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Planes, Designs and List Designs

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Abstract

Existence and construction of combinatorial designs, projective and affine planes, nets has been a topic, extensively studied during last 8-10 decades. Main interest arose from classical projective geometry, group theory and applications in Statistics, in designs of experiments, computer science and digital electronics *etc.* The paper gives a short survey of trends in Discrete Mathematics focused on topics of planes, nets, designs and list designs (designs with multisets as blocks). Main methods used during these decades have been algebraic methods, graph-theoretic methods, probabilistic methods and combinatorial techniques of forming bigger designs by pasting together smaller ones.

Key words: Designs; Planes; Algebraic methods; BIBD; Latin squares.

AMS Subject Classifications: 05B05, 05B25

1. Basic definitions

Let X be a finite set of v elements, $X = \{x_1, x_2, \dots, x_v\}$. We will denote by $\mathbb{P}(X)$, the set all subsets of X and by $\mathbb{P}_k(X)$, the set of all k-subsets of X, $0 \le k \le v$. We will denote by $V(X)$, the set of all rational valued functions $f : \mathbb{P}(X) \to \mathbb{Q}$. Clearly $V(X)$ is a vector space over $\mathbb{Q}$, of dimension 2^v. The set $M(X) \subseteq V(X)$ of all integral valued functions, is clearly a module of rank 2^v over the ring of integers $\mathbb{Z}$. By $\mathbb{N}(X)$ we will denote the set of all nonnegative integral valued functions $f : \mathbb{P}(X) \to \mathbb{N}$. Thus $N(X) \subseteq M(X) \subseteq V(X)$.

Similarly we will denote by $V_k(X)$ the subspace of $V(X)$ of dimension $\binom{v}{k}$ of all rational valued functions $f \in V(X)$ such that $f(B) = 0$ if $|B| \neq k$. Thus when $f \in V_k(X)$, we can also think of f also as a function $f : \mathbb{P}_k(X) \to \mathbb{Q}$. We will denote by $M_k(X)$ the submodule of $M(X)$ of rank $\binom{v}{k}$ of all $f \in M(X)$ such that $f(B) = 0$ if $|B| \neq k$. Thus $f \in M_k(X)$ can also be thought of as an integral valued functions $f : \mathbb{P}_k(X) \to \mathbb{Z}$. Similarly by $N_k(X)$, we will denote the subset of $N(X)$ of all $f \in N(X)$ such that $f(B) = 0$, if $|B| \neq k$. Thus $f \in N_k(X)$, we will also think of as a nonnegative integral valued function $f : \mathbb{P}_k(X) \to \mathbb{N}$. Thus $N_k(X) \subseteq M_k(X) \subseteq V_k(X)$.

For any real valued function $f : X \to \mathbb{R}$, the subset $supp(f) \subseteq X$ defined by $supp(f) = \{x \in X \mid f(x) \neq 0\}$ is called the **support** of the function f. A **list** on X (also called a

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frequency function on X) is a map $\ell : X \to \mathbb{N}$. For each $x \in X$, $\ell(x)$ is called the **multiplicity** or the **frequency** of x in the list ℓ. The subset $supp(\ell) \subseteq X$, is called the **support** of the list ℓ.

A list ℓ on X is essentially a **multiset** on X. We can also visualize the list ℓ on X as a multiset ℓ on X, where for each $x \in X$, $\ell(x)$ gives the number of times the element x occurs in the multiset ℓ.

Example 1: The multiset $\ell = [x, x, y, y, y]$ is the same as the list ℓ defined by $\ell(x) = 2$, $\ell(y) = 3$ and $\ell(z) = 0$ for $z \neq x, y$. Also, the multiset $[x, x, y, y, y]$ is the same as the multiset $[x, y, x, y, y]$ or $[y, y, y, x, x]$ *etc.* In general we can visualize a multiset or a list as an indexed family $(x_i)_{i \in I}$ or an unordered tuple $[x_i | i \in I]$. For example the multiset ℓ considered here is an unordered tuple $[x_i | 1 \leq i \leq 5]$ with $x_1 = x_2 = x$ and $x_3 = x_4 = x_5 = y$. When the indexing set $I = I_n = \{1, 2....n\}$, we may also use the notation $[x_1, x_2, \ldots, x_n]$ for the multiset $[x_i | i \in I_n]$.

