Statistics and Applications {ISSN 2454-7395 (online)}
Volume 19, No. 1, 2021 (New Series), pp 467–481

Investigation into The Robustness of Balanced Incomplete Block Designs

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Received: 05 April 2021; Revised: 23 May 2021; Accepted: 25 May 2021

Abstract

A set of measures is developed which indicate the robustness of a Balanced Incomplete Block Design (BIBD) against yielding a disconnected eventual design in the event of observation loss. The measures have uses as a pilot procedure and as a tool to aid in design selection in situations in which significant observation loss is thought possible. The measures enable non-isomorphic BIBDs with the same parameters to be ranked. Investigation of a class of BIBDs suggests there is some correspondence between robustness against becoming disconnected and rankings associated with A-efficiency.

Key words: Connected; Efficiency; Observation loss; Optimality.

AMS Subject Classifications: 62D10, 62K05, 62K10

1. Introduction

Consider D, a binary connected incomplete block design. During experimentation, some observations may be lost and the properties of the eventual design, D_e, will be different from those of D. The eventual design may be far less efficient than the original design. In an extreme situation, D_e may be disconnected, resulting in serious damage to the aims of the experiment. When selecting a design for experimentation, it is prudent to assess the potential for observation loss to result in a disconnected design.

The universal optimality properties of BIBDs make such designs appealing when available within the practical constraints of an experiment. Where it is non-empty, the class of non-isomorphic BIBDs with v treatments arranged in b blocks of size k is denoted $\mathcal{D}(v, b, k)$. See Mathon and Rosa (1996) for sets of relatively small non-isomorphic designs. All designs in a $\mathcal{D}(v, b, k)$ have treatments replicated $r = bk/v$ times and each pair of treatments occurs together in $\lambda = r(k-1)/(v-1)$ blocks. Designs in $\mathcal{D}(v, b, k)$ are usually considered as having equal merit. In particular, optimality criteria cannot be used to distinguish between designs in a class. However, in the event of observation loss from a BIBD, the property of balance is destroyed and the A-efficiency of the eventual design may be small.

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Many authors investigate observation loss in binary connected incomplete block designs, which are not necessarily balanced. A design is said to be Criterion-1 robust against a specific pattern of observation loss, if a connected eventual design is guaranteed in the event of such observation loss. The Criterion-1 robustness of designs against the loss of t observations is investigated in Ghosh (1979), and results are given on the maximum number of blocks and the maximum number of observations that can be lost whilst ensuring a connected eventual design in Ghosh (1982) (see Kageyama (1990), for a review of other related work up to 1988). Baksalary and Tabis (1987) and Godolphin and Warren (2011) give sufficient conditions for Criterion-1 robustness against the loss of a subset of blocks. Bailey *et al.* (2013) investigate the Criterion-1 robustness of classes of universally optimal and D-optimal designs. Tsai and Liao (2013) look into the Criterion-1 robustness of designs with blocks of size two. Conditions on the number of individual observations and on the number of whole blocks that a design is Criterion-1 robust against losing, are given in terms of the E-value of the design and of the design support, in Godolphin (2016, 2019).

The A-efficiency of eventual designs following a specified pattern of observation loss, for which it is known that D_e will be connected, provides a second measure of design robustness. A design is Criterion-2 robust against a pattern of observation loss if the A-efficiency for any potential D_e is *not too small*. Dey (1993) investigates the Criterion-1 and Criterion-2 robustness of a design according to two patterns of loss: the loss of t observations on the same treatment; the loss of all observations in a single block. Lal *et al.* (2001) develop conditions for Criterion-1 robustness against the loss of any t observations, and give expressions for the A-efficiencies of eventual designs resulting from the loss of some configurations of observation pairs, any pair of blocks and for sets of disjoint blocks. Related work by Bhar (2014) advocates the advantages of the E-efficiencies of potential eventual designs as an alternative criterion to assess design robustness in the event of observation loss.

For results specific to Criterion-1 robustness of BIBDs see, for example, Ghosh (1982), where it is established that a BIBD is Criterion-1 robust against the loss of any $r-1$ observations and against the loss of any $r-1$ blocks. Key work associated with the Criterion-2 robustness of BIBDs includes Bhaumik and Whittinghill (1991), who consider the loss of complete blocks, and Whittinghill (1995) who considers the effect of losing any two observations on optimality criteria. Das and Kegeyama (1992) investigate the Criterion-2 robustness of a BIBD against observation loss in one block. Results of Lal *et al.* (2001) on observation loss in BIBDs mirror those of Whittinghill (1995), with Whittinghill's case 3 omitted. Prescott and Mansson (2001) investigate properties of eventual designs arising from the loss of observation pairs, with reference to a design in $\mathcal{D}(8, 14, 4)$. The Intersection Aberration criterion of Morgan and Parvu (2008) ranks members of $\mathcal{D}(v, b, k)$ according to efficiency properties of eventual designs arising from the loss of two blocks.

A Rank Reducing Observation Set (RROS) in D is a set of observations, the removal of which yields D_e with $\text{Rank}(X) > \text{Rank}(X_e)$, where X and X_e are the design matrices of D and D_e, respectively. In this work, the focus is on identifying the sizes and numbers of RROSs for designs in $\mathcal{D}(v, b, k)$ that are most damaging to the aims of the experiment. The work is closely aligned to the concept of Criterion-1 robustness: if D is not Criterion-1 robust against a specific pattern of observation loss, then there will be at least one set of observations corresponding to this pattern that comprise a RROS. Using an approach closely aligned to the treatment partitioning processes of Godolphin and Warren (2011), the smallest RROSs for different treatment partitions are determined. Expressions for the measure $(S_u,\, T_u)$ are developed, where S_u is the smallest number of observations in a RROS of specific type and T_u is the number of such RROSs. Observation loss is assumed to be random, that is, each observation has the same probability of being lost, independent of any other. For designs in a $\mathcal{D}(v,b,k)$ the $(S_u,\, T_u)$ measure has uses:

(i) as a pilot procedure to provide information on the robustness of a design;

(ii) to aid selection of a design from a $\mathcal{D}(v,b,k)$ having cardinality greater than one.

