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The Existence of 2-pairwise Additive Cyclic BIB Designs of Block Size Two

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Abstract

The existence of pairwise additive cyclic BIB designs with $k = 2$ and $\lambda = 1$ has been discussed in the literature. In this paper, new classes of 2-pairwise additive BIB designs are mainly constructed through methods of block trades, and then the existence of 2-pairwise additive cyclic BIB designs with $k = 2$ and $\lambda \geq 1$ is shown entirely.

Key words: BIB design; Pairwise additive cyclic BIB design (PACB); Cyclic relative difference family (CDF); Pairwise additive CDF (PACDF); Trade.

1 Introduction

A *balanced incomplete block* (BIB) *design* is a system $(V, \mathcal{B})$, where V is a set of v points and $\mathcal{B}$ ($|\mathcal{B}| = b$) is a family of k-subsets (blocks) of V, such that each point of V appears in r different blocks of $\mathcal{B}$ and any two different points of V appear in exactly λ blocks in $\mathcal{B}$ (Raghavarao, 1988). This is denoted by BIBD(v, b, r, k, λ) or B(v, k, λ).

For a BIB design $(V, \mathcal{B})$, let σ be a permutation on V. For a block $B = \{v_1, \ldots, v_k\} \in \mathcal{B}$ and a permutation σ on V, let $B^\sigma = \{v_1^\sigma, \ldots, v_k^\sigma\}$. When $\mathcal{B} = \{B^\sigma | B \in \mathcal{B}\}$, σ is called an *automorphism* of $(V, \mathcal{B})$. If there exists an automorphism σ of order $v = |V|$, then the BIB design is said to be *cyclic*.

For a cyclic BIB design $(V, \mathcal{B})$, the set V of v points can be identified with $Z_v = \{0, 1, \ldots, v-1\}$. In this case, the design has an automorphism $\sigma : a \mapsto a+1 \pmod v$. The *block orbit* containing $B = \{v_1, v_2, \ldots, v_k\} \in \mathcal{B}$ is a set of distinct blocks $B + a = \{v_1 + a, v_2 + a, \ldots, v_k + a\} \pmod v$ for $a \in Z_v$. A block orbit is said to be *full* or *short* according as $|\{B + a \mid 0 \leq a \leq v - 1\}| = v$ or not.

Choose an arbitrary block from each block orbit and call it an *initial block*. The initial block in a full block orbit and a short block orbit is called a full initial block and a short initial block, respectively.

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Let $s = v/k$, where s need not be an integer unlike other parameters. A set of ℓ BIBD(v, b, r, k, λ)s, namely, $(V, \mathcal{B}_1), (V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_\ell)$, is called an *$\ell$-pairwise additive BIB design*, denoted by ℓ-PAB(v, k, λ), if it is possible to pair the designs $(V, \mathcal{B}_1), (V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_\ell)$, in such a way that every pair $(V, \mathcal{B}_{i_1}), (V, \mathcal{B}_{i_2})$, where $1 \leq i_1, i_2 \leq \ell$, $i_1 \neq i_2$, gives rise to a new design $(V, \mathcal{B}^*_{i_1 i_2})$ with parameters $v^* = v = sk$, $b^* = b$, $r^* = 2r$, $k^* = 2k$, $\lambda^* = 2r(2k-1)/(sk-1)$, where the family $\mathcal{B}^*_{i_1 i_2}$ is given by $\mathcal{B}^*_{i_1 i_2} = \{B_{i_1 j} \cup B_{i_2 j} \mid 1 \leq j \leq b\}$ with B_{ij} being the jth block of an ith block family $\mathcal{B}_i$. An ℓ-PAB(v, k, λ) is said to be *cyclic*, denoted by ℓ-PACB(v, k, λ), if (i) each of designs $(V, \mathcal{B}_1)$, $(V, \mathcal{B}_2), \ldots, (V, \mathcal{B}_\ell)$ is cyclic, and (ii) every design $(V, \mathcal{B}^*_{i_1 i_2})$ arising from the pair $(V, \mathcal{B}_{i_1}), (V, \mathcal{B}_{i_2})$ is also cyclic and its initial blocks are obtained by joining an initial block in $(V, \mathcal{B}_{i_1})$ to an initial block in $(V, \mathcal{B}_{i_2})$, where two orbits given by initial blocks $B_{i_1 j}$ and $B_{i_2 j}$ have the same cardinality for each j of $1 \leq j \leq b$. Note that when $k = 2$ every short orbit coincides with the orbit of length $v/2$ given by $\{0, v/2\}$, and for convenience the orbit of length v given by $\{0, v/2\}$ is also regarded as a full orbit which contains each block exactly twice.

