)}{\widetilde{S}_{W,T}(0)} = \frac{S_{Y,T}(1)}{S_{Y,T}(0)} + \frac{1}{S_{Y,T}(0)}(S_{W,T}(1) - S_{Y,T}(1)) - \frac{S_{Y,T}(1)}{S^2_{Y,T}(0)}(\widetilde{S}_{W,T}(0) - S_{Y,T}(0)) + O_p(T^{-2}).$$

Therefore,

$$\begin{aligned}
\text{Var}\left(\frac{S_{W,T}(1)}{\widetilde{S}_{W,T}(0)}\middle|\boldsymbol{Y}\right) &= \frac{1}{S^2_{Y,T}(0)}\text{Var}\left(S_{W,T}(1)|\boldsymbol{Y}\right) + \frac{S^2_{Y,T}(1)}{S^4_{Y,T}(0)}\text{Var}\left(\widetilde{S}_{W,T}(0)\middle|\boldsymbol{Y}\right) \\
&\quad - \frac{2S_{Y,T}(1)}{S^3_{Y,T}(0)}\text{Cov}\left(S_{W,T}(1), \widetilde{S}_{W,T}(0)\middle|\boldsymbol{Y}\right) + O(T^{-2}) \\
&:= A_1 + A_2 + A_3 + O_p(T^{-2}).
\end{aligned}$$

To bound $\mathbb{E}(A_1)$, we notice that

$$\begin{aligned}
\text{Var}\left(S_{W,T}(1)|\boldsymbol{Y}\right) &= \sum_{t=2}^{T}(Y_t^2\sigma^2_{u,t-1} + Y^2_{t-1}\sigma^2_{u,t} + \sigma^2_{u,t}\sigma^2_{u,t-1} + 2Y_tY_{t-2}\sigma^2_{u,t-1}) \\
&\leq \sum_{t=2}^{T}(2Y_t^2 + Y^2_{t-1} + Y^2_{t-2})\sigma^2_u + T\sigma^4_u \\
&\leq 6S_{Y,T}(0)\sigma^2_u + T\sigma^4_u.
\end{aligned}$$

From Lemma 1, we have $\mathbb{E}(A_1) = \mathbb{E}\left(\dfrac{6\sigma^2_u}{S_{Y,T}(0)} + \dfrac{T\sigma^4_u}{S^2_{Y,T}(0)}\right) = O(T^{-2})$.

For the term A_2, we have

$$\text{Var}\left(\widetilde{S}_{W,T}(0)\middle|\boldsymbol{Y}\right) = 2\sum_{t=2}^{T}\sigma^4_{u,t-1} + 4\sum_{t=2}^{T}Y^2_{t-1}\sigma^2_{u,t-1} \leq 2T\sigma^4_u + 4\sigma^2_uS_{Y,T}(0).$$

From Lemma 1, $\mathbb{E}(A_2) = \mathbb{E}\left(\dfrac{2S^2_{Y,T}(1)}{S^4_{Y,T}(0)}T\sigma^4_u + \dfrac{4S^2_{Y,T}(1)}{S^3_{Y,T}(0)}\sigma^2_u\right) = O(T^{-2})$. For the last term A_3, we notice that

$$\text{Cov}\left(S_{W,T}(1), \widetilde{S}_{W,T}(0)\middle|\boldsymbol{Y}\right) = 2\sum_{t=2}^{T}(Y_tY_{t-1} + Y_{t-1}Y_{t-2})\sigma^2_{u,t-1} \leq 10\sigma^2_uS_{Y,T}(0).$$

Hence, $\mathbb{E}(A_3) = \mathbb{E}\left(\dfrac{20\sigma^2_uS_{Y,T}(1)}{S^2_{Y,T}(0)}\right) = O(T^{-2})$. This implies that $\mathbb{E}\left(\text{Var}(\hat{\rho}_{Adj}|\boldsymbol{Y})\right) = O(T^{-2})$.