In §2, the different types of RROS are defined and illustrated via an example. Formulae for the $(S_u,\, T_u)$ measures are developed in §3. These are of two types: $(S_u,\, T_u)$ depending only on v,b,k,u, which are fixed for all designs in a $\mathcal{D}(v,b,k)$; $(S_u,\, T_u)$ that can vary within a $\mathcal{D}(v,b,k)$. The former can be used to assess the general robustness of designs in the class against giving rise to a disconnected D_e. The latter provide a means of design comparison and aid in design selection. In §4, results are illustrated by reference to $\mathcal{D}(8,14,4)$. The design ranking obtained by the $(S_u,\, T_u)$ measure is found to be consistent with ranking according to worst A- and E-efficiencies according to the loss of between two and five observations, and to the Intersection Aberration criterion of Morgan and Parvu (2008).

2. Preliminaries

Consider D, a planned incomplete block design, that is both binary and connected, with n observations on v treatments in b blocks of size k. The observations are assumed to be uncorrelated each with variance σ^2, and the observation vector $\mathbf{Y}$ is assumed to follow the additive model.

$$E(\mathbf{Y}) = \mu\mathbf{1_n} + X_1\boldsymbol{\tau} + X_2\boldsymbol{\beta}.$$

Here, μ is a scalar constant, $\mathbf{1_n}$ is the vector of length n with all elements unity, and $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_v)^T$ and $\boldsymbol{\beta}$ are vectors of the treatment and block effects. Matrices X_1 and X_2, of orders $n \times v$ and $n \times b$, relate to the treatment and block components of D, each row of X_i, $i = 1, 2$, having one element unity and remaining elements zero. The design has design matrix $X = (1_n\, X_1\, X_2)$ and $v \times v$ information matrix:

$$C = X_1^T X_1 - X_1^T X_2 (X_2^T X_2)^{-1} X_1.$$

Since D is connected, $\mathrm{Rank}(C) = v - 1$ and the positive eigenvalues of C are expressed as:

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{v-1}.$$

Any RROS of D can be categorised as being of Types I to III. These types are not mutually exclusive. Brief details are given below.

Type I: If observations comprising a Type I RROS are lost from D then $\mathcal{B}_e$, the set of blocks of D_e, can be partitioned into non-empty sets $\mathcal{B}_0$ and $\mathcal{B}_e \setminus \mathcal{B}_0$ with the treatments in $\mathcal{B}_0$ being distinct from those in $\mathcal{B}_e \setminus \mathcal{B}_0$.

Type II: A Type II RROS contains all observations from one or more blocks.

Type III: A Type III RROS contains all replicates of one or more treatments.

The most extreme consequence of observation loss arises if not all $v(v-1)/2$ pairwise treatment contrasts are estimable from D_e. The loss of a RROS that is of Type II only will not restrict the capacity to estimate treatment contrasts, but will affect the overall efficiency. Such RROSs are not the focus of this work. However, in the event of the loss of observations comprising a RROS that is Type I and/or Type III then $\text{Rank}(C_e) < v-1$, where C_e denotes the information matrix of D_e, and not all treatment contrasts will be estimable. A Type III RROS contains all replicates of a subset of v_0 treatments. Such RROSs are immediately evident from the treatment replications of D. In the event that a RROS that is Type III and not Type I is lost from D, then $\text{Rank}(C_e) = v - v_0 - 1$ and all contrasts in the $v - v_0$ treatments occurring in D_e will be estimable. If a Type I RROS is lost from D then no pairwise treatment contrast involving one treatment occurring in a block of $\mathcal{B}_0$ and one occurring in a block of $\mathcal{B}_e \setminus \mathcal{B}_0$ will be estimable. The available data comprise two observation sets which cannot be analysed as a single entity, although they can be analysed separately, see Searle (1971, §7.4), to gain limited information.

Whilst the aims of the experiment are seriously compromised by the loss of a Type I or a Type III RROS from D, the Type I RROSs are not easily identifiable from the planned design and these are the main focus of this work. For more extensive discussion of the types of RROS see Godolphin and Warren (2014). Type I and Type III RROSs are demonstrated in the following example.

Example: The design D has seven treatments, each with replication three, arranged in seven blocks of size three. The design is depicted below, with columns as blocks numbered 1 to 7.

$$D = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 3 & 5 & 5 & 4 & 5 \\ 7 & 4 & 6 & 7 & 7 & 6 & 6 \end{array}$$

The following six potential eventual designs, labelled $D_{e1}, \ldots, D_{e6}$, result from different configurations of observation loss, with each pattern of loss corresponding to a RROS of Type I and/or Type III.

$$D_{e1} = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & * & * & 3 & 4 \\ * & 3 & 3 & 5 & 5 & 4 & 5 \\ 7 & * & 6 & 7 & 7 & 6 & 6 \end{array} \quad D_{e2} = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 2 & 2 & * & 4 \\ 2 & * & * & 5 & 5 & 4 & 5 \\ 7 & 4 & 6 & 7 & 7 & 6 & 6 \end{array} \quad D_{e3} = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline * & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 3 & 5 & 5 & 4 & * \\ 7 & 4 & 6 & 7 & 7 & 6 & 6 \end{array}$$

$$D_{e4} = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 1 & 2 & 2 & 3 & 4 \\ * & 3 & 3 & 5 & 5 & 4 & * \\ * & 4 & 6 & 7 & 7 & 6 & 6 \end{array} \quad D_{e5} = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline * & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & * & 3 & 5 & 5 & 4 & * \\ 7 & 4 & 6 & 7 & 7 & 6 & 6 \end{array} \quad D_{e6} = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline * & 1 & 1 & 2 & 2 & 3 & 4 \\ 2 & 3 & 3 & * & * & 4 & * \\ 7 & 4 & 6 & 7 & 7 & 6 & 6 \end{array}$$

The RROSs giving rise to D_{e1} and D_{e2} are Type III only. In both cases all replicates of one treatment have been lost. The eventual designs are connected designs in six treatments.

Thus $\text{Rank}(C_{e1}) = \text{Rank}(C_{e2}) = 5$. All pairwise treatment contrasts in the six treatments of D_{e1} and D_{e2} are estimable. The RROS giving rise to D_{e2} is minimal in the sense that, if any observation of the RROS is reinstated then the resulting design is a connected design in seven treatments. The RROS of D_{e2} is not minimal: if the replicate of treatment 4 in block 2 is reinstated then the observation loss incurred still corresponds to a Type III RROS.