Example 1.1. A 2-PACB$(6, 2, 2)$ on Z_6 has two block families:

$$\begin{aligned} \mathcal{B}_1 &: \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\} \mod 6 \\ \mathcal{B}_2 &: \{4,5\}, \{3,5\}, \{4,5\}, \{2,5\}, \{2,4\} \mod 6. \end{aligned}$$

Example 1.2. A 2-PACB$(18, 2, 2)$ on Z_{18} has two block families:

$$\begin{aligned} \mathcal{B}_1 : \ & \{0,1\}, \{0,2\}, \{0,3\}, \{0,4\}, \{0,5\}, \{0,6\}, \{0,7\}, \\ & \{0,8\}, \{0,9\}, \{0,10\}, \{0,11\}, \{0,12\}, \{0,13\}, \{0,14\}, \\ & \{0,15\}, \{0,16\}, \{0,17\} \mod 18 \\ \mathcal{B}_2 : \ & \{5,8\}, \{8,12\}, \{9,14\}, \{1,17\}, \{3,4\}, \{3,16\}, \{4,14\}, \\ & \{2,13\}, \{12,14\}, \{3,17\}, \{2,10\}, \{7,10\}, \{2,14\}, \{5,6\}, \\ & \{6,17\}, \{4,13\}, \{1,7\} \mod 18. \end{aligned}$$

Direct and recursive constructions of a 2-PACB$(v, k, 1)$ are given in (Matsubara and Kageyama, 2013; Matsubara *et al.*, 2015). Especially, some results on the existence of an ℓ-PACB(v, k, λ) are known for $\ell = 2, k = 2$ and $\lambda = 1$ as the following shows.

Lemma 1.3. (Matsubara and Kageyama, 2013) *There exists a* 2*-PACB*$(v, 2, 1)$ *for any odd integer* $v \geq 5$ *such that* $\gcd(v, 9) \neq 3$.

Lemma 1.4. (Matsubara *et al.*, 2015) *There exists a* 2*-PACB*$(2^m t, 2, 1)$ *for any integer* $m \geq 2$ *and any odd integer* $t \geq 1$ *such that* $\gcd(t, 27) \neq 3, 9$.

However, even if $\ell = 2, k = 2$ and $\lambda = 1$, the existence of some classes of 2-PACB(v, k, λ)s has not been shown. In this paper, by further elaboration of the results given in Lemmas 1.3 and 1.4, through new methods of construction, the complete existence of a 2-PACB$(v, 2, \lambda)$ will be shown as follows.

Theorem 1.5. *A* 2*-PACB*$(v, 2, \lambda)$ *exists if and only if* $v \geq 4$ *and* $\lambda \geq 1$, *except for* $v \equiv 2 \ (mod\ 4)$ *and* $\lambda \equiv 1 \ (mod\ 2)$.

2 Known Results on Constructions

Many types of combinatorial structures with a cyclic automorphism can be constructed by use of cyclic difference matrices and cyclic relative difference families (Buratti, 1998; Jimbo, 1993; Yin, 1998). Especially, some useful constructions of a 2-PACB(v, k, λ) are available in Matsubara and Kageyama (2013); Matsubara *et al.* (2015).

A *cyclic difference matrix* on Z_v, denoted by CDM$(4, v)$, is defined as a $4 \times v$ array (a_{mn}), $a_{mn} \in Z_v$, $1 \le m \le 4$, that satisfies

$$Z_v = \{a_{m_1 n} - a_{m_2 n} \pmod v | 1 \le n \le v\}$$

for each m_1, m_2 of $1 \le m_1 < m_2 \le 4$, that is, the differences of any two distinct rows contain every element of Z_v exactly once (cf. Ge, 2005).

Lemma 2.1. (Ge, 2005) *There exists a CDM*$(4, v)$ *for any odd integer* $v \ge 5$ *and* $\gcd(v, 27) \ne 9$.

Let G be a group and N be a subgroup of G. Then a family $\mathcal{F} = \{F_j \mid j \in J\}$ of k-subsets of G is called a *relative difference family*, denoted by (G, N, k, λ)-DF, if the multiset $\Delta\mathcal{F} = \{d - d' \mid d, d' \in F_j, d \ne d', j \in J\}$ of differences contains each element of $G \setminus N$ exactly λ times and each element of N zero time. When G is the cyclic group Z_{vg} and N is the subgroup of Z_{vg} of order g, denoted by vZ_g, the relative difference family is said to be *cyclic*, and it is denoted by (vg, g, k, λ)-CDF (cf. Buratti, 1998; Yin, 1998).

