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On a Process of Rumour Propagation

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Abstract

In recent years there has been a vast amount of work to model the spread of rumour. Here we review some of these mathematical models and present some of the main results.

Key words: Rumour process; Firework processes; Renewal processes; Double coverage.

AMS Subject Classifications: 60K35

1. Introduction

The Oxford Dictionary defines rumour as 'a statement or report circulating in a community, of the truth of which there is no evidence'. Mathematically, Gilbert (1961) used the Poisson Boolean model and Maki and Thompson (1973) used a slight variant of this model to study the transmission of information/rumour. This model consisted of a signal being transmitted through a relay of transmitters to its recipient. Two such versions are the Poisson Boolean model and the rumour processes. We present a brief description of these processes here.

Poisson Boolean model: Let $\Xi := (\xi_1, \xi_2, \ldots)$ on $\mathbb{R}^d$ be a homogeneous Poisson point process of intensity λ and $\{\rho_1, \rho_2, \ldots\}$ an independent collection of i.i.d. positive real valued random variables. This is the Poisson Boolean model and its covered region is defined to be the random set $C := \cup_{i=1}^{\infty} B(\xi_i, \rho_i)$, where $B(\xi, \rho)$ is the closed ball centred at ξ and of radius ρ in the Euclidean norm. Geometric properties of this Boolean model has been studied by Matheron (1968), Hall (1988) and Chiu, *et al.* (2013). Kertesz and Vicsek (1982) used this model to study a continuum version of percolation whose parameter is the intensity λ with the radius random variables $\rho_1, \rho_2, \ldots$ being either constants or of a fixed distribution (see Meester and Roy (1996) and Penrose (2003) for a review of the percolation properties of this model). Gupta and Kumar (1998) used this model to study questions of signal-to-interference-ratio (SINR) and other such problems in wireless transmission, see Franciscceti and Meester (2007) for a review.

Rumour process: Sudbury (1985) studied the variant of the information-transmission model introduced in Maki and Thompson (1973). Subsequently, Junior, *et al.*(2011) renamed

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the rumour process as the 'firework process' and introduced a different variant the 'reverse firework process'.

Firework process: Let $\{R_i : i \geq 0\}$ be a sequence of non-negative integer valued i.i.d. random variables . At time 0 the origin starts a rumour and passes it onto all individuals in the interval $[0, R_0]$. At time t, all individuals who heard the rumour for the first time at time $t-1$ spread the rumour, with the individual at site j spreading it among all individuals in the region $[j, j+R_j]$. Note that allowing $\mathbb{P}\{R_j = 0\} > 0$ ensures that there are individuals who are inactive.

Reverse firework process: The reverse firework process consists of the origin who knows the rumour at time 0, and at time t an individual located at site j listens to individuals in the interval $[j - R_j, j]$. If there is an individual at a site in this interval who has heard the rumour by time $t-1$, then the individual at site j gets to know the rumour. Here the random variables $\{R_i : i \geq 0\}$ are as in the firework process.

1.1. Definitions

For each individual at site $i \in \mathbb{N}$ associate the pair (X_i, ρ_i) where $(X_i)_{i\geq 1}$, is a sequence of Bernoulli (p) random variables, i.e.,

$$X_{\mathbf{i}} = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p. \end{cases} \tag{1}$$

and $(\rho_i)_{i\geq 1}$ a sequence of i.i.d. copies of some $\mathbb{N}$-valued random, independent of the random variables $(X_i)_{i\geq 1}$. Let ρ denote a generic random variable with the same distribution as ρ_i. In addition, let ρ_0 an independent $\mathbb{N}$ valued random variables, independent of the collections $(X_i)_{i\geq 1}$ and $(\rho_i)_{i\geq 1}$, with ρ_0 having the same distribution as ρ. Whenever $X_i = 1$, the individual at site i starts to spread rumour within a random distance to its right (an interval of length ρ_i). Coverage occurs if every site of $\mathbb{N}$ is covered by some interval. Set $X_0 \equiv 1$ and let

$$C := \bigcup_{\{i\geq 0: X_i = 1\}} [i, i+\rho_i],$$

and

$$\begin{aligned} D := \{x \in \mathbb{R} : & \text{ there exist } j,\ k \geq -1 \text{ with } j \neq k,\ X_j = X_k = 1 \\ & \text{and } x \in ([j, j+\rho_j] \cap [k, k+\rho_k])\}. \end{aligned}$$