The RROSs giving rise to D_{e3} to D_{e5} are Type I only. All three eventual designs are disconnected with the blocks partitioned so that each block contains treatments from exactly one set of $\{1, 3, 4, 6\}$ and $\{2, 5, 7\}$. The eventual designs have $\text{Rank}(C_{e3}) = \text{Rank}(C_{e4}) = \text{Rank}(C_{e5}) = 5$. Pairwise treatment contrasts are estimable within each treatment set, but the 12 pairwise treatment contrasts involving one treatment from each set, such as $\tau_1 - \tau_2$, are not estimable. The Type I RROSs leading to D_{e3} and D_{e4}, containing two and three observations respectively, are minimal. The Type I RROS leading to D_{e3} is of particular interest, since no smaller Type I RROS exists for D. The Type I RROS leading to D_{e5} is not minimal: reinstatement of the replicate of treatment 3 in block 2 gives D_{e3}.

Finally, the RROS giving rise to D_{e6} is both Type I and Type III. The eventual design D_{e6} is disconnected with block partition such that treatments in $\{2, 7\}$ are contained in $\mathcal{B}_0$, say, and treatments in $\{1, 3, 4, 6\}$ are contained in $\mathcal{B}_e \setminus \mathcal{B}_0$. Only seven pairwise treatment contrasts are estimable. The eventual design has $\text{Rank}(C_{e6}) = 4$.

Some basic properties can be established for minimal Type I RROSs, such as those RROSs leading to D_{e3} and D_{e4} in Example 1.

Theorem 1: Consider a Type I RROS for design D such that no subset is also a RROS. Then, the eventual design D_e arising due to the loss of the RROS from D has the properties:

(i) D_e, has b blocks;

(ii) D_e, has v treatments.

Proof: By the definition of a Type I RROS, the blocks of D_e, can be partitioned into non-empty sets $\mathcal{B}_0$ and $\mathcal{B}_e \setminus \mathcal{B}_0$ with the treatments in blocks of $\mathcal{B}_0$ being distinct from those in the blocks of $\mathcal{B}_e \setminus \mathcal{B}_0$. For (i): assume D_e has fewer than b blocks. Then the RROS contains all observations from at least one block of D. Reinstate any one observation in such a block to form $D_e^{\dagger}$. The additional block of $D_e^{\dagger}$, over those of D_e, can be allocated to one of $\mathcal{B}_0$ and $\mathcal{B}_e \setminus \mathcal{B}_0$ to form a partition in $D_e^{\dagger}$. Thus the observation loss resulting in $D_e^{\dagger}$ corresponds to a Type I RROS, but the missing observations are a subset of those lost to form D_e. This provides a contradiction and it follows that D_e has b blocks, as required. For (ii), assume D_e has fewer than v treatments. Then the RROS contains all replicates of one or more treatments of D. Reinstate one observation of such a treatment to form $D_e^{\dagger\dagger}$. By (i) the reinstated observation will be in a block in either $\mathcal{B}_0$ and $\mathcal{B}_e \setminus \mathcal{B}_0$. In either case, there is partition in $D_e^{\dagger\dagger}$. As with (i), the observations lost from D to form $D_e^{\dagger\dagger}$ comprise a Type I RROS, but are a subset of those lost to form D_e. This gives a contradiction. Hence, D_e has v treatments, as required.

Such RROSs are summarised in the following definition, where $\mathcal{V}$, $\mathcal{B}$ denote the sets of treatments and blocks of D, respectively:

Definition: A RROS(u) for D, for $1 \leq u \leq v/2$, is a Type I RROS with the following properties:

(i) no subset is also a RROS;

(ii) in D_e, the treatments are partitioned into sets $\mathcal{V}_u$, $\mathcal{V}_{v-u}$, with cardinalities u and $v-u$, and the blocks are partitioned into non-empty sets $\mathcal{B}_1$, $\mathcal{B}_2$, with treatments from $\mathcal{V}_u$ arranged exclusively in blocks of $\mathcal{B}_1$ and those from $\mathcal{V}_{v-u}$ exclusively in blocks of $\mathcal{B}_2$.

The partitioning of $\mathcal{V}$ and $\mathcal{B}$ induced by the loss of a Type I RROS is termed a *consistent treatment and block partition*.

3. Robustness Measures for BIBDs

Henceforth, any planned design, will be taken as being a BIBD, that is, $D \in \mathcal{D}(v, b, k)$. It is evident that D contains v Type III RROSs of size r. The aim is to add to this information by finding RROS(u)s of smallest size for $1 \leq u \leq v/2$. From Ghosh (1982), D is Criterion-1 robust against the loss of any $r-1$ observations. Thus, a RROS(u) must be of size at least r. A RROS(u) consists of all replicates of treatments in $\mathcal{V}_{v-u}$ contained in blocks of B_1 and all replicates of treatments in $\mathcal{V}_u$ contained in blocks of B_2. The consequence of losing all observations in the RROS(u) is that in $\mathcal{B}_1$ only treatments from $\mathcal{V}_u$ are preserved and it is precisely treatments in $\mathcal{V}_u$ that are lost from $\mathcal{B}_2$. For a given $\mathcal{V}_u$, denote the smallest number of observations in a RROS(u) by s_u. Further, define S_u to be $\min_{\mathcal{V}_u}\{s_u\}$, the minimisation being over all $v!/[u!(v-u)!]$ sets of u treatments, and define T_u to be the number of RROS(u)s of size S_u. The pair (S_u, T_u) forms the robustness measure. It gives the smallest number of observations that must be lost, and the number of observation sets of this size, for the possibility of an eventual design with a consistent treatment and block partition, with the treatment sets being of sizes u and $v-u$.

Relationships associated with the distribution of subsets of treatments in $\mathcal{V}_u$ amongst the blocks of D arise as a consequence of the properties of BIBDs. These relationships are given below without proof.

Theorem 2: For given $\mathcal{V}_u$, let b_j be the number of blocks in D containing exactly j elements from $\mathcal{V}_u$, for $0 \leq j \leq w$, where $w = \min\{u, k\}$. Then:

$$\sum_{j=0}^{w} b_j = b \tag{1}$$

$$\sum_{j=0}^{w} j b_j = ur \tag{2}$$

$$\sum_{j=0}^{w} \binom{j}{2} b_j = \binom{u}{2}\lambda. \tag{3}$$

Corollary 1: For any $D \in \mathcal{D}(v, b, k)$:

(i) Each of the v sets $\mathcal{V}_1$ has $(b_0, b_1) = (b-r, r)$;

(ii) Each of the $v(v-1)/2$ sets $\mathcal{V}_2$ has $(b_0, b_1, b_2) = (b - 2r + \lambda, 2r - 2\lambda, \lambda)$.