A set of two families $\mathcal{F}_1$ and $\mathcal{F}_2$ is called a 2-*pairwise additive* (vg, g, k, λ)-CDF, denoted by 2-(vg, g, k, λ)-PACDF, if both $\mathcal{F}_1$ and $\mathcal{F}_2$ are (vg, g, k, λ)-CDFs and the family of set-unions of the jth k-subsets $F_j \in \mathcal{F}_1$ and $F'_j \in \mathcal{F}_2$, $1 \le j \le |\mathcal{F}_1| = |\mathcal{F}_2|$, is also a $(vg, g, 2k, \lambda')$-CDF with $\lambda' = 2\lambda(2k-1)/(k-1)$, that is, $\Delta\mathcal{F}_i$ contains every element of $Z_{vg} \setminus vZ_g$ exactly λ times for each $i = 1, 2$, and $\Delta(\mathcal{F}_1, \mathcal{F}_2) = \{\pm(d - d') \mid F_j \in \mathcal{F}_1, F'_j \in \mathcal{F}_2, d \in F_j, d' \in F'_j, 1 \le j \le |\mathcal{F}_1| = |\mathcal{F}_2|\}$ contains every element of $Z_{vg} \setminus vZ_g$ exactly $2k\lambda/(k-1)$ times.

Throughout the paper, the above "2-(vg, g, k, λ)-PACDF" is simply denoted by "(vg, g, k, λ)-PACDF".

Note that a B$(v, 2, \lambda)$ may contain some short orbits of length $v/2$ given by $\{0, v/2\}$, and a family of initial blocks of a 2-PACB$(v, 2, \lambda)$ with no short orbit coincides with a $(v, 1, 2, \lambda)$-PACDF. Now, the multisets of differences of short initial blocks of size two are newly defined by

$$\begin{aligned} \Delta_s\mathcal{S} &= \{a_j - b_j \mid \{a_j, b_j\} \in \mathcal{S}, 1 \le j \le |\mathcal{S}|\}, \\ \Delta_s(\mathcal{S}_1, \mathcal{S}_2) &= \{\pm(a_j - c_j), \pm(a_j - d_j) \mid \{a_j, b_j\} \in \mathcal{S}_1, \{c_j, d_j\} \in \mathcal{S}_2, \\ &\quad\ 1 \le j \le |\mathcal{S}_1| = |\mathcal{S}_2|\}, \end{aligned}$$

where $\mathcal{S}$ and $(\mathcal{S}_1, \mathcal{S}_2)$ are a set of short initial blocks in a B$(v, 2, \lambda)$ and a pair of sets S_1, S_2 of short initial blocks in a 2-PACB$(v, 2, \lambda)$, respectively. Actually, since $b_j = a_j + v/2$ and $d_j = c_j + v/2$, $\Delta_s\mathcal{S}$ is composed of $|\mathcal{S}|$ same elements $v/2$.

If a 2-PACB$(v, 2, \lambda)$ contains short orbits, then $\Delta\mathcal{F}_i \cup \Delta_s\mathcal{S}_i$ contains every element of $Z_{vg} \setminus \{0\}$ exactly λ times for each $i = 1, 2$ and $\Delta(\mathcal{F}_1, \mathcal{F}_2) \cup \Delta_s(\mathcal{S}_1, \mathcal{S}_2)$ contains every element of $Z_{vg} \setminus \{0\}$ exactly 4λ times.

Two symbols Δ and Δ_s defined here are used in Sections 3 and 4.

Lemma 2.2. (Matsubara and Kageyama, 2017) *There exists a* $(27, 3, 2, 1)$*-PACDF.*

A fundamental construction of a 2-PACB(vg, k, λ) from a PACDF(vg, g, k, λ) is at first provided as follows.

Lemma 2.3. (Matsubara and Kageyama, 2017) *The existence of a* $(vg, g, 2, \lambda)$*-PACDF and a* $(g, 1, 2, \lambda)$*-PACDF (or* 2*-PACB*$(g, 2, \lambda)$*) implies the existence of a* 2*-PACB*$(vg, 2, \lambda)$.

Recursive constructions on a 2-PACB$(v, 2, \lambda)$ and a PACDF$(vg, g, 2, \lambda)$ are next reviewed.

Lemma 2.4. (Matsubara and Kageyama, 2013) *Let* $v \geq 5$ *and* $v' \geq 5$ *be odd integers. Then the existence of a* 2*-PACB*$(v, 2, \lambda)$*, a* 2*-PACB*$(v', 2, \lambda)$ *and a CDM*$(4, v')$ *implies the existence of a* 2*-PACB*$(vv', 2, \lambda)$.

Lemma 2.5. (Matsubara and Kageyama, 2017) *The existence of a* $(vg, g, 2, \lambda)$*-PACDF and a CDM*$(4, v')$ *implies the existence of a* $(vv'g, v'g, 2, \lambda)$*-PACDF.*

Lemma 2.6. (Matsubara and Kageyama, 2017) *The existence of a* $(vg, g, 2, \lambda)$*-PACDF implies the existence of a* $(2^m vg, 2^m g, 2, \lambda)$*-PACDF for any* $m \geq 1$.