To consider $\mathbb{E}\left(\mathbb{E}^2(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y})\right)$, we apply (12) and Cauchy-Schwartz inequality to obtain

$$\begin{aligned}
\mathbb{E}^2\left(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y}\right) &\leq 3\left(\frac{2|S_{Y,T}(1)|}{S^3_{Y,T}(0)}T\sigma^4_u\right)^2 + 3\left(\frac{4|S_{Y,T}(1)|}{S^2_{Y,T}(0)}\sigma^2_u\right)^2 + 3\left(\frac{5}{S_{Y,T}(0)}\sigma^2_u\right)^2 \\
&= \frac{12S^2_{Y,T}(1)}{S^6_{Y,T}(0)}T^2\sigma^8_u + \frac{48S^2_{Y,T}(1)}{S^4_{Y,T}(0)}\sigma^4_u + \frac{75}{S^2_{Y,T}(0)}\sigma^4_u.
\end{aligned}$$

From Lemma 1, $\mathbb{E}\left(\mathbb{E}^2(\hat{\rho}_{Adj} - \hat{\rho}_Y|\boldsymbol{Y})\right) = O(T^{-4})$. Hence, from (13), $\text{Var}(\hat{\rho}_{Adj} - \hat{\rho}_Y) = O(T^{-2})$.

From (1) and (2), we have $\mathbb{E}(\hat{\rho}_{Adj} - \hat{\rho}_Y)^2 = O(T^{-2})$. Therefore, $\hat{\rho}_{Adj} - \hat{\rho}_Y = o_p(1)$ as T goes to infinity. Moreover, since $\hat{\rho}_Y - \rho = o_p(1)$, we have $\hat{\rho}_{Adj} - \rho = o_p(1)$ as T goes to infinity. □

Having proved the asymptotic property of $\hat{\rho}_{Adj}$, we will prove the asymptotic distribution of the test statistics $\hat{\tau}_{Adj}$ by first obtaining some important lemmas as follows.

Lemma 2: Under the assumption that $\rho = 1$,

$$\frac{1}{T^2}\widetilde{S}_{W,T}(0) - \sum_{i=1}^{\infty} \gamma_i^2 Z_i^{*2} = o_p(1)$$

as T goes to infinity, where $\gamma_i = (-1)^{i+1}\dfrac{2}{(2i-1)\pi}$ and $Z_i^* \overset{iid}{\sim} N(0, \sigma_e^2)$.

Proof: We know from Dickey (1976) that

$$\frac{1}{T^2}S_{Y,T}(0) - \sum_{i=1}^{\infty} \gamma_i^2 Z_i^{*2} = o_p(1),$$

as T goes to infinity. To prove this lemma, we will show that

$$\frac{1}{T^2}\widetilde{S}_{W,T}(0) - \frac{1}{T^2}S_{Y,T}(0) = o_p(1) \tag{14}$$

as T goes to infinity.
First, we notice that

$$\frac{\widetilde{S}_{W,T}(0)}{T^2} - \frac{S_{Y,T}(0)}{T^2} = \frac{1}{T^2}\sum_{t=2}^{T} 2Y_{t-1}u_{t-1} + \frac{1}{T^2}\sum_{t=2}^{T}(u_{t-1}^2 - \sigma_{u,t-1}^2).$$

Since $\mathbb{E}(Y_t u_t)$ and $\mathbb{E}\left(u_t^2 - \sigma_{u,t}^2\right)$ are equal to zero for all t,

$$\mathbb{E}\left(\frac{\widetilde{S}_{W,T}(0)}{T^2} - \frac{S_{Y,T}(0)}{T^2}\right) = \frac{1}{T^2}\sum_{t=2}^{T} 2\,\mathbb{E}\left(Y_{t-1}u_{t-1}\right) + \frac{1}{T^2}\sum_{t=2}^{T}\mathbb{E}(u_{t-1}^2 - \sigma_{u,t-1}^2) = 0. \tag{15}$$