For any $D \in \mathcal{D}(v, b, 2)$:

(iii) Every $\mathcal{V}_u$ has $(b_0, b_1, b_2) = (b - ur + u(u-1)\lambda/2, ur - u(u-1)\lambda, u(u-1)\lambda/2)$.

Proof: (i) follows through application of (1) and (2) with $w = 1$. Similarly, (ii) and (iii) follow through use of (1) to (3), with $u = 2$, $k \geq 2$ for (ii), and $u \geq 2$, $k = 2$ for (iii).

From Corollary 1, the values of b_j are dependent only on the design parameters and u for $w = \min\{u, k\} \leq 2$. For many $D \in \mathcal{D}(v, b, k)$, the elements of $(b_0, b_1, \ldots, b_w)$ will depend on the particular $\mathcal{V}_u$, for $w \geq 3$. For example, some sets of three treatments may occur together in more blocks than other sets. Consider $D \in \mathcal{D}(v, b, k)$, with $k \geq 3$. Using (1), (2) and (3), a given $\mathcal{V}_3$ has:

$$(b_0, b_1, b_2, b_3) = (b - 3r + 3\lambda - b_3,\ 3r - 6\lambda + 3b_3,\ 3\lambda - 3b_3,\ b_3). \tag{4}$$

Further, for given u, the distributions of $(b_0, b_1, \ldots, b_w)$ may differ between designs within a $\mathcal{D}(v, b, k)$.

We now use the properties of $\mathcal{D}(v, b, k)$ design classes to obtain expressions for the (S_u, T_u) measures.

3.1. (S_u, T_u) measures for $\mathcal{D}(v, b, 2)$

Any non-empty $\mathcal{D}(v, b, 2)$ has cardinality one and thus the (S_u, T_u) measures provide a pilot process to check the Criterion-1 robustness properties of the design.

Let $D \in \mathcal{D}(v, b, 2)$. First consider the trivial case $u = 1$. For any $\mathcal{V}_1$, exactly r of the blocks of D contain the treatment in $\mathcal{V}_1$. There are $2^r - 1$ ways of selecting one observation from each of these blocks to form a RROS(1), that is, to yield a treatment disconnected eventual design in v treatments. There are v ways of selecting $\mathcal{V}_1$. Thus $S_1 = r$ and $T_1 = v(2^r - 1)$. Now consider any set $\mathcal{V}_u$ with $2 \leq u \leq v/2$. A RROS(u) is formed by selecting one observation from each of the b_1 blocks containing one element from $\mathcal{V}_u$. Using Corollary 1 (iii), $b_1 = ur - u(u-1)\lambda = ru(v-u)/(v-1)$. This is independent of the particular set of u treatments, indicating that $s_u = ur(v-u)/(v-1)$, for every $\mathcal{V}_u$. There are $v!/(u!(v-u)!)$ sets of u treatments. It follows that the robustness measures are:

$$(S_1, T_1) = (r, v(2^r - 1)) \tag{5}$$

$$(S_u, T_u) = \left(\frac{ru(v-u)}{v-1}, \frac{2^{S_u} v!}{u!(v-u)!}\right), \quad \text{for } 2 \leq u \leq v/2. \tag{6}$$

From (5) and (6) S_u increases monotonically with u. Thus a pilot procedure starts by evaluation of S_1. Hence, in addition to the v Type III RROSs of size r, by (5) there are $v(2^r - 1)$ Type I RROSs, also of size r. Then, by (6) there are many Type I RROSs of size $2r(v-2)/(v-1)$, and so on.

3.2. (S_u, T_u) measures for $\mathcal{D}(v, b, 3)$

Many $\mathcal{D}(v,b,3)$ design classes have cardinality greater than one. For example, $\mathcal{D}(7,14,3)$, $\mathcal{D}(7,21,3)$ and $\mathcal{D}(7,28,3)$ have cardinalities 4, 10 and 35 respectively.

Consider $D \in \mathcal{D}(v,b,3)$. For any set $\mathcal{V}_1$, exactly r blocks of D contain the treatment in $\mathcal{V}_1$. For a valid RROS(1), the eventual design must contain all v treatments, thus a smallest RROS(1) contains the replicate of the treatment of $\mathcal{V}_1$ from $r-1$ blocks and the replicates of the two treatments from $\mathcal{V}_{v-1}$ from the rth block. This gives $s_1 = r+1$, independent of the particular $\mathcal{V}_1$. For given $\mathcal{V}_1$ there are r ways of selecting the replicate of the treatment in $\mathcal{V}_1$ that is preserved in D_e and there are v ways of selecting $\mathcal{V}_1$. Thus, $S_1 = r+1$ and $T_1 = rv$. Now let $\mathcal{V}_u$ be any set with $2 \le u \le v/2$. For $\mathcal{V}_u$, a RROS(u) of smallest size comprises the observation of a treatment contained in $\mathcal{V}_u$ from each of the b_1 blocks containing one element of $\mathcal{V}_u$ and the observation of a treatment in $\mathcal{V}_{v-u}$ from each of the b_2 blocks containing two elements of $\mathcal{V}_u$. Using (2) and (3), $s_u = b_1 + b_2 = ur - \lambda u(u-1)/2 = ru(v-u)/(v-1)$. Again, the value of s_u is independent of the particular $\mathcal{V}_u$. Given $\mathcal{V}_u$, the RROS(u) of size s_u is unique. Thus, the robustness measures for $\mathcal{D}(v,b,3)$ are:

$$(S_1, T_1) = (r+1, rv) \tag{7}$$

$$(S_u, T_u) = \left(\frac{ru(v-u)}{v-1}, \frac{v!}{u!(v-u)!}\right), \quad \text{for} \quad 2 \le u \le v/2. \tag{8}$$

As with $k=2$, the value of S_u increases with u for $1 \le u \le v/2$.

Results for $k=3$ merit special comment. In some $\mathcal{D}(v,b,3)$, designs in the class differ in the number of repeated blocks. For example, the four designs in $\mathcal{D}(7,14,3)$ have support sizes (i.e. number of distinct blocks) of 7, 11, 13 and 14 respectively. Several authors, including Bhaumik and Whittinghill (1991), recommend avoiding BIBDs with repeated blocks when observation loss is possible. Also, see Raghavarao *et al.* for an investigation of designs in $\mathcal{D}(7,21,3)$ with emphasis on the relationship between the support size and the estimation of contrasts of the block effects. Conversely, Foody and Hedayat (1977) discuss some experimental situations in which deliberate use of designs with repeated blocks gives practical advantages. In assessing robustness within a $\mathcal{D}(v,b,3)$ via (S_u, T_u) measures, no advantage is gained by the avoidance of designs with repeated blocks, since the formulae of (7) and (8) are the same for all designs in a class. Thus, all designs in a $\mathcal{D}(v,b,3)$ are equally vulnerable to becoming disconnected through random observation loss.