In fact, Matsubara and Kageyama (2013, 2017) show the above Lemmas 2.3 to 2.6 for $\lambda = 1$. Furthermore, by taking copies of each structure, it is clear that the results can also be shown for any $\lambda \geq 2$. Finally note that the existence of a $(v, 1, 2, 1)$-PACDF is equivalent to the existence of a 2-PACB$(v, 2, 1)$ for odd $v \geq 5$. Each of Lemmas 2.1 to 2.6 is used to have Theorem 1.5.

3 Methods of Block Trades

Trades discussed in Hedayat and Khosrovshahi (2007) are useful in study of t-designs and its related structures. In this section, by use of families similar to the trades, we provide two methods of constructing a 2-PACB$(vg, 2, 1)$ from a PACDF$(vg, g, 2, 1)$.

Now, we consider the following families of blocks of size two on Z_{3p} throughout the paper for some positive integer m:

$$\begin{aligned}
\mathcal{H}_1 &= \{\{0,1\},\{0,2mp+2\},\{2mp+1,mp\},\{mp+2,mp\},\\
&\quad \{2mp+1,mp-1\}\},\\
\mathcal{H}_2 &= \{\{mp+2,mp+3\},\{2mp+4,mp+6\},\{3,2mp+2\},\{6,4\},\\
&\quad \{4,2mp-4\}\},\\
\mathcal{H}_1^* &= \{\{0,1\},\{0,2mp+2\},\{2mp+1,mp\},\{mp+2,mp\},\\
&\quad \{2mp+1,3\},\{mp,2mp\}\},\\
\mathcal{H}_2^* &= \{\{2mp+4,2mp+5\},\{mp+2,mp\},\{mp+5,4\},\{0,2mp+2\},\\
&\quad \{mp,2mp\},\{mp+2,-6\}\},
\end{aligned}$$

where $p \geq 7$ is an odd prime and $p \equiv m \pmod 3$.

Now, we have the following result by trading $\mathcal{H}_1, \mathcal{H}_2$ for $\mathcal{H}_1^*, \mathcal{H}_2^*$, respectively.

Lemma 3.1. *Let $p \geq 7$ be an odd prime. Then the existence of a $(3p, 3, 2, 1)$-PACDF with two families $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfying*

$$\begin{aligned}&\{(B_{1j}, B_{2j}) \mid B_{1j} \in \mathcal{H}_1, B_{2j} \in \mathcal{H}_2, 1 \leq j \leq 5\}\\ &\subset \{(B_{1j}, B_{2j}) \mid B_{1j} \in \mathcal{F}_1, B_{2j} \in \mathcal{F}_2, 1 \leq j \leq |\mathcal{F}_1| = |\mathcal{F}_2|\}\end{aligned} \quad (3.1)$$

implies the existence of a 2-PACB$(3p, 2, 1)$.

Proof. For the differences arising from the above families $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_1^*$ and $\mathcal{H}_2^*$, it holds that

$$\begin{aligned}&\Delta\mathcal{H}_i^* = \Delta\mathcal{H}_i \cup \{p, 2p\},\ i = 1, 2,\\ &\Delta(\mathcal{H}_1^*, \mathcal{H}_2^*) = \Delta(\mathcal{H}_1, \mathcal{H}_2) \cup \{p^{(4)}, (2p)^{(4)}\},\end{aligned}$$

where $g^{(t)}$ denotes that an element g is contained t times in the multiset on Z_{3p}.

On the other hand, since the families $\mathcal{F}_1$ and $\mathcal{F}_2$ yield a $(3p, 3, 2, 1)$-PACDF, $\Delta\mathcal{F}_i$ contains every element of $Z_{3p} \setminus \{0, p, 2p\}$ exactly once for each $i = 1, 2$, and $\Delta(\mathcal{F}_1, \mathcal{F}_2)$ contains every element of $Z_{3p} \setminus \{0, p, 2p\}$ exactly four times.