Since $\{Y_t u_t\}_{1\le t\le T}$ and $\{u_t^2 - \sigma_{u,t}^2\}_{1\le t\le T}$ are uncorrelated random sequences,

$$\begin{aligned}\text{Var}\left(\frac{\widetilde{S}_{W,T}(0)}{T^2} - \frac{S_{Y,T}(0)}{T^2}\right) &= \frac{1}{T^4}\sum_{t=2}^{T} 4\,\text{Var}(Y_{t-1}u_{t-1}) + \frac{1}{T^4}\sum_{t=2}^{T}\text{Var}(u_{t-1}^2 - \sigma_{u,t-1}^2) \\ &\le \frac{1}{T^4}\sigma_e^2\sigma_u^2 \cdot \frac{1}{2}T(T-1) + \frac{2}{T^4}T\sigma_u^4 \qquad (16) \\ &= O(T^{-2}).\end{aligned}$$

Hence, from (15) and (16), (14) is proved. Consequently,

$$\frac{1}{T^2}\widetilde{S}_{W,T}(0) - \sum_{i=1}^{\infty} \gamma_i^2 Z_i^{*2} = o_p(1),$$

as T goes to infinity. □

Theorem 2: Under the assumption that $\rho = 1$, the statistics $T(\hat{\rho}_{adj} - 1)$ has the same limiting distribution as $T(\hat{\rho}_Y - 1)$ as T goes to infinity. In a particular,

$$T(\hat{\rho}_{Adj} - 1) \xrightarrow{d} \frac{\left(\sum_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sqrt{\sum_{i=1}^{\infty} \gamma_i^2 Z_i^2}},$$

where $\gamma_i = (-1)^{i+1}\frac{2}{(2i-1)\pi}$ and $Z_i \overset{iid}{\sim} N(0,1)$.

Proof: From the definition of $\hat{\rho}_{Adj}$, $T(\hat{\rho}_{Adj} - 1)$ can be simplified as

$$T(\hat{\rho}_{Adj} - 1) = T\left(\frac{S_{W,T}(1) - \widetilde{S}_{W,T}(0)}{\widetilde{S}_{W,T}(0)}\right) = \left(\frac{1}{T^2}\widetilde{S}_{W,T}(0)\right)^{-1}\left(\frac{1}{T}\left(S_{W,T}(1) - \widetilde{S}_{W,T}(0)\right)\right). \quad (17)$$

From (1) and (5), we have

$$\begin{aligned}
\frac{1}{T}\left(S_{W,T}(1) - \widetilde{S}_{W,T}(0)\right) &= \frac{1}{T}\sum_{t=2}^{T}\left((Y_{t-1} + u_{t-1})(Y_t + u_t - Y_{t-1} - u_{t-1}) + \sigma^2_{u,t-1}\right) \\
&= \frac{1}{T}\sum_{t=2}^{T}\left((Y_{t-1} + u_{t-1})(e_t + u_t - u_{t-1}) + \sigma^2_{u,t-1}\right) \\
&= \frac{1}{T}\sum_{t=2}^{T} Y_{t-1}e_t + \frac{1}{T}\sum_{t=1}^{T-1} e_t u_T - \frac{Y_1 u_1}{T} - \frac{1}{T}\sum_{t=2}^{T-1} e_t u_{t-1} \\
&\quad + \frac{1}{T}\sum_{t=2}^{T} e_t u_{t-1} + \frac{1}{T}\sum_{t=2}^{T} u_t u_{t-1} - \frac{1}{T}\sum_{t=2}^{T}(u^2_{t-1} - \sigma^2_{u,t-1}). \quad (18)
\end{aligned}$$

Notice that each of the terms in (18) except $\frac{1}{T}\sum_{t=2}^{T} Y_{t-1}e_t$ is a sum of uncorrelated random variables with zero means and finite variances. Therefore, by the law of large number, each of those terms converges in probability to zero.