3.3. (S_u, T_u) measures for $\mathcal{D}(v, b, k)$, with $k \ge 4$

Let $D \in \mathcal{D}(v,b,k)$, with $k \ge 4$.

For $1 \le u < k/2$, choose any u treatments from any one block for $\mathcal{V}_u$. From the same block, select the $k-u$ treatments in $\mathcal{V}_{v-u}$ for removal. From the $b-1$ remaining blocks, select the treatments from $\mathcal{V}_u$ for removal. The $(k-u)+(r-1)u = u(r-2)+k$ selected observations comprise a RROS(u) and, by the process used, there is no smaller RROS(u) for that $\mathcal{V}_u$, giving $s_u = (k-u)+(r-1)u = u(r-2)+k$. The value of s_u does not depend on the chosen block or on the treatments used from the block for $\mathcal{V}_u$. Thus

$$(S_u, T_u) = \left(u(r-2)+k, \frac{k!b}{u!(k-u)!}\right), \quad \text{for} \quad 1 \le u < k/2. \tag{9}$$

Now, for even k, consider $u = k/2$. For a $\mathcal{V}_{k/2}$ with $b_{k/2} > 0$, select the observations from $\mathcal{V}_{v-k/2}$ for removal from at least one of the $b_{k/2}$ blocks containing all $k/2$ treatments of $\mathcal{V}_{k/2}$. From all other blocks select the observations from $\mathcal{V}_{k/2}$ for removal. Every observation on a treatment in $\mathcal{V}_{v-k/2}$ occurs in a block with no more than $k/2$ treatments from $\mathcal{V}_{k/2}$, thus, the selected observations comprise a RROS($k/2$) of minimal size: $s_{k/2} = kr/2$. For the particular $\mathcal{V}_{k/2}$, there will be $2^{b_{k/2}} - 1$ RROS($k/2$)s of this size, giving

$$(S_{k/2}, T_{k/2}) = \left(\frac{kr}{2}, \sum_{\Psi_0}\left(2^{b_{k/2}} - 1\right)\right), \tag{10}$$

where, the summation is over Ψ_0, the set of $\mathcal{V}_{k/2}$ sets with $b_{k/2} > 0$.

Now consider $k/2 < u \leq k$. For every $\mathcal{V}_u$, perform a scan of D in the following way. For blocks containing fewer than $k/2$ members of $\mathcal{V}_u$, select the members of $\mathcal{V}_u$ for removal. Conversely, for blocks containing more than $k/2$ members of $\mathcal{V}_u$, select the members of $\mathcal{V}_{v-u}$ for removal. For even k, for blocks containing exactly $k/2$ members of $\mathcal{V}_u$, select either treatment set for removal. Let the number of selected observations be N. Then

$$N = \sum_{i=1}^{[k/2]} ib_i + \sum_{i=[k/2]+1}^{k-1} (k-i)b_i,$$

where $[k/2]$ denotes the integer part of $k/2$. Using (2):

$$N = ur - \sum_{i=[k/2]+1}^{k-1} (2i-k)b_i. \tag{11}$$

The selected observations form a RROS of Type I and/or Type III. Any $\mathcal{V}_u$ with $b_j = 0$ for all $j > k/2$ has $N = ur$, and any $\mathcal{V}_u$ with $b_j > 0$ for at least one $j > k/2$ has $N < ur$. Thus, for any $\mathcal{V}_u$ yielding the minimum value for N, there is at least one block containing more than $k/2$ treatments from $\mathcal{V}_u$ in D. Suppose a RROS obtained by the scan for a $\mathcal{V}_u$ for which N is minimised comprises a Type III RROS. Call this $\mathcal{V}_u$ set $\mathcal{V}_u^*$. Then all replicates of a member of $\mathcal{V}_u^*$, say u_0, are selected by the scan and at least one treatment from $\mathcal{V} \setminus \mathcal{V}_u^*$, say u_1, is in a block containing more than $k/2$ treatments from $\mathcal{V}_u^*$. Now consider the set $\mathcal{V}_u^\dagger$ with u_1 replacing u_0 but with the other $u-1$ treatments common to those of $\mathcal{V}_u^*$. This has smaller N, providing a contradiction. Thus, $\mathcal{V}_u$ sets corresponding to the smallest value of N only yield RROS(u)s by the scan, and, by the process, no smaller RROS(u) exists for that $\mathcal{V}_u$. It follows that $S_u = \min_{\mathcal{V}_u}\{N\} < ur$. Let Ψ_1 be the set of $\mathcal{V}_u$ sets achieving S_u. Then $T_u = \sum_{\Psi_1} 2^{b_{k/2}}$, where $b_{k/2}$ is taken to be zero for odd k. Thus,

$$(S_u, T_u) = \left(ur - \max_{\mathcal{V}_u}\left\{\sum_{i=[k/2]+1}^{k-1}(2i-k)b_i\right\}, \sum_{\Psi_1} 2^{b_{k/2}}\right), \quad \text{for} \quad k/2 < u \leq k.$$

Now consider $k < u \leq v/2$. As for $k/2 < u \leq k$, the approach is to conduct a scan for each $\mathcal{V}_u$ and to obtain N as given by (11). However, in this case the minimum value of N can arise for sets of selected observations corresponding to Type III RROSs. An additional step is required in order to identify RROS(u)s of smallest size.