We also note that the set $N(X)$ is the set of all lists (or all multisets) on the set $\mathbb{P}(X)$.

We will denote by $|\boldsymbol{\ell}|$ the sum $\sum \ell(x)$, summed over all $x \in X$ and call it the **size** of the list ℓ. If $|\ell| = k$, we will say that ℓ is a ***k*-list** or a ***k*-multiset**. A list ℓ is clearly a **subset** of X if $\ell(x) \in \{0, 1\}$ for all $x \in X$.

We will denote by $\boldsymbol{L(X)}$ the set of all lists on X and by $\boldsymbol{L_k(X)}$ the set of all k-lists on X. We note that $|L_k(X)|$ is the same as the number of ways to choose k elements from the set X with repetitions allowed and is thus given by

$$|L_k(X)| = \binom{m+k-1}{m-1} = \binom{m+k-1}{k}, \; m = |X|. \tag{1}$$

Now suppose $\ell, \ell_1 \in L(X)$. We will say that ℓ_1 is a **sublist** (or a **submultiset** when we consider ℓ as a multiset) of ℓ and denote it by $\boldsymbol{\ell_1 \subseteq \ell}$, if and only if $\ell_1(x) \leq \ell(x)$ for all $x \in X$.

Unlike sets, a multiset or a list ℓ_1 can occur as a submultiset of ℓ in several ways. In fact the number of ways in which the multiset ℓ_1 occurs as a submultiset ℓ is precisely $\boldsymbol{c(\ell, \ell_1)}= \prod_{x \in X} \binom{\ell(x)}{\ell_1(x)}$. We note that a product over an empty set is defined to be 1.

Example 2: Let $\ell = [x_i \mid i \in I_{10}]$ be a multiset on a set $X = \{a, b, c\}$. Suppose $x_1 = x_2 = x_3 = a, x_4 = x_5 = x_6 = b$ and $x_7 = x_8 = x_9 = x_{10} = c$. Thus $\ell = [a, a, a, b, b, b, c, c, c, c]$. Now suppose $\ell_1 = [y_j \mid j \in I_7]$ is the multiset with $y_1 = a,\ y_2 = y_3 = b,\ y_4 = y_5 = y_6 = c,\ y_7 = a$. It can be easily seen that the multiset $[x_i \mid i \in A]$ is same as the multiset ℓ_1 if and only if $A = \{j_n \mid 1 \leq n \leq 7\}$, where $j_1, j_2 \in I_3,\ j_3, j_4 \in \{4, 5, 6\}$ and $j_5, j_6, j_7 \in \{7, 8, 9, 10\}$. Thus $c(\ell, \ell_1) = \binom{3}{2}\binom{3}{2}\binom{4}{3} = 36$

A **design** is a pair $D = (X, f)$, where X is finite set and $f \in N(X)$, *i.e.*, f is nonnegative integral valued function on $\mathbb{P}(X)$. Thus f is just a list on $\mathbb{P}(X)$. The elements of the set X are called **points** or **treatments** of the design D. When $f(B) \neq 0$, the subset$B$ of X is called a **block** of the design D. For a block B of D, $f(B)$ gives the number of times the block B is **repeated** in the design D, it is also called the **frequency** of the block B in the design D. The number $|B|$ is said to be the **size** of the block B. If all blocks of a design D have size k, then k is said to be the **block-size** of the design D. When the design has the block size k, clearly $f \in N_k(X)$.