Following the results of Fuller (1976) that

$$\frac{1}{T}\sum_{t=2}^{T} Y_{t-1}e_t \xrightarrow{d} \frac{1}{2}\left(\sum_{i=1}^{\infty}\sqrt{2}\gamma_i Z_i^*\right)^2 - \frac{\sigma_e^2}{2},$$

where $\gamma_i = (-1)^{i+1}\dfrac{2}{(2i-1)\pi}$ and $Z_i^* \overset{iid}{\sim} N(0, \sigma_e^2)$, we can show that

$$\frac{1}{T}\left(S_{W,T}(1) - \widetilde{S}_{W,T}(0)\right) \xrightarrow{d} \frac{1}{2}\left(\sum_{i=1}^{\infty}\sqrt{2}\gamma_i Z_i^*\right)^2 - \frac{\sigma_e^2}{2}. \quad (19)$$

From Lemma 2, (17), and (19),

$$T(\hat{\rho}_{Adj} - 1) \xrightarrow{d} \frac{\left(\sum_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sqrt{\sum_{i=1}^{\infty} \gamma_i^2 Z_i^2}}, \quad (20)$$

where $\gamma_i = (-1)^{i+1}\dfrac{2}{(2i-1)\pi}$ and $Z_i \overset{iid}{\sim} N(0,1)$. □

Lemma 3: Define the statistic $\hat{\sigma}^2_{Adj,e}$ as

$$\hat{\sigma}^2_{Adj,e} = |\hat{\sigma}^2_{W,e,1} - \hat{\sigma}^2_{W,e,2}|,$$

where

$$\hat{\sigma}^2_{W,e,1} = \frac{1}{T-2}\sum_{t=2}^{T}(W_t - \hat{\rho}_{Adj}W_{t-1})^2,$$

and

$$\hat{\sigma}^2_{W,e,2} = \frac{1}{T-2}\sum_{t=2}^{T}\left(\sigma^2_{u,t} + \hat{\rho}^2_{Adj}\sigma^2_{u,t-1}\right).$$

Then, under the assumption that $\rho = 1$,

$$\hat{\sigma}^2_{Adj,e} - \hat{\sigma}^2_e = o_p(1).$$

In particular, $\hat{\sigma}^2_{Adj,e} - \sigma^2_e = o_p(1)$.

Proof: Notice that

$$\begin{aligned}
(T-2)\hat{\sigma}^2_{W,e,1} &= \sum_{t=2}^{T}(Y_t - \hat{\rho}_Y Y_{t-1} + (\hat{\rho}_Y - \hat{\rho}_{Adj})Y_{t-1} + u_t - \hat{\rho}_{Adj}u_{t-1})^2 \\
&= (T-2)\hat{\sigma}^2_e + (\hat{\rho}_Y - \hat{\rho}_{Adj})^2 S_{Y,T}(0) + \sum_{t=2}^{T}(u_t - \hat{\rho}_{Adj}u_{t-1})^2 \\
&\quad + 2(\hat{\rho}_Y - \hat{\rho}_{Adj})\sum_{t=2}^{T}Y_{t-1}(u_t - \hat{\rho}_{Adj}u_{t-1}) + 2\sum_{t=2}^{T}(Y_t - \hat{\rho}_Y Y_{t-1})(u_t - \hat{\rho}_{Adj}u_{t-1}) \\
&= (T-2)\hat{\sigma}^2_e + (\hat{\rho}_Y - \hat{\rho}_{Adj})^2 S_{Y,T}(0) + \sum_{t=2}^{T}(u_t - \hat{\rho}_{Adj}u_{t-1})^2 \\
&\quad + 2\sum_{t=2}^{T}\left(e_t + (\rho - \hat{\rho}_{Adj})Y_{t-1}\right)(u_t - \hat{\rho}_{Adj}u_{t-1}).
\end{aligned}$$