Definition: A covering for $\mathcal{V}_u$ comprises two sets of blocks from D, denoted $\mathcal{B}_{1c}$ and $\mathcal{B}_{2c}$, such that together the blocks of $\mathcal{B}_{1c}$ contain all the treatments of $\mathcal{V}_u$, and together the blocks of $\mathcal{B}_{2c}$ contain all the treatments of $\mathcal{V}_{v-u}$. The weight of the covering is

$$\sum_{\substack{j=1\\ \mathcal{B}_{1c}}}^{[k/2]-1} (k-2b_j) + \sum_{\substack{j=[k/2]+1\\ \mathcal{B}_{2c}}}^{k-1} (2b_j-1).$$

Consider a scan of D conducted with treatment set $\mathcal{V}_u$ in the usual way, but with an adjustment for the blocks of a covering. Treatments from $\mathcal{V}_{v-u}$ are selected from the blocks of $\mathcal{B}_{1c}$, and treatments from $\mathcal{V}_u$ are selected from the blocks of $\mathcal{B}_{2c}$, regardless of the numbers of treatments from $\mathcal{V}_u$ in blocks of either set. Then, the number of observations selected in total exceeds N by the weight of the covering. For a given $\mathcal{V}_u$, let W be the minimum weight of all coverings for $\mathcal{V}_u$. Then, the RROS(u) of smallest size for that $\mathcal{V}_u$ contains $W+N$ observations. These are: observations from $\mathcal{V}_{v-u}$ in blocks of $\mathcal{B}_{1c}$; observations from $\mathcal{V}_u$ in blocks of $\mathcal{B}_{2c}$; observations selected from the scan in the usual way for all other blocks. Thus $S_u = \min_{\mathcal{V}_u}\{W+N\}$. Let Ψ_2 be the set of $\mathcal{V}_u$ sets achieving S_u. Then

$$(S_u, T_u) = \left(\min_{\mathcal{V}_u} \{W+N\}, \sum_{\Psi_2} 2^{b_{k/2}} \right), \quad \text{for} \quad k < u \leq v/2. \tag{12}$$

3.4. A lower bound for S_u

For $k < u \leq v/2$, the process of obtaining minimal coverings for each $\mathcal{V}_u$, before running the scan, to obtain (S_u, T_u) via (12) can be computer intensive. The following result gives a lower bound for S_u. For u moderate in size, the magnitude of this bound might indicate that S_u is sufficiently large that the identification of the exact value is not of concern, given understanding of the expected level of observation loss for the particular experimental situation.

Theorem 3: For $D \in \mathcal{D}(v,b,k)$ and $1 \leq u \leq v/2$, a lower bound for S_u is given by:

$$\left\lceil \frac{u(v-u)r}{v-1} \right\rceil$$

Proof: For any set $\mathcal{V}_u$, the sum of concurrences between treatments in $\mathcal{V}_u$ and treatments in $\mathcal{V}_{v-u}$ is $u(v-u)\lambda$. To induce a Type I RROS through observation loss, the concurrence between any treatment in $\mathcal{V}_u$ and a treatment in $\mathcal{V}_{v-u}$ must be reduced to zero. The greatest reduction in the sum of the concurrences between treatments in $\mathcal{V}_u$ and $\mathcal{V}_{v-u}$ caused through the loss of a single observation occurs if the observation is in a block containing exactly one or $k-1$ treatments from $\mathcal{V}_u$. The loss of such an observation reduces the sum of the concurrences by $k-1$. Thus the number of observations in a RROS(u) is at least

$$\frac{u(v-u)\lambda}{k-1} = \frac{u(v-u)r}{v-1}$$

as required.

4. Investigation of Designs in $\mathcal{D}(8,14,4)$

We use the $\mathcal{D}(8,14,4)$ design class, which has cardinality four, to demonstrate the results of §3, and compare the design ranking produced with design comparisons focused on Criterion-2 robustness. A set of four non-isomorphic designs in $\mathcal{D}(8,14,4)$ is obtained by combining pairs of four base designs. Base designs D_a and D_b contain treatments $1,2,\ldots,7$ and base designs D_c and D_d contain treatments $1,2,\ldots,8$. Each base design is obtained via a cyclic construction, modulo 7: D_a and D_b are members of $\mathcal{D}(7,7,4)$ with initial blocks containing $1,3,4,5$ and $1,2,3,5$, respectively; D_c has initial block containing $1,2,4$ and each block is augmented with treatment 8; D_d is obtained from D_c with treatments 1 and 2 interchanged. The base designs are displayed below:

$$D_a = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \end{array} \qquad D_b = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \end{array}$$

$$D_c = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 \end{array} \qquad D_d = \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 2 & 1 & 3 & 4 & 5 & 6 & 7 \\ 1 & 3 & 4 & 5 & 6 & 7 & 2 \\ 4 & 5 & 6 & 7 & 2 & 1 & 3 \\ 8 & 8 & 8 & 8 & 8 & 8 & 8 \end{array}$$

Members of $\mathcal{D}(8,14,4)$, denoted by $D1, D2, D3$ and $D4$, comprise the base design pairs:

$D1$: D_a and D_d, $D2$: D_b and D_d, $D3$: D_b and D_c, $D4$: D_a and D_c

The labelling of the designs as $D1$ to $D4$ is consistent with Morgan and Parvu (2008). Design $D3$ is the design given careful consideration in Prescott and Mansson (2001).

4.1. (S_u, T_u) measures for $\mathcal{D}(8,14,4)$

Robustness measures for $u=1$ and $u=2$ are common to all designs in the class. By (9), $(S_1,T_1)=(9,56)$. Every $\mathcal{V}_2$ in each design has $b_2=\lambda=3$, giving $(S_2,T_2)=(14,196)$, by (10). These measures indicate the extent of observation loss required from designs in $\mathcal{D}(8,14,4)$ to result in an eventual design in which the treatments are partitioned into sets of size one and seven, and into sets of size two and six, respectively. The lowest value of u enabling discrimination between the four designs is $u=3$. The measure (S_3,T_3) is different for each design and ranks the designs in terms of robustness against incurring a RROS(3). For each design and each $\mathcal{V}_3$, (11) gives $s_3 = 3r - 2b_3 = 21 - 2b_3$. Designs $D1$ to $D3$ each have some $\mathcal{V}_3$ sets with $b_3=2$. For example $D1$ has $b_3=2$ for the sets $\{1,3,5\}$, $\{1,6,7\}$, $\{2,3,7\}$ and $\{2,5,6\}$. Thus, designs $D1$ to $D3$ each have $S_3=17$. By contrast, $D4$ has $b_3=1$ for every set $\mathcal{V}_3$, giving $S_3=19$. The values of T_3 depend on the numbers of $\mathcal{V}_3$ with maximum b_3. Using (4),

$$(b_0,b_1,b_2,b_3) = (2-b_3,\ 3+3b_3,\ 9-3b_3,\ b_3). \tag{13}$$

For designs $D1$ to $D3$, the $\mathcal{V}_3$ sets with $b_3=2$ each have $b_2=3$, by (13) and contribute $2^3=8$ to T_3. For $D4$, each $\mathcal{V}_3$ set has $b_2=6$, and contributes 64 to T_3. The (S_3,T_3) measures