Similarly we define a **signed design** to be a pair $D = (X, f)$, where $f \in M(X)$ and a **rational** design to be a pair $D = (X, f)$, where $f \in V(X)$. The blocks, frequency, block-size are similarly defined for these designs too. Note that the frequency of a block of a signed design is an integer thus it may be even negative and for a rational design it is a rational number. Signed designs or rational designs are useful, as a tool to study and construct designs.

When the set $X = \{x_1, x_2, \ldots, x_v\}$ of points is fixed, we may consider f, it self as the design (X, f).

Designs have been one of the main focus of studies in discrete mathematics, since 1940's at least. Specially studies of projective and affine planes, nets and t-designs have been a dominating factor in the field of discrete mathematics for last several decades. These studies have also influenced many other areas. In particular a lot of developments in the study of 2-designs, also called BIBD, was done by Statisticians. Graph and Hypergraph theory, Group theory, Computer science, Applied Algebra, Digital Electronics are some other areas which have been influenced by studies in designs and vice verse.

In the next section we will give a short survey of developments in projective planes and nets in this era. While in section 3 we will do the same in the case of more general t-designs. Note that BIBD's are particular case of t-designs. in fact they are exactly 2-designs. Also symmetric BIBD's with $\lambda = 1$ are precisely Projective planes.

We will give some references in these sections. But let us end this section with mention of three good books discussing these aspects, by Hughes and Piper on projective planes Hughes and Piper (1970), by Beth Jugnickel and Lenz on Design Theory Beth *et al.* (2000) and by Raghavarao on designs and their applications in designs of experiments Raghavarao (1971). Another classic book is Finite Geometries book by Peter Dembowski, a great reference book for both geometries and designs Dembowski (1968)

2. Projective planes and nets

In this section we will study basically finite geometries. These are special cases of designs. In a geometry, generally treatments are called points and blocks are called **lines**. We will also use usual terms from geomtry. For example if two or more points are on a line, they are also called **collinear** and similarly if three or more lines are on the same point, they are called **concurrent**.

A **partial linear space** is a design $D = (X, f)$, such that any pair of distinct points $x, y \in X$ is on at most one line of D. Such a space is called a **linear space**, if every pair of points $x, y \in X$ is on a unique line of D. Note that a linear space is essentially the same as partially balanced design (PBD) with $\lambda = 1$ (see Section 2 for definition of PBD) .

A **projective space** is a linear space $D = (X, f)$, containing four points no three of which are collinear and which satisfies the following Pasch's axiom.

Pasch's axiom: Suppose ℓ_1 and ℓ_2 are two distinct intersecting lines of D, *i.e.*, $\ell_1 \neq \ell_2$ and there is a point $x \in \ell_1 \cap \ell_2$. Also suppose ℓ_3 and ℓ_4 are two lines of D, which are **transversal** to ℓ_1 and ℓ_2, *i.e.*, both of them are not on x but each of them intersects both ℓ_1 and ℓ_2. Then ℓ_3 and ℓ_4 are also intersecting lines.

Each projective space has a unique **dimension** (see for more details Hughes and Piper (1970) or Veblen and Young (1938)). A classical theorem of projective geometry states that every projective space of dimension 3 or more is essentially coordinatized by a field.

The result is not true for a **projective plane**, *i.e.*, a projective space of dimension 2. For many years, people have believed that a finite projective plane with no proper subplane is coordinatized by a prime field (Example below describes, what generally one means by coordinatizing a plane) and that the order of a finite projective plane (order is defined below), is a power of a prime number. Axiomatizing and classifying projective planes and related structures has been a very active field. Included among a large number of mathematicians, who have made significant contributions are Pasch, Hilbert, Dickson, Albert, Hall and Bose. Some good sources for the results and theory are Albert (1961),Hall (1943) Hughes and Piper (1970) and Veblen and Young (1938).