Then,

$$\begin{aligned}
(T-2)(\hat{\sigma}^2_{W,e,1} - \hat{\sigma}^2_{W,e,2} - \hat{\sigma}^2_e) &= (\hat{\rho}_Y - \hat{\rho}_{Adj})^2 S_{Y,T}(0) + \sum_{t=2}^{T}(u_t - \hat{\rho}_{Adj}u_{t-1})^2 \\
&\quad + 2\sum_{t=2}^{T}\left(e_t + (\rho - \hat{\rho}_{Adj})Y_{t-1}\right)(u_t - \hat{\rho}_{Adj}u_{t-1}) \\
&\quad - \sum_{t=2}^{T}\left(\sigma^2_{u,t} + \hat{\rho}^2_{Adj}\sigma^2_{u,t-1}\right) \\
&= (\hat{\rho}_Y - \hat{\rho}_{Adj})^2 S_{Y,T}(0) + \sum_{t=2}^{T}(u_t^2 - \sigma^2_{u,t}) + \hat{\rho}^2_{Adj}\sum_{t=2}^{T}(u^2_{t-1} - \sigma^2_{u,t-1}) \\
&\quad - 2\hat{\rho}_{Adj}\sum_{t=2}^{T}u_t u_{t-1} + 2\sum_{t=2}^{T}e_t u_t + 2(\rho_Y - \hat{\rho}_{Adj})\sum_{t=2}^{T}Y_{t-1}u_t
\end{aligned}$$

$$
\begin{aligned}
&- 2\hat{\rho}_{Adj} \sum_{t=2}^{T} e_t u_{t-1} - 2\hat{\rho}_{Adj}(\rho_Y - \hat{\rho}_{Adj}) \sum_{t=2}^{T} Y_{t-1} u_{t-1} \\
&= o_p(T),
\end{aligned}
$$

where we use Theorem 1, Lemma 2, and the weak law of large number to obtain the last equation. Therefore, $\hat{\sigma}^2_{W,e,1} - \hat{\sigma}^2_{W,e,2} - \hat{\sigma}^2_e = o_p(1)$. Consequently, $\hat{\sigma}^2_{Adj,e,1} - \sigma^2_e = o_p(1)$. □

Applying Lemma 2 - Lemma 3, we obtain the asymptotic distribution of the proposed statistic $\hat{\tau}_{Adj}$ in the following theorem.

Theorem 3: Let $\hat{\tau}_{Adj}$ be a statistic defined by

$$
\hat{\tau}_{Adj} = \frac{(\hat{\rho}_{Adj} - 1)\sqrt{\widetilde{S}_{W,T}(0)}}{\sqrt{\hat{\sigma}^2_{Adj,e}}}.
$$

Then $\hat{\tau}_{Adj}$ has the same asymptotic distribution as $\hat{\tau}$ in (4). That is

$$
\hat{\tau}_{Adj} \xrightarrow{d} \frac{\left(\sum\limits_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sqrt{\sum\limits_{i=1}^{\infty} \gamma_i^2 Z_i^2}},
$$

where $\gamma_i = (-1)^{i+1} \dfrac{2}{(2i-1)\pi}$ and $Z_i \overset{iid}{\sim} N(0,1)$.

Proof: From Lemma 2 and Lemma 3, we have

$$
\frac{1}{T^2}\widetilde{S}_{W,T}(0) \cdot \frac{1}{\hat{\sigma}^2_{Adj,e}} \xrightarrow{p} \sum_{i=1}^{\infty} \gamma_i^2 \frac{Z_i^{*2}}{\sigma_e^2},
$$

where $\gamma_i = (-1)^{i+1} \dfrac{2}{(2i-1)\pi}$ and $Z_i^* \overset{iid}{\sim} N(0, \sigma_e^2)$.

Then,

$$
\sqrt{\frac{1}{T^2}\widetilde{S}_{W,T}(0) \cdot \frac{1}{\hat{\sigma}^2_{Adj,e}}} \xrightarrow{p} \sqrt{\sum_{i=1}^{\infty} \gamma_i^2 Z_i^2}, \tag{21}
$$

where $Z_i \overset{iid}{\sim} N(0,1)$.

From (20) and (21), we can conclude that

$$
\hat{\tau}_{Adj} = T(\hat{\rho}_{Adj} - 1) \cdot \sqrt{\frac{1}{T^2}\widetilde{S}_{W,T}(0) \cdot \frac{1}{\hat{\sigma}^2_{Adj,e}}} \xrightarrow{d} \frac{\left(\sum\limits_{i=1}^{\infty} \sqrt{2}\gamma_i Z_i\right)^2 - 1}{2\sqrt{\sum\limits_{i=1}^{\infty} \gamma_i^2 Z_i^2}}.
$$

□