Table 1: (S_3, T_3) measures for designs in $\mathcal{D}(8, 14, 4)$

Design	$\max_{\mathcal{V}_3} b_3$	(S_3, T_3)	Rank
$D1$	2	$(17, 32)$	2
$D2$	2	$(17, 48)$	3
$D3$	2	$(17, 56)$	4
$D4$	1	$(19, 3584)$	1

are displayed in Table 1. Design $D4$ is ranked highest with $S_3 = 19$. The other designs have $S_3 = 17$ and are ranked according to T_3. To summarise, of designs in $\mathcal{D}(8, 14, 4)$, design $D4$ is the most robust against becoming disconnected through a consistent treatment and block partition with treatments separated into sets of sizes three and five. Two more observations need to be lost from $D4$ than from the other designs, before there is a possibility of a D_e with a consistent block and treatment partition, with the treatments partitioned into sets of cardinalities 3 and 5.

4.2. A- and E-efficiencies of eventual designs

It would be hoped that the loss of as many as 17 observations from a design in $\mathcal{D}(8, 14, 4)$ would be considered a remote possibility in most experimental situations. It is interesting to investigate the quality of eventual designs arising from the loss of much smaller numbers of observations from $D1$ to $D4$. To compare the designs with regards to Criterion-2 robustness, the A-efficiencies of eventual designs are considered and, in line with suggestions of Bhar (2014), the E-efficiencies are also obtained.

For $D \in \mathcal{D}(\upsilon, b, k)$, all non-zero eigenvalues of C are $\upsilon\lambda/k$. Let D_e be a connected eventual design arising from the loss of observations from D and let the eigenvalues of C_e be $0 < \mu_{1e} \leq \mu_{2e} \leq \cdots \leq \mu_{(\upsilon-1)e}$. The A- and E-efficiencies of D_e, denoted $E_A(D_e)$ and $E_E(D_e)$, have formulae:

$$E_A(D_e) = \frac{\sum_{i=1}^{\upsilon-1} \frac{1}{\mu_i}}{\sum_{i=1}^{\upsilon-1} \frac{1}{\mu_{ie}}} = \frac{(\upsilon-1)^2 k}{\upsilon r(k-1)\sum_{i=1}^{\upsilon-1} \frac{1}{\mu_{ie}}} \quad \text{and} \quad E_E(D_e) = \frac{\mu_{1e}}{\mu_1}.$$

Hence, for designs in $\mathcal{D}(8, 14, 4)$, the A- and E-efficiencies of D_e are $E_A(D_e) = 7/(6\sum_{i=1}^{\upsilon-1} \frac{1}{\mu_{ie}})$ and $E_E(D_e) = \mu_{1e}/6$. In Table 2 results are given on the lowest A- and E-efficiencies of D_e arising from the loss of up to five observations from designs in $\mathcal{D}(8, 14, 4)$. As established in the literature, for example see Whittinghill (1995), all designs are equivalent when only one observation is lost. For the loss of between 2 and 5 observations the ranking of $D1$ to $D4$, according to the lowest A- and E-efficiences of eventual designs, is consistent with the design ranking according to (S_3, T_3). Design $D4$ consistently demonstrates better performance. Designs $D1$ to $D3$ have the same values for the lowest A- and E-efficiences, but the number of eventual designs with worst properties is consistent with T_3 measure. It is notable that, in each case, the eventual designs with lowest A-efficiency are exactly those with lowest E-efficiency.

4.3. Intersection Aberration

The Intersection Aberration criterion of Morgan and Parvu (2008) extends results of Bhaumik and Whittinghill (1991) to enable the comparison of designs in a $\mathcal{D}(\upsilon, b, k)$

Table 2: Smallest A-efficiency values following the loss of up to five observations from designs in $\mathcal{D}(8, 14, 4)$

Design	No.of missing observations	min{A-efficiency}	min{E-efficiency}	No. of eventual designs
$D1$	1	0.9722	0.8333	36
$D2$	1	0.9722	0.8333	36
$D3$	1	0.9722	0.8333	36
$D4$	1	0.9722	0.8333	36
$D1$	2	0.9354	0.6806	12
$D2$	2	0.9354	0.6806	18
$D3$	2	0.9354	0.6806	21
$D4$	2	0.9373	0.6944	168
$D1$	3	0.8885	0.5462	36
$D2$	3	0.8885	0.5462	54
$D3$	3	0.8885	0.5462	63
$D4$	1	0.8909	0.5556	280
$D1$	4	0.8216	0.4096	36
$D2$	4	0.8216	0.4096	54
$D3$	4	0.8216	0.4096	63
$D4$	4	0.8249	0.4167	280
$D1$	5	0.7155	0.2722	12
$D2$	5	0.7155	0.2722	18
$D3$	5	0.7155	0.2722	21
$D4$	5	0.7206	0.2778	168

according to lowest A-efficiency on the loss of two blocks. This criterion enables the ranking of designs within a $\mathcal{D}(v, b, k)$ in order of their robustness against suffering the most damage on the loss of any two blocks. For $D \in \mathcal{D}(v, b, k)$, let $\eta_g(D)$ denote the number of pairs of blocks such that blocks in a pair have exactly g common treatments. These design properties can be summarised by the intersection aberration vector $\boldsymbol{\eta}(D) = (\eta_0(D), \eta_1(D), \cdots, \eta_k(D))$. Following Morgan and Parvu (2008):

Definition: Let designs $D\dagger$, $D\ddagger \in \mathcal{D}(v, b, k)$, and let p be the largest integer such that $\eta_p(D\dagger) \neq \eta_p(D\ddagger)$. Then $D\dagger$ is described as having less intersection aberration than $D\ddagger$ if $\eta_p(D\dagger) < \eta_p(D\ddagger)$.

A design with less intersection aberration has greater Criterion-2 robustness against the loss of two blocks than one with more intersection aberration.