We say that $\mathbb{N}$ is *eventually covered* by C if there exists a $t \geq 1$ such that $[t, \infty) \subseteq C$. We say that $\mathbb{N}$ is *eventually doubly covered* by D if it contains a region $[t, \infty)$, for some $t \geq 1$.

Let $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^d\}$ be a collection of Bernoulli (p) random variables and $\{\rho_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^d\}$ a collection of i.i.d. $\mathbb{N}$ valued random variables, independent of the collection $\{X_{\mathbf{i}} : \mathbf{i} \in \mathbb{N}^d\}$. Let ρ denote a generic random variable with the same distribution as $\rho_{\mathbf{i}}$ and

$$\mathbf{C} := \cup_{\{\mathbf{i}: X_{\mathbf{i}}=1\}} (\mathbf{i} + [0, \rho_{\mathbf{i}}]^d)$$

denote the *covered region* of $\mathbb{N}^d$; here and subsequently $\mathbf{i}+[0,\rho_{\mathbf{i}}]^d = [i_1, i_1+\rho_{\mathbf{i}}]\times\cdots\times[i_d, i_d+\rho_{\mathbf{i}}]$, where $\mathbf{i} = (i_1,\ldots,i_d)$. We say that $\mathbb{N}^d$ is *eventually covered* if there exists $\mathbf{t}\in\mathbb{N}^d$ such that $\mathbf{t}+\mathbb{N}^d \subseteq \mathbf{C}$. Note that this definition may be seen to be equivalent to percolation of the homogenous firework process for $d = 1$, and in that sense, it extends the definition of percolation for a homogenous firework process in $\mathbb{N}^d$. We say that $\mathbb{N}^d$ is *eventually doubly covered* if there exists $\mathbf{t}\in\mathbb{N}^d$ such that $\mathbf{t}+\mathbb{N}^d \subseteq \mathbf{D}$, where

$$\begin{aligned}\mathbf{D} := \{\mathbf{x}\in\mathbb{R}^d : &\text{there exist } \mathbf{i},\ \mathbf{j}\in\mathbb{N}^d \text{ with } \mathbf{i}\neq\mathbf{j} \text{ and } X_{\mathbf{i}} = X_{\mathbf{j}} = 1\\ &\text{such that } x\in(\mathbf{i}+[0,\rho_{\mathbf{i}}]^d)\cap(\mathbf{j}+[0,\rho_{\mathbf{j}}]^d)\}.\end{aligned}$$

The probability of coverage in terms of stochastic geometry or probability of survival for the original rumour process depends on both, the marginal distribution of the radius of influence ρ, and the joint distribution of the X_i's. There are three types of scenarios for random variables X_i's for which we have a necessary and sufficient condition to guarantee a positive probability of survival of the rumour:

(1): X_i's are i.i.d. random variables.

(2): X_i's are a $\{0,1\}$-valued Markov chain.

(3): X_i's are a one-dimensional undelayed discrete renewal point process.