Though the problem of classifying projective planes has been studied for more than 200 years, a spurt in the activity during last few decades was caused by Marshal Hall's via his paper Hall (1943) and by R.C. Bose via his Paper in 1939 Bose (1939). While Marshal Hall connected the problem with many algebraic structures, groups, permutation groups, fields, near fields, nonassociative rings ternary rings *etc.*, Bose was interested in looking at constructing designs, specially BIBD's from projective planes, affine planes, nets*etc.* and even from higher dimensional geometries. He used these designs for the designs of experiments, a branch of statistics, which was just evolving then.These papers made many researchers from all these areas, finite group theory, number theory, algebra, nonassociative algebras, statistics, graph theorists, computer scientists and digital electronics engineers interested in these geometric and designs problems. Perhaps more than 1000 remarkable papers may have evolved on planes nets and t-designs, as a result of these two exceptional path-breaking papers. We will describe some of the results which evolved as a result of these two papers. We will also discuss some recent work of the author (Singhi (2010), Singhi (2009).

We will restrict our discussion in this section essentially to projective and affine planes and nets. As already remarked projective planes, also affine planes are examples of BIBD's, which will be studied in the next section.

It is not too difficult to see (Hughes and Piper (1970)) from the definition of a projective space that a projective space of dimension 2, *i.e.*, a projective plane is a design $D = (X, f)$, satisfying the following conditions and conversely every such design is a projective plane.
(A). D is a linear space, *i.e.*, given any two distinct points $x, y \in X$, there is a unique line (block) ℓ of D such that $x, y \in \ell$.
(B). Any two lines of D intersect in a unique point.

(C). There is exist 4 points in X, no three of which are collinear.
(D). All lines are on the same number $n+1$ points.
(E). All points are on exactly the same number $n+1$ lines.
(F). Total number of points or lines of D are n^2+n+1.

The common number n is called **order** of the plane D.

An **affine plane** is obtained from a projective plane of order n by removing a line and all the points on it. The number n is also called the **order** of the affine plane. It can be easily seen (see Hughes and Piper (1970)) that an affine plane of order n is also a linear space, in which every line is on exactly n points and every point is on exactly $n+1$ lines. Conversely every linear space satisfying these conditions is an affine plane of order n. An affine plane of order n has exactly n^2 points and n^2+n lines.

A **parallel class** is a partial linear space $D=(X,f)$ is a set of lines of D such that every point of D is on exactly one line of this class. Thus a parallel class of partial linear space D is actually a partition of X into lines. It can be easily seen that in an affine plane of order n there are exactly $n+1$ parallel classes, which are mutually disjoint and they partition the set of lines of the affine plane.

A **net**, is a partial linear space $D=(X,f)$ on n^2 points, such that each line of D is on exactly n points and in all there are nr lines, $r\geq 2$, partitioned into r, parallel classes. The net is said to have **order** n and r parallel classes. We will also say that D is $\mathbf{Net}(n,r)$. It can be easily seen that $r\leq n+1$ for a $\mathrm{Net}(n,r)$. When $r=n+1$, one can see that the net is actually an affine plane. It is well-known that a $\mathrm{Net}(n,r)$ gives rise to $r-2$ mutually orthogonal latin squares and conversely. Nets behave as if $n+1-r$ parallel classes are removed from an affine plane of order n. Though not all nets can be completed to an affine plane. Nets were formally defined by Bruck , who studied general problem of embedding a net into an affine plane (seeBruck (1963)). Though as mutually orthogonal latin squares they were studied much earlier, (see Bose (1939)). In particular the embedding problem was solved for the case when $r=n-1$ by Marshal Hall and Connor and by Shrikhande. Bruck proved a much more general result. Bruck's paper is also well known for describing a basic technique, started by Hoffman, of using maximal claws in a graph to find large cliques. Such cliques correspond to adding more lines to the net. Bruck's paper resulted in a lot of activity in studying such problems and connected studies of designs with graphs. Almost the same time R.C. Bose generalized Bruck's ideas to define a strongly regular graph and also generalizing Bruck's nets to a much more general class of partial linear spaces. He called them partial geometries Bose (1963). Strongly regular graphs were studied earlier by statisticians as 2 class association schemes. But looking at them as graphs gave a new thrust to this area and many researchers both in mathematics and statistics started looking at such problems. Later these ideas were further generalized to multigraphs and partial geometric designs by Bose Shrikhande and Singhi and used to solve a much more general problem of embedding of a residual BIBD into a symmetric BIBD Bose *et al.* (1976). Note that projective planes are particular case of symmetric BIBD's.