Designs in $\mathcal{D}(8, 14, 4)$, investigated in Morgan and Parvu (2008), have intersection aberration vectors:

$$\begin{aligned}
\boldsymbol{\eta}(D1) &= (3, 12, 72, 4, 0) \\
\boldsymbol{\eta}(D2) &= (1, 18, 66, 6, 0) \\
\boldsymbol{\eta}(D3) &= (0, 21, 63, 7, 0) \\
\boldsymbol{\eta}(D4) &= (7, 0, 84, 0, 0)
\end{aligned}$$

The designs are ranked by Intersection Aberration from most to least robust in the order $D4$, $D1$, $D2$, $D3$, according to $\eta_3(.)$ values. This ranking is consistent with the ranking obtained by consideration of (S_3, T_3) in §4.1. Note that the $\eta_3(.)$ values are precisely the number of $\mathcal{V}_3$ sets having $b_3 = 2$ for each design. Thus within $\mathcal{D}(8, 14, 4)$ the robustness of a design against incurring a RROS(3) through the loss of random observations corresponds to its robustness against lowest A-efficiency from the loss of two blocks.

See Thornewell (2011) for further investigation into coincidence between rankings of designs according to Intersection Aberration and (S_3, T_3).

5. Conclusion

For $D \in \mathcal{D}(v, b, k)$, the (S_u, T_u) measures developed in §3 give the smallest number of observations that comprise a specific kind of RROS and the number of such observation sets. Loss of observations in such a RROS yields an eventual design in which the treatments are partitioned into sets of size u and $v - u$ respectively, and the usual analysis to compare the treatments cannot be conducted.

For $u \in \{1, 2\}$ and $u < [k/2]$, both S_u and T_u are fixed for all designs in $\mathcal{D}(v, b, k)$. Also, for $\mathcal{D}(v, b, 2)$ and $\mathcal{D}(v, b, 3)$ design classes, all S_u and T_u are functions of the basic design parameters. Information obtained from these measures complements information on the Type III RROSs to give a full picture of the vulnerability of a design to become disconnected through observation loss. Prior to experimentation, calculation of fixed measures, and knowledge of the potential level of observation loss, provide the experimenter with a pilot procedure to check that the eventual design is likely to be connected.

Other measures are dependent on properties of the particular design. For k even and at least six, $S_{k/2}$ is fixed but $T_{k/2}$ is design dependent. For $k \geq 4$ and $u > k/2$, both S_u and T_u can vary within a $\mathcal{D}(v, b, k)$ indicating that consequences of observation loss may vary within the design class. For a $\mathcal{D}(v, b, k)$ with cardinality greater than one, comparison of measures for the lowest value of u for which $(S_u,\ T_u)$ vary, enables the designs to be ranked according to vulnerability.

Investigation of designs in $\mathcal{D}(8, 14, 4)$ indicates that designs which are ranked high according to (S_u, T_u) also perform well with regards to Criterion-2 robustness in the event of different patterns of observation loss.

Acknowledgement

Helen Warren (nee Thornewell)'s PhD research at the University of Surrey was supported by EPSRC grant KB9021.

References

Bailey, R. A., Schiffl, K. and Hilgers, R-D. (2013). A note on robustness of D-optimal block designs for two-colour microarray experiments. *Journal of Statistical Planning and Inference*, **143** 1195–1202.

Baksalary, J. K. and Tabis, Z. (1987). Conditions for the robustness of block designs against the unavailability of data. *Journal of Statistical Planning and Inference*, **16** 49–54.

Bhar, L. (2014). Robustness of variance balanced designs. *Sankhya B*, **76**, 305–316.

Bhaumik, K. and Whittinghill, D. C. (1991). Optimality and robustness to the unavailability of blocks in block designs. *Journal of the Royal Statistical Society*, **B53**, 399–407.

Das, A. and Kageyama, S. (1992). Robustness of BIB and extended BIB designs against the nonavailability of any number of observations in a block. *Computational Statistics and Data Analysis*, **14**, 343–358.

Dey, A. (1993). Robustness of block designs against missing data. *Statistica Sinica*, **3**, 219–231.

Foody, W. and Hedayat, A. (1977). On theory and applications of BIB designs with repeated blocks. *Annals of Statistics*, **5** 932–945.

Ghosh, S. (1979). On robustness of designs against incomplete data. *Sankhya B*, **40**, 204–208.

Ghosh, S. (1982). Robustness of BIBD against the unavailability of data. *Journal of Statistical Planning and Inference*, **6**, 29–32.

Godolphin, J. D. (2016). A link between the E-value and the robustness of block designs. *Journal of the American Statistical Association*, **111**, 1736–1745.

Godolphin, J. D. (2019). Conditions for connectivity of incomplete block designs. *Quality and Reliability Engineering International*, **35**, 1279–1287.

Godolphin, J. D. and Warren, H. R. (2011). Improved conditions for the robustness of binary block designs against the loss of whole blocks. *Journal of Statistical Planning and Inference*, **141**, 3498–3505.

Godolphin, J. D. and Warren, H. R. (2014). An efficient procedure for the avoidance of disconnected incomplete block designs. *Computational Statistics and Data Analysis*, **71**, 1134–1146.

Kageyama, S. (1990). Robustness of block designs. *Probability Statistics and Design of Experiments* (R. R. Bahadur, ed.). Wiley Eastern, New Delhi, 425–438.

Lal, K., Gupta, V. K. and Bhar, L. (2001). Robustness of designed experiments against missing data. *Journal of Applied Statistics*, **28**, 63–79.

Mathon, R. and Rosa, A. (1996). 2-(v, k, λ) designs of small order. *CRC Handbook of Combinatorial Designs*, Ed. (C. J. Colbourn and J. H. Dinitz, ed.). CRC Press: Boca Raton, 3–41.

Morgan, J. P. and Parvu V. (2008). Most robust BIBDs. *Statistica Sinica*, **18**, 689–707.

Prescott, P. and Mansson, R. (2001). Robustness of balanced incomplete block designs to randomly missing observations. *Journal of Statistical Planning and Inference*, **92**, 283–296.

Raghavarao, D., Federer, W. T. and Schwager, S. J. (1986). Characteristics for distinguishing among balanced incomplete blocks with repeated blocks. *Journal of Statistical Planning and Inference*, **13**, 151–163.

Searle, M. (1971). *Linear Models.* Wiley, New York.

Shah, K. R. and Sinha, B. K. (1989). *Theory of Optimal Designs.* Springer, New York.

Thornewell, H. (2011). *Robustness Criteria for Vulnerability of Block Designs in the Event of Observation Loss.* Ph.D. thesis, University of Surrey, Mathematics.

Tsai, S-F and Liao, C-T. (2013). Minimum breakdown designs in blocks of size two. *Journal of Statistical Planning and Inference*, **143**, 202–208.

Whittinghill, D. C. (1995). A note on the optimality of block designs resulting from the unavailability of scattered observations. *Utilitas Mathematica*, **47**, 21–31.