2. The i.i.d Case

Suppose $\{X_{\mathbf{i}} : \mathbf{i}\in\mathbb{N}^d\}$ is a collection of $\{0,1\}$-valued i.i.d. random variable with $p = \mathbb{P}(X_i = 1)$. We assume that this collection is independent of the the collection of i.i.d. positive integer-valued random variables $\{\rho_{\mathbf{i}} : \mathbf{i}\in\mathbb{N}^d\}$. Let $\mathbb{P}_p$ denote the product probability law of X and ρ. When the individuals are not sceptical we have:

Proposition 1: (Athreya, *et al.* (2004))

(i): For $d = 1$

$$\mathbb{P}_p(C\,\text{eventually covers}\,\mathbb{N}) = \begin{cases}1 & \text{if } p > 1/l\\ 0 & \text{if } p < 1/L.\end{cases}$$

where

$$l := \liminf_{j\to\infty} j\mathbb{P}(\rho > j) > 1 \text{ and } L := \limsup_{j\to\infty} j\mathbb{P}(\rho > j) < \infty.$$

(ii): For $d > 1$ and $0 < p < 1$, we have

$$\mathbb{P}_p(\mathbf{C}\,\text{eventually covers}\,\mathbb{N}^d) = \begin{cases}1 & \text{if } \liminf_{j\to\infty} j\mathbb{P}(\rho > j) > 0\\ 0 & \text{if } \lim_{j\to\infty} j\mathbb{P}(\rho > j) = 0.\end{cases}$$

A priori it may be the case that 'single coverage' occurs, i.e. $\mathbf{C}\supseteq\mathbf{t}+\mathbb{N}^d$, but double coverage does not occur. Equivalently, in terms of the rumour process, a rumour may have a positive probability of spreading in a population consisting of only disbelievers or gullible persons. While if among the gullible persons there is also a further group who are sceptics, then the rumour may not spread with positive probability. However, the following proposition shows that this is not the case:

Proposition 2: (Sajadi and Roy (2019))

(i): For $d = 1$,

$$\mathbb{P}_p(D \,\text{eventually covers}\, \mathbb{N}) = \begin{cases} 1 & \text{if } p > 1/l \\ 0 & \text{if } p < 1/L. \end{cases}$$

where

$$l := \liminf_{j\to\infty} j\mathbb{P}(\rho \geq j) > 1 \text{ and } L := \limsup_{j\to\infty} j\mathbb{P}(\rho \geq j) < \infty.$$

(ii): For $d > 1$ and $p > 0$, we have

$$\mathbb{P}_p(\mathbf{D} \,\text{eventually covers}\, \mathbb{N}^d) = \begin{cases} 1 & \text{if } \liminf_{j\to\infty} j\mathbb{P}(\rho \geq j) > 0 \\ 0 & \text{if } \lim_{j\to\infty} j\mathbb{P}(\rho \geq j) = 0. \end{cases}$$

The key to the proof of Proposition 1 (i) is to note that, for $d = 1$, the coverage process forms a renewal process, with renewal happening at every site $i \in \mathbb{N}$ such that $i \notin C$. Part (ii) of the above two propositions exhibits a dichotomy in the behaviour of the process in dimension 1 and in dimensions 2 or more. If $\mathbb{P}(\rho \geq j) = O(j)$ as $j \to \infty$, then in dimension 1, depending on the value of p, there may not be coverage or double coverage, with probability 1. However, for dimensions 2 or more, the only case when there is no coverage (and hence no double coverage) with probability 1 when $p = 0$, i.e. there are no gullible people in the population.

In particular, for $p > 0$ and $i \geq 1$, let

$$A_i := \{i \notin C\} \text{ and } B_i := \{i \notin D\}.$$

Taking $G(i) = \mathbb{P}(\rho \geq i)$ and $g_p(i) = 1 - pG(i)$, we observe that

$$\begin{aligned}
&\mathbb{P}_p(B_i) \\
&\quad = \mathbb{P}\left(A_i \cup \{(\text{there exists exactly one } j \text{ with } X_j = 1 \text{ such that } i \in [j, j+\rho_j]\}\right) \\
&\quad = \mathbb{P}_p(A_i) + \sum_{l=0}^{i-1} \mathbb{P}_p(X_{i-l} = 1,\ i \leq i - l + \rho_{i-l},\ i \notin \cup_{\{j \neq i-l, X_j = 1\}}[j, j+\rho_j]) \\
&\quad = \mathbb{P}_p(A_i) + p\prod_{l=1}^{i-1} g_p(l) + p(1-p)\sum_{k=1}^{i-1} G(k) \prod_{l \neq k, l=1}^{i-1} g_p(l). \qquad (2)
\end{aligned}$$