Example 3: (a). Let F be a finite field of order n. Let X be the set of all ordered pairs of F, $X=\{(x,y)|x,y\in F\}$. Let $m,c\in F$. Define $\ell(m,c)=\{(x,y)\in X|y=mx+c\}$. Also define for each $d\in F,[d]=\{(x,y)\in X|x=d\}$. Let $D=(X,f)$, be the design, where f is defined as follows. For any $B\in\mathbb{P}(X)$, $f(B)=1$ if $B=\ell(m,c)$, $m,c\in F$ or $B=[d]$, $d\in F$

and $f(B) = 0$ in all other cases. It can be easily checked that D is an affine plane of order n, Hughes and Piper (1970). For the line $\ell(m, c)$, m is called the **slope** of the line and c the ***y*-intercept**. All lines of D with a given slope m are parallel and they form a parallel class. Similarly all lines $[d]$, $d \in F$, also form a parallel class, the so called lines parallel to y-axis. The line $[0]$ may be thought of as y-axis and the line $\ell(0, 0)$ is the x-axis. We say that the affine plane D is **coordinatized** by the field F.

One can also use other algebraic structures like quasi fields, near fields, nonassociative division rings *etc.* to construct an affine plane in quite similar manner.

(b). Note that in case we use real field $\mathbb{R}$ instead of a finite field, the above construction exactly gives us the usual real affine plane which we study in high school geometry.

(c). Instead of pairs, now we take a set X_1 of all triplets (x, y, z), $x, y, z \in F$. For each triplet ℓ, m, s of elements of F, define $[\ell, m, s] = \{(x, y, z) \in X_1 | \ell x + my + sz = 0\}$. Let $D_1 = (X_1, f_1)$ be a design, where f_1 is defined by $f_1(B) = 1$ if $B = [\ell, m, s]$ for some $\ell, m, s \in F$ and $f_1(B) = 0$ in all other cases. It can be easily seen that D_1 is a projective plane of order n, coordinatized by the field F.

Let $D = (X, f)$ be a projective plane (or affine plane). A projective plane (resp. affine plane) $D_1 = (Y, g)$ is said to be **subplane** of D if every line of D_1 is a subset of a line of D. A projective plane (or affine plane) is said to be a **prime plane** if it has no proper subplane.

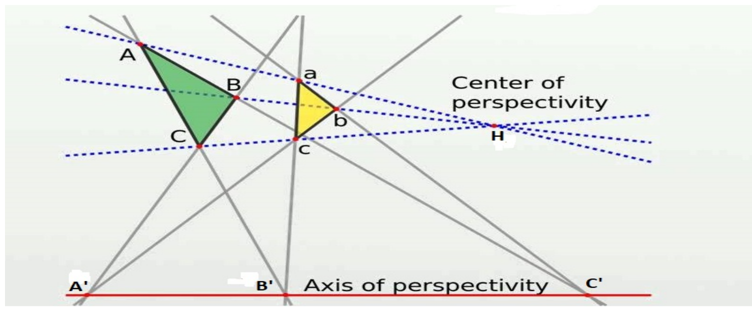
As remarked earlier, apart from the field plane, *i.e.*, the projective or affine plane coordiatized by a finite field, there are many other examples of projective planes for example coordinatized by quasi fields or near fields *etc.* But all known planes so far have order a power of prime. Also all prime planes so far known, have a prime order and are in fact the prime field plane . This gives rise to the following Conjecture.