Hence, two families $(\mathcal{F}_i \setminus \mathcal{H}_i) \cup \mathcal{H}_i^*$ with $i = 1, 2$ yield a $(3p, 1, 2, 1)$-PACDF. Thus, the required 2-PACB$(3p, 2, 1)$ is obtained. □

Next, we consider the following families of blocks of size two on Z_{2p} throughout the paper:

$$\begin{aligned}\mathcal{I}_1 &= \{\{0, 1\}, \{0, p+2\}, \{p+2, p+4\}, \{p+4, p+8\}, \{0, 12\},\\ &\quad \{0, p+12\}\},\\ \mathcal{I}_2 &= \{\{p+2, p+4\}, \{p+4, p+8\}, \{0, 1\}, \{0, p+2\}, \{p+4, 8\},\\ &\quad \{p+4, p+8\}\},\\ \mathcal{I}_1^* &= \{\{0, 1\}, \{0, p+2\}, \{p+1, p+3\}, \{2, 6\}, \{0, 12\},\\ &\quad \{0, p\}, \{4, p-8\}\},\\ \mathcal{I}_2^* &= \{\{p+1, p+3\}, \{2, 6\}, \{0, 1\}, \{0, p+2\}, \{p+4, p+8\},\\ &\quad \{4, p+8\}, \{0, p\}\},\end{aligned}$$

where $p \geq 5$ is an odd prime.

Now, we have the following result by trading $\mathcal{I}_1, \mathcal{I}_2$ for $\mathcal{I}_1^*, \mathcal{I}_2^*$, respectively.

Lemma 3.2. *Let $p \geq 5$ be an odd prime.Then the existence of a $(2p, 2, 2, 2)$-PACDF with two families $\mathcal{F}_1$ and $\mathcal{F}_2$ satisfying*

$$\begin{aligned}&\{(B_{1j}, B_{2j}) \mid B_{1j} \in \mathcal{I}_1, B_{2j} \in \mathcal{I}_2, 1 \leq j \leq 6\}\\ &\subset \{(B_{1j}, B_{2j}) \mid B_{1j} \in \mathcal{F}_1, B_{2j} \in \mathcal{F}_2, 1 \leq j \leq |\mathcal{F}_1| = |\mathcal{F}_2|\}\end{aligned} \quad (3.2)$$

implies the existence of a 2-PACB$(2p, 2, 2)$.

Proof. For the differences arising from the above families $\mathcal{I}_1, \mathcal{I}_2, \mathcal{I}_1^*$ and $\mathcal{I}_2^*$, it holds that

$$\Delta\mathcal{I}_i^* = \Delta\mathcal{I}_i \cup \{p^{(2)}\},\ i = 1, 2,$$
$$\Delta(\mathcal{I}_1^*, \mathcal{I}_2^*) = \Delta(\mathcal{I}_1, \mathcal{I}_2) \cup \{p^{(8)}\},$$

where $g^{(t)}$ denotes that an element g is contained t times in the multiset on Z_{2p}.

On the other hand, since the families $\mathcal{F}_1$ and $\mathcal{F}_2$ yield a $(2p, 2, 2, 2)$-PACDF, $\Delta\mathcal{F}_i$ contains every element of $Z_{2p} \setminus \{0, p\}$ exactly twice for each $i = 1, 2$ and $\Delta(\mathcal{F}_1, \mathcal{F}_2)$ contains every element of $Z_{2p} \setminus \{0, p\}$ exactly eight times.

Hence, two families $(\mathcal{F}_i \setminus \mathcal{I}_i) \cup \mathcal{I}_i^*$ with $i = 1, 2$ yield a $(2p, 1, 2, 2)$-PACDF. Thus, the required 2-PACB$(2p, 2, 2)$ can be obtained. □

Lemmas 3.1 and 3.2 are effective for argument in Section 4.

4 Proof of Theorem 1.5

At first, we provide the new results on the existence of a 2-PACB$(v, 2, \lambda)$ as follows.

Lemma 4.1. *There exists a* 2*-PACB*$(3p, 2, 1)$ *for any odd prime* $p \geq 7$.

Proof. Let $p \geq 7$ be any odd prime and m be some positive integer. Also let $\mathcal{F}_1$ and $\mathcal{F}_2$ be a set of two families on Z_{3p} with

$$\begin{aligned}
\mathcal{F}_1 &= \{\{0, at + mp\}, \{at, mp\}, \{\alpha^{a-1}t, -\alpha^{a-1}t\} \mid 1 \leq a \leq (p-1)/2\},\\
\mathcal{F}_2 &= \{\{2at, 3at + mp\}, \{3at, 2at + mp\}, \{4\alpha^{a-1}t + mp, -4\alpha^{a-1}t + mp\} \mid\\
&\quad 1 \leq a \leq (p-1)/2\}
\end{aligned}$$

for $p \equiv m \pmod 3$, $t = 2mp + 1$ and a primitive element α of GF(p). Then, it can be checked that the $\mathcal{F}_1$ and $\mathcal{F}_2$ form a $(3p, 3, 2, 1)$-PACDF satisfying $\mathcal{H}_1 \subset \mathcal{F}_1$ and $\mathcal{H}_2 \subset \mathcal{F}_2$. Hence Lemma 3.1 shows the existence of a 2-PACB$(3p, 2, 1)$. □

The following example illustrates Lemma 4.1 with $p = 7, m = 1, t = 15$ and $\alpha = 3$.