We next show that for $p > 1/l$, where l is as in Proposition 1

$$\sum_i \mathbb{P}_p(B_i) < \infty, \qquad (3)$$

which, by Borel-Cantelli lemma yields $\mathbb{P}_p(B_i$ occurs finitely often$) = 1$, i.e. there is a random variable T, with $T < \infty$ almost surely, such that $\mathbb{P}_p\{D \supseteq [T, \infty)\} = 1$.

Also, for $p < 1/L$, where L is as in Proposition 1 we have

$$\sum_i \mathbb{P}_p(A_i) = \infty. \tag{4}$$

However, since A_i's are not independent events so we cannot apply the converse of the Borel–Cantelli lemma. Our observation A_i's are renewal events (in the sense that, for $k > i$, $P_p(A_k \cap A_i) = P_p(A_{k-i})P_p(A_i)$) allows us to use Theorem 3, on page 312 of Feller (1971) to conclude that (4) implies that A_i occurs for infinitely many i's. Thus single coverage (and hence double coverage) does not occur almost surely

3. The Markovian Case

Suppose $(X_i)_{i\geq 1}$ is a $\{0,1\}$-valued time-homogeneous Markov chain with $p_{ij} = \mathbb{P}(X_{n+1} = j|X_n = i)$, for $i, j \in \{0,1\}$ and $n \geq 0$. Also suppose $(\rho_i)_{i\geq 1}$ is an independent and identically distributed sequence of random variables assuming values on $\mathbb{N}$, independent of the Markov chain. Let $l := \liminf_{j\to\infty} j\mathbb{P}(\rho > j) > 1$ and $L := \limsup_{j\to\infty} j\mathbb{P}(\rho > j) < \infty$. We have

Theorem 1: (Athreya, *et al.* (2004)) For $0 < p_{00}, p_{10} < 1$, we have

$$\mathbb{P}(C \,\text{eventually covers}\, \mathbb{N}) \begin{cases} > 0 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} > 1/l \\ = 0 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} < 1/L. \end{cases}$$

Theorem 2: (Esmaeeli and Sajadi (2020)) For $0 < p_{00}, p_{10} < 1$, we have

$$\mathbb{P}(D \,\text{eventually covers}\, \mathbb{N}) \begin{cases} > 0 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} > 1/l \\ = 0 & \text{if } \frac{p_{01}}{p_{10}+p_{01}} < 1/L. \end{cases}$$

The proofs of the above two theorems require intricate analysis using probability generating functions. In particular, for $k \geq 1$ and the event A_i as before, let $\mathbb{P}_0(A_k) = \mathbb{P}(A_k|X_1 = 0)$ and $\mathbb{P}_1(A_k) = \mathbb{P}(A_k|X_1 = 1)$. We observe that

$$\mathbb{P}_0(A_{k+1}) = p_{00}\mathbb{P}_0(A_k) + p_{01}\mathbb{P}_1(A_k),\ \mathbb{P}_1(A_{k+1}) = \mathbb{P}(\rho_0 \leq k-1)[p_{10}\mathbb{P}_0(A_k) + p_{11}\mathbb{P}_1(A_k)]. \tag{5}$$

Taking k_0 such that $k_0 + (1 - L) > 0$, and considering the functions $A(s) := \sum_{k\geq k_0} \mathbb{P}_0(A_k)s^k$ and $B(s) := \sum_{k\geq k_0} \mathbb{P}_1(A_k)s^k$ we show that

$$A(1) = \sum_{k\geq k_0} \mathbb{P}_0(A_k) = B(1) = \sum_{k\geq k_0} \mathbb{P}_1(A_k) = \infty.$$

This along with the observation that A_k's are delayed renewal events allow us to use the theorem in Feller (1971). The proof of the double coverage case involves the same ideas, except that the relation (5) is more complicated and as such the calculations are more intricate.