Projective plane conjecture:
(a). Order of any projective plane is power of a prime number.
(b). A prime projective plane is coordinatized by a prime field.

A lot of research in the area of projective planes has been motivated by these conjectures and related problems. Though the problem is hard.There are very interesting examples of planes, which defy all possibilities of relationship with a field but so far no example of prime planes has been found which is not a prime field plane. Even though there are many planes, for example the so called Hughes planes, which have order q^2, where q is a power of an odd prime, which have subplanes of order 2 or 3 Caliskan and Moorhouse (2011). Such examples show difficulty in solving the problem of classifying projective planes.

As noted earlier Projective spaces of dimension more than 2 are unique and coordinatized by fields. One reason they are coordinatized by fields, is that they satisfy Desargues's Thoerem, which we describe now. Two triangles ABC and abc in a projective space are said to be **centrally perspective** if the lines Aa, Bb, Cc are concurrent, at a point say H. The point H is called the **centre of perspectivity** of the triangles ABC and abc. Similarly dually consider the three points A', B' and C' of intersection of three pairs of lines

(BC, bc), (AC, ac) and (AB, ab) respectively. Suppose the points A', B' and C' are collinear. Then The triangles ABC and abc are said to be **axially perspective** and the line $A'B'C'$ is called **axis of perspectivity**. Desargues's theorem says that if any two triangles in a projective space of dimension 3 or more are centrally perspective then they are also axially perspective.



The above figure with 10 Points A, B, C, a, b, c centre of perspectivity H, and three points A', B' and C' on the axis of perspectivity, together with the 10 lines on these points as given in the above figure is called a **Desargues's Configuration**, Hughes and Piper (1970), ChapterIV. Note that some times H may be on the axis of perspectivity too.

In a general projective plane, it is not true that two triangles which are centrally perspective are also axially perspective. But it is known that every projective plane has several pairs of centrally perspective triangles, which are also axially perspective. Desargues's configuarations play an important role in classification projective planes Hughes and Piper (1970), Chapter IV. We will discuss this later in this section.

Marshal Hall's 1943 paper was a landmark. Among many interesting ideas in the paper, one of them, idea of associating a ternary ring with a projective plane gave a completely new color to the study of these planes. If you look at Example 3(a) above, the affine plane coordinatized by a finite field F, the equation of the line $\ell(m, c)$ not parallel to the y-axis is $y = mx + c$. Marshal Hall had a bright idea that in the case of looking at projective plane, instead of looking at product and addition as operations in field or other such structures, it seems more natural to look at a ternary operation $\tau(m, x, c)$, instead of addition and product in the field, where $\tau(m, x, c) = mx + c$. He was indeed right. He defined a general such ternary ring, called it, planar ternary ring, which we will describe below and he showed that Every projective or affine plane can be coordinatized by such a ring. This insight opened up study of projective planes to a new areas. A problem which looked so far to be of geometry type, suddenly started looking equally as an algebraic problem. We now define planar ternary rings, defined by Hall in this paper.

Let S be a finite set. An ordered pair $R = (S, \tau)$ is said to be a **ternary ring**, if $\tau : S \times S \times S \to S$. When the ternary operation τ is fixed, we will call S itself, the ternary ring. Thus by a ternary ring S, we will mean, a finite set S, with a ternary operation τ on it.

A ternary ring S is said to be a **planar ternary ring** if there are two special elements $0, 1 \in S$ and the following conditions are satisfied.

(A).$\tau(x, 0, c) = \tau(0, x, c) = k$ for all $x, c \in S$.
(B). $\tau(x, 1, 0) = \tau(1, x, 0) = x$ for all $x \in S$
(C). Given $x, y, m \in S$, there is a unique $c \in S$ such that $\tau(x, m, c) = y$.
(D) Given $m, c, k, p \in S,\ m \neq k$, there is unique $x \in S$ such that $\tau(x, m, c) = \tau(x, k, p)$.