Example 4.2. A 2-PACB$(21, 2, 1)$ is obtained by use of Lemma 4.1 with the following families:

$$\begin{aligned}
\mathcal{H}_1 &= \{\{0,1\}, \{0,16\}, \{15,7\}, \{9,7\}, \{15,6\}\},\\
\mathcal{H}_2 &= \{\{9,10\}, \{18,13\}, \{3,16\}, \{6,4\}, \{4,10\}\},\\
\mathcal{H}_1^* &= \{\{0,1\}, \{0,16\}, \{15,7\}, \{9,7\}, \{15,3\}, \{7,14\}\},\\
\mathcal{H}_2^* &= \{\{18,19\}, \{9,7\}, \{12,4\}, \{0,16\}, \{7,14\}, \{9,15\}\},\\
\mathcal{F}_1 &= \{\{0,1\}, \{0,16\}, \{0,10\}, \{15,7\}, \{9,7\}, \{3,7\}, \{15,6\},\\
&\quad \{3,18\}, \{9,12\}\},\\
\mathcal{F}_2 &= \{\{9,10\}, \{18,13\}, \{6,16\}, \{3,16\}, \{6,4\}, \{9,13\}, \{4,10\},\\
&\quad \{19,16\}, \{1,13\}\}.
\end{aligned}$$

It can be checked that (i) $\mathcal{H}_i$ and $\mathcal{F}_i$ satisfy the condition (3.1) for $i = 1, 2$, (ii) $\mathcal{F}_i^* = (\mathcal{F}_i \setminus \mathcal{H}_i) \cup \mathcal{H}_i^*$ yields a cyclic B$(21, 2, 1)$ for each $i = 1, 2$ and (iii) the pair of $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$ yields the 2-PACB$(21, 2, 1)$.

Lemma 4.3. *There exists a* 2*-PACB*$(2p, 2, 2)$ *for any odd prime* $p \geq 5$.

Proof. Let $p \geq 5$ be any odd prime. Further let $\mathcal{F}_1$ and $\mathcal{F}_2$ be a set of two families on Z_{2p} with

$$\begin{aligned}\mathcal{F}_1 &= \{\{0, a+(a-1)p\}, \{2a+p, 4a+p\}, \{0, 12a\}, \{0, 12a+p\} \mid 1 \leq a \leq \\ &\quad (p-1)/2\},\\ \mathcal{F}_2 &= \{\{2a+p, 4a+p\}, \{0, a+(a-1)p\}, \{4a+p, 8a\}, \{4a+p, 8a+p\} \mid \\ &\quad 1 \leq a \leq (p-1)/2\}.\end{aligned}$$

Then, it can be also checked that the $\mathcal{F}_1$ and $\mathcal{F}_2$ form a $(2p, 2, 2, 2)$-PACDF satisfying $\mathcal{I}_1 \subset \mathcal{F}_1$ and $\mathcal{I}_2 \subset \mathcal{F}_2$. Hence by Lemma 3.2 the required 2-PACB$(2p, 2, 2)$ is obtained. □

The following example illustrates Lemma 4.3 with $p = 5$.

Example 4.4. A 2-PACB$(10, 2, 2)$ is obtained by use of Lemma 4.3 with the following families:

$$\begin{aligned}\mathcal{I}_1 &= \{\{0,1\}, \{0,7\}, \{7,9\}, \{9,3\}, \{0,2\}, \{0,7\}\},\\ \mathcal{I}_2 &= \{\{7,9\}, \{9,3\}, \{0,1\}, \{0,7\}, \{9,8\}, \{9,3\}\},\\ \mathcal{I}_1^* &= \{\{0,1\}, \{0,7\}, \{6,8\}, \{2,6\}, \{0,2\}, \{0,5\}, \{4,7\}\},\\ \mathcal{I}_2^* &= \{\{6,8\}, \{2,6\}, \{0,1\}, \{0,7\}, \{9,3\}, \{4,3\}, \{0,5\}\},\\ \mathcal{F}_1 &= \{\{0,1\}, \{0,7\}, \{7,9\}, \{9,3\}, \{0,2\}, \{0,4\}, \{0,7\}, \{0,9\}\},\\ \mathcal{F}_2 &= \{\{7,9\}, \{9,3\}, \{0,1\}, \{0,7\}, \{9,8\}, \{3,6\}, \{9,3\}, \{3,1\}\}.\end{aligned}$$

It can be checked that (i) $\mathcal{I}_i$ and $\mathcal{F}_i$ satisfy the condition (3.2) for $i = 1, 2$, (ii) $\mathcal{F}_i^* = (\mathcal{F}_i \setminus \mathcal{I}_i) \cup \mathcal{I}_i^*$ yields a cyclic B$(10, 2, 2)$ for each $i = 1, 2$ and (iii) the pair of $\mathcal{F}_1^*$ and $\mathcal{F}_2^*$ yields the 2-PACB$(10, 2, 2)$.