4. The Renewal Case

Let $(T_i)_{i\geq 1}$ be a sequence of independent copies of some $\mathbb{N}$-valued random variable T. Taking $X_0 = 1$, define the $\{0,1\}$-valued random variables $\mathbf{X} = (X_i)_{i\geq 1}$ as follows:

$$X_{\mathbf{i}} = \begin{cases} 1 & \text{if and only if there exists}\, n \geq 1\, \text{such that } \sum_{k=1}^{n} T_k = i \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Observe that $\mathbf{X}$ is a binary undelayed renewal sequence with inter-arrival times T. Let $(\rho_i)_{i\in\mathbb{N}}$ be a sequence of independent copies of some $\mathbb{N}$-valued random variable ρ, independent of the sequence $\mathbf{X}$ and satisfying $\mathbb{P}(\rho = 0) > 0$. We say that there is coverage if the event

$$\mathcal{A} := \Big(\bigcup_{\{i\geq 0: X_i = 1 \& \rho_i \geq 1\}} [i+1, i+\rho_i] = \mathbb{N} \Big),$$

occurs. The main objective is to study $\mathbb{P}(\mathcal{A})$. We start with conditions under which this probability is null.

Proposition 3: (Gallo and Garcia (2018)) If $\mathbb{E}[T] = \infty$, then $\mathbb{P}(\mathcal{A}) = 0$.

Also

Proposition 4: (Gallo and Garcia (2018)) If $\limsup_{n\to\infty} \dfrac{n\mathbb{P}(\rho > n)}{\mathbb{E}[T]} < 1$, then $\mathbb{P}(\mathcal{A}) = 0$.

But the case where $\mathbb{E}[T] = \infty$, is not interesting and usually it assumes that the process $(X_i)_{i\geq 1})$ is positive recurrent. If we further assume that $\mathbf{X}$ is aperiodic, we have the following formula from Gallo and Garcia (2018), for the probability of coverage:

Theorem 3: (Gallo and Garcia (2018))

$$\mathbb{P}(\mathcal{A}) = \left(1 + \sum_{n\geq 1} \mathbb{E} \prod_{i=0}^{n-1} [\mathbb{P}(\rho \leq i)]^{X_{i+1}} \right)^{-1}.$$

This above quantity is difficult to handle in general and Gallo and Garcia (2018) presented explicit bounds for the probability of coverage.

To guarantee positive probability of coverage, Gallo and Garcia (2018) needed an extra assumption.

Proposition 5: (Gallo and Garcia (2018)) Let $q_i := \max_{n\leq i} \mathbb{P}(T \geq n+2 | T \geq n+1)$, $i \geq 0$. If

$$\sum_{j=1}^{k} \prod_{i=k}^{k+j-2} q_i = o(k) \tag{7}$$

and $\limsup_{n\to\infty} \dfrac{n\mathbb{P}(\rho > n)}{\mathbb{E}[T]} > 1$ then $\mathbb{P}(\mathcal{A}) > 0$.

If the rumour process satisfies condition (7), then by Propositions 3 and 4, we have a sharp phase transition between null and positive probability of coverage.

Gallo and Garcia (2018) conjectured that if the renewal process is positive recurrent, then $\liminf_{n\to\infty} \dfrac{n\mathbb{P}(\rho > n)}{\mathbb{E}[T]} > 1$ should guarantee that $\mathbb{P}(\mathcal{A}) > 0$.