Example 4: (A). Let S be a planar ternary ring. Construct a Design $D = (X, f)$, in exactly the same manner as we did, while constructing Affine plane from a field in the previous Example. Only this time equations of lines not parallel to y-axis will be $y = \tau(x, m, c)$, instead of $y = mx + c$. It can be easily seen that the design we get is an Affine plane. A projective plane can always be obtained from Affine plane. This construction was given by Hall in his paper in 1943 (See Hughes and Piper (1970), ChapterV).

(B). Conversely given any four points in a projective plane D of order n, no three of which are collinear, using them we can construct a planar ternary ring $S, |S| = n$, such that D is coordinatized by S, as described in (A) above and further all points on the line with slope 1 are of the type $(x, x),\ x \in S$. For more details see Hughes and Piper (1970), Chapter V, Hall's method.

This example shows that studying projective planes and planar ternary rings is essentially the same thing. Thus the problem can be studied as a geometric problem or algebraic problem. We will use word **PTR** for a planar ternary ring.

Two PTR S_1 and S_2 are said to be **isotopic** if they coordinatize the same projective plane. Unlike the field case, isotopic here does not imply isomorphic.

Let us define a few more terms for a PTR to understand them better and also to see that how they behave almost like fields and yet are very different too. Let S be a planar ternary ring. Let $x, y \in S$. Define $x + y = \tau(x, 1, y)$ and $xy = \tau(x, y, 0)$. When S is a field, *i.e.*, $\tau(x, m, k) = xm + k$ where addition and multiplication are the field operation, clearly the above definitions of addition and multiplication in this case, are the same as the addition and multiplication in the field. Also, When S is a field, S is a group under addition, the additive group and $S^* = S/\{0\}$ is a group, the multiplicative group of the field. This is not true when S is not a field, addition or multiplication may not be associative, in a general PTR. However in every PTR S, both $(S, +)$ and $(S^*, .)$ are loops under the addition and multiplication, as defined above. We will call $(S, +)$, the **additive loop** of the planar ternary ring S and similarly $(S^*, .)$, the **multiplicative loop** of S. A PTR S is said to be **linear**, if $\tau(x, m, c) = xm + c$, for all $x, m, c \in S$. A PTR S is called a **quasifield** if the additive loop (S,+) is a group, S is linear and satisfies left distributive law, *i.e.* $a(b + c) = ab + ac$ for all $a, b, c \in S$. A quasifield satisfying right distributive law also is called **division ring**. Planes coordinatized by quasifields are very special, they are called **translation planes**. We will describe group theoretic and geometric significance of them.

Let us first see in terms of desarguesian configurations. Let H be a point of a projective plane and ℓ be a line. A projective plane is said to be (H, ℓ)**- desarguesian** if for every pair of triangles ABC and abc which are centrally perspsective with H, as the centre of perspectivity (see above figure of desargues's configuration) and ℓ as the possible axis of perspectivity, *i.e.*, any two of the points A', B', C' are on ℓ, the two triangles ABC and abc are also axially perspective and the third point is also on ℓ. Thus ℓ is the axis of perspectivity.

Thus being (H, ℓ)-desarguesian, essentially says that any two centrally perspective triangles with H as centre of perspectivity, and ℓ as "possible" axis of perspectivity, are actually also axially perspective, with ℓ as axis of perspectivity.