On the other hand, some nonexistence result can be shown here. Especially, the nonexistence of a 2-PACB$(4m+2, 2, 1)$ for any integer $m \geq 1$ is given in Matsubara and Kageyama (2013). More generally, we have the following.

Lemma 4.5. *There does not exist a* 2*-PACB*$(4m+2, 2, \lambda)$ *for any integer* $m \geq 1$ *and any odd integer* $\lambda \geq 1$.

Proof. Assume that there exists a 2-PACB$(4m+2, 2, \lambda)$ with families $\mathcal{F}_i$ of full initial blocks and families $\mathcal{S}_i$ of short initial blocks, $i = 1, 2$. Then the fact that λ is odd implies that $|\mathcal{S}_i|$ is odd.

For each pair of full initial blocks $\{a_j, b_j\} \in \mathcal{F}_1$ and $\{c_j, d_j\} \in \mathcal{F}_2$, $1 \leq j \leq |\mathcal{F}_1| = |\mathcal{F}_2|$, $a_j - c_j$ and $a_j - d_j$ have the same parity if and only if $b_j - c_j$ and $b_j - d_j$ have the same parity. Also any two elements $-e, e \in Z_{4m+2}$ have the same parity. Hence, for each j, the number of even elements in $\{\pm(a_j - c_j), \pm(a_j - d_j), \pm(b_j - c_j), \pm(b_j - d_j)\}$ can be divided by 4, that is, the number of even elements in $\Delta(\mathcal{F}_1, \mathcal{F}_2)$ can be divided by 4.

On the other hand, for each pair of short initial blocks $\{a_j, a_j + 2m + 1\} \in \mathcal{S}_1$ and $\{c_j, c_j + 2m + 1\} \in \mathcal{S}_2$, $1 \leq j \leq |\mathcal{S}_1| = |\mathcal{S}_2|$, $\{\pm(a_j - c_j), \pm(a_j - c_j + 2m + 1)\}$ contains two even elements and two odd elements of Z_{4m+2}. Since $|\mathcal{S}_i|$ is odd, the number of even elements in $\Delta_s(\mathcal{S}_1, \mathcal{S}_2)$ cannot be divided by 4.

However, since every element of $Z_{4m+2} \setminus \{0\}$ must appear in the multiset $\Delta(\mathcal{F}_1, \mathcal{F}_2) \cup \Delta_s(\mathcal{S}_1, \mathcal{S}_2)$ exactly 4λ times, the number of even elements in $\Delta(\mathcal{F}_1, \mathcal{F}_2) \cup \Delta_s(\mathcal{S}_1, \mathcal{S}_2)$ must be divided by 4. This is a contradiction. □

We are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. Let $P \geq 5$ be an odd integer with $\gcd(P, 6) = 1$.

Case I : v is odd. By Lemma 1.3 it is sufficient to show the existence of a 2-PACB$(v, 2, 1)$ with $\gcd(v, 9) = 3$. For an odd integer v with $\gcd(v, 9) = 3$, we can let $v = 3P$. If P is an odd prime $p \geq 7$, then Theorem 4.1 gives the required design. When $P = 5$, the 2-PACB$(15, 2, 1)$ is given in Matsubara and Kageyama (2013, Example 3.8). Also if P is not an odd prime, then P has at least two odd prime factors. Hence, since $\gcd(P, 6) = 1$, we can let $P = pq$ with an odd prime $p \geq 5$ and an odd integer $q \geq 5$ satisfying $\gcd(q, 6) = 1$. Then Lemma 2.4 with the 2-PACB$(3p, 2, 1)$, the 2-PACB$(q, 2, 1)$ and the CDM$(4, q)$ obtained by Lemmas 4.1, 1.3 and 2.1, respectively, gives the required 2-PACB$(v, 2, 1)$. By taking copies of the design, a 2-PACB$(v, 2, \lambda)$ can be obtained for any $\lambda \geq 1$.

Case II : v is even. For $m = 1$, any $n \geq 0$ and any odd integer λ, Lemma 4.5 shows that there are no 2-PACB$(2^m 3^n, 2, \lambda)$ and no 2-PACB$(2^m 3^n P, 2, \lambda)$. For any $m \geq 2$, any nonnegative integer $n \neq 1, 2$ and any $\lambda \geq 1$, there are a 2-PACB$(2^m 3^n, 2, \lambda)$ and a 2-PACB$(2^m 3^n P, 2, \lambda)$ by taking copies of the design given in Lemma 1.4.