5. Fireworks Version of Rumour Processes

To study rumour propagation in terms of the fireworks process, the main goal is to find out the probability of having an infinite set of individuals knowing the rumour is positive. Junior, *et al.* (2011) presented the survival event as a limit of an increasing sequence of events whose probability can be bounded by a use of the FKG inequality. To find conditions under which the process dies out, they used a non-standard version of the Borel-Cantelli lemma. Gallo, *et al.* (2014) used a technique based on the relationship between the rumour process and a certain discrete time renewal process to obtain more precise results for the homogeneous versions of the fireworks process.

Suppose at time 0, the origin spreads a rumour to all individuals in the interval $[0, \rho_0]$. At time t all individuals, who received the rumour at time $t-1$, spread the rumour, with an individual j spreading the rumour to all individuals in the interval $[j, j+\rho_j]$ who have not been activated before. Define the following monotone decreasing event and its limit:

$$V_n := \{\text{the vertex } n \text{ is hit by an explosion}\} \text{ and } V = \lim_{n\to\infty} V_n.$$

Theorem 4: (Junior, *et al.* (2011)) For the homogeneous firework process we have

$$\sum_{n=1}^{\infty}\prod_{i=0}^{n}\mathbb{P}(\rho \le i) = \infty \text{ if and only if } \mathbb{P}(V) = 0\,.$$

Let $\mu := 1 + \sum_{n\ge 1}\prod_{i=0}^{n-1}\mathbb{P}(\rho \le i)$.

Theorem 5: (Gallo, *et al.* (2014)) For the homogeneous fireworks process

$$\mathbb{P}(V) = \frac{1}{\mu}\,.$$

Esmaeeli and Sajadi (2021) extended this result for the propogation of rumour among sceptics. Suppose that at the beginning, only two individuals $\{0,1\}$ are active and set $B_0 := \{0,1\}$. Define the sequence of events $(B_n)_{n\ge1}$ as

$$B_n := \{i \ge 2 \;:\; \exists j_1 \ne j_2 \in \cup_{i=0}^{n-1} B_i \quad \text{such that} \quad i \in [j_1, j_1+\rho_{j_1}] \cap [j_2, j_2+\rho_{j_2}] \cap \mathbb{N}\}.$$

Let $B := \bigcup_{n\ge1} B_n$. B is the set of all sceptic individuals who have heard the rumour.Let $\bar{\mathcal{A}}$ be the event that the rumour survives among sceptic individuals. We have the following result.

Theorem 6: (Esmaeeli and Sajadi (2021)) $\mathbb{P}(\bar{\mathcal{A}}) = \dfrac{1}{\bar{\mu}}$, where

$$\bar{\mu} = 2 + \sum_{k=2}^{\infty}\prod_{i=2}^{k}\bar{\alpha}_i \text{ and } \bar{\alpha}_i = \sum_{l=1}^{i}(-1)^{l-1}\sum_{I\subset\{1,\dots,i\},|I|=l}\prod_{r\in I}\prod_{k=1,k\ne r}^{i}\mathbb{P}(\rho \le k-1), \quad i \ge 2. \qquad (8)$$

They also showed that $\bar{\mu} < \infty$ if and only if $\mu < \infty$ and from that they concluded the rumour dies out among sceptics under the same conditions presented in Gallo, *et al.* (2014) for non-sceptics.

Theorem 7: (Esmaeeli and Sajad (2021))

$$\mathbb{P}(\bar{\mathcal{A}}) = 0 \Longleftrightarrow \mathbb{P}(\mathcal{A}) = 0.$$

6. Further Questions

Bertacchi and Zucca (2013) studied the spread of rumour in a random environment on $\mathbb{N}$ and on Galton-Watson trees. Also, as in Mukhopadhyay, *et al.* (2020), a natural question is to ask for the rate of the spread of a rumour in a complete graph of N individuals, when every individual samples a fixed k number of individuals. The mean field limit of this model may suggest the rate of spread. Also if there are competing rumours, then a majority rule mechanism may also be used to find which rumour survives and which does not. This approach may provide rigorous answers to the simulation based observations of Zanette (2001, 2002).

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