Now we consider the case of $m \geq 2$ and $n = 1, 2$. Since a 2-PACB$(9, 2, 1)$ (or a $(9, 1, 2, 1)$-PACDF) and a $(12, 2, 2, 1)$-PACDF are given in Matsubara and Kageyama (2013, Example 3.6) and Matsubara and Kageyama (2017, Example A.8), respectively, a $(2^m 3^2, 2^m, 2, 1)$-PACDF and a $(2^{m+1} 3, 2^m, 2, 1)$-PACDF can be obtained by Lemma 2.6 for $m \geq 2$. Furthermore, a 2-PACB$(12, 2, 1)$ is given in Matsubara and Kageyama (2013, Example 3.7). Hence, for $m \geq 2$ and $n = 1, 2$, a 2-PACB$(2^m 3^n, 2, 1)$ can be obtained by Lemma 2.3 with the 2-PACB$(2^m, 2, 1)$ given in Lemma 1.4. Also, for $m \geq 2$ and $n = 1, 2$, a 2-PACB$(2^m 3^n P, 2, 1)$ can be obtained by Lemma 2.4 with the 2-PACB$(P, 2, 1)$ and the CDM$(4, P)$ obtained by Lemmas 1.3 and 2.1, respectively. By taking copies of the design, a 2-PACB$(2^m 3^n, 2, \lambda)$ and a 2-PACB$(2^m 3^n P, 2, \lambda)$ can be obtained for any $m \geq 2$, $n = 1, 2$ and any $\lambda \geq 1$.

For even v, the remaining case is that $m = 1, n \geq 0$ and λ is even. It is sufficient to show the existence of a 2-PACB$(2 \cdot 3^n, 2, 2)$ for $n \geq 1$ and a 2-PACB$(2 \cdot 3^n P, 2, 2)$ for $n \geq 0$. At first, for $n = 0$ and an odd prime $P \geq 5$, there exists a 2-PACB$(2P, 2, 2)$ by Lemma 4.3. If P is not an odd prime, then P has at least two odd prime factors. Hence, we can let $n = 0$ and $P = pq$ with an odd prime $p \geq 5$ and an odd integer $q \geq 5$ satisfying $\gcd(q, 6) = 1$. Then a 2-PACB$(q, 2, 2)$ can be obtained by taking a copy of the design given in Lemma 1.3. Hence, a 2-PACB$(2P, 2, 2)$ can be obtained through Lemma 2.4 with the 2-PACB$(2p, 2, 2)$, the 2-PACB$(q, 2, 2)$ and the CDM$(4, q)$ obtained by Lemmas 4.3, 1.3 and 2.1, respectively. By taking copies of the design, a 2-PACB$(2P, 2, \lambda)$ can be obtained for any even λ.

Next, for $m = 1, n \geq 1$, a 2-PACB$(6, 2, 2)$ and a 2-PACB$(18, 2, 2)$ are given in Examples 1.1 and 1.2. On the other hand, a $(54, 6, 2, 2)$-PACDF can be obtained by Lemma 2.6 with the $(27, 3, 2, 2)$-PACDF obtained by taking a copy of the family given in Lemma 2.2. Then a 2-PACB$(54, 2, 2)$ can be obtained by Lemma 2.3 with the 2-PACB$(6, 2, 2)$. Moreover, a 2-PACB$(6, 2, 2)$ gives a $(2 \cdot 3^n, 3^{n-1}, 2, 2)$-PACDF with $n \geq 4$, by taking Lemma 2.5 with the CDM$(4, 3^{n-1})$ given in Lemma 2.1. Hence, for any $n \geq 4$, a 2-PACB$(2 \cdot 3^n, 2, 2)$ can be obtained by Lemma 2.3 with the $(2 \cdot 3^n, 3^{n-1}, 2, 2)$-PACDF and the 2-PACB$(3^{n-1}, 2, 1)$ obtained by Lemma 1.3. Thus, a 2-PACB$(2 \cdot 3^n, 2, 2)$ can be obtained for any $n \geq 1$. Furthermore, Lemma 2.4 with the above 2-PACB$(2 \cdot 3^n, 2, 2)$, the 2-PACB$(P, 2, 2)$ and the CDM$(4, P)$ obtained by Lemmas 1.3 and 2.1, respectively, show the existence of a 2-PACB$(2 \cdot 3^n P, 2, 2)$ for any $n \geq 1$.

Thus, a 2-PACB$(2 \cdot 3^n, 2, 2)$ for any $n \geq 1$ and a 2-PACB$(2 \cdot 3^n P, 2, 2)$ for any $n \geq 0$ have been obtained. By taking copies of the design, the proof is complete. □

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