We now describe what is meant by (H, ℓ)-transitive. For a projective plane $D = (X, f)$, we will denote by $aut(D)$, the group of all automorphisms of D, i,e., permutations of the set X which take lines into lines. For any $x \in X,\ \sigma \in aut(D)$, we will denote by $\sigma(x)$, the image of x under σ. and similarly for a line ℓ of D, we will denote by $\sigma(\ell)$ the line, which is image of ℓ under σ. If $\sigma(x) = x$, the point x is said to be **fixed** by σ. If further $\sigma(m) = m$ for all lines m of the plane which are on x, then we say that point x is **fixed line-wise** by σ. We can similarly define a line ℓ to be **fixed point-wise**, if $\sigma(\ell) = \ell$ and $\sigma(x) = x$ for all $x \in \ell$. An automorphism $\sigma \in Aut(D)$ is said to be a (x, ℓ)**-perspectivity**, if σ fixes x line-wise and line ℓ point-wise.

Now the projective plane D is said to be $\boldsymbol{(H, \ell)}$**-transitive**, if for all points $y, w \in X$ such that $x \neq y,\ x \neq w,\ y \notin \ell,\ w \notin \ell$ and x, y, w are collinear, there is an (H, ℓ)-perspectivity σ such that $\sigma(y) = \sigma(w)$. Thus (H, ℓ)-transitive essentially says that (H, ℓ)-perspectivities act transitively. A line ℓ is said to be a **translation line** of the projective plane D, if D is (H, ℓ)-transitive for all points $H \in \ell$. If D has a translation line then D is called a **translation plane**. The following two theorems show how closely projective geometry and algebra are related. They show properties of algebraic structures coordinatizing the plane, transitivity properties of automorphism groups and geometric properties like deasarguessian configurations occur mutually together, Hughes and Piper (1970), Chapter IV, Dembowski (1968)

Theorem 1: A projective Plane D is (H, ℓ)-transitive if and only if D is (H, ℓ)-desarguesian.

Theorem 2: A plane is coordinatized by a quasifield, if and only if it is a translation plane.

Almost every algebraic structure which coordinatizes a projective plane can similarly be related with some similar transitivity of an automorphism group as well as occurrence of desarguesian configurations. This relationship inspired a lot of work in this area during the second half of last century. Still many interesting research papers appear regularly studying such aspects. Another idea which similarly resulted in a lot of activity was started by Lenz and Barlotti. Now it is known as Lenz-Barlotti classification Dembowski (1968) pages 123-126. They looked at non-desarguesian planes, *i.e.*, those not coordinatized by a field. It has clearly then a non-desarguesian configuration. They classified all such non-desarguesian configurations, into different classes, which could occur in a plane. Assuming such structures they classified all planes in many of these classes. Many others also completed similar work for different such classes.

Still the basic problem I described above as conjectures remains. We will end this section with another a very different style of studying projective planes via PTR. The problem to classify the planes is the same as classifying PTR. These ternary rings have many properties similar to fields, for example any subring of a PTR is a PTR. Classifying finite fields has a very direct path, one starts with the ring of integers $\mathbb{Z}$ which may be considered as a free ring generated by 1 subject to the usual rules of commutativity, associativity, distributivity, linearity (one can think of $\mathbb{Z}$ as ternary ring, satisfying linearity as we defined earlier for fields). We will write cadl rules in short for these 4 rules. One looks at maximal ideals in this free ring $\mathbb{Z}$, to get prime fields as quotients. Other finite fields are obtained then by considering polynomial rings over these prime fields. The polynomial rings also may be thought of as free rings generated by starting variables subject to cadl laws. One difficulty with general PTR is that even though they are very similar to fields, yet many of them satisfy none of these cadl laws. In the papers Singhi (2010) and Singhi (2009), some more general structures than PTR which are more like ring of integers, instead fields were defined. Free such structures were defined and constructed, which in a way corresponded to some kind of generalized ring of "Integers" and "polynomials" which did not satisfy cadl laws. Actual ring of integers or polynomials are quotients of these general ternary rings quotiented by ideals whose elements are all cadl laws. It was shown that every such PTR is quotient of a maximal ideal in these rings. Though it is not clear that all maximal ideals give PTR. The idea is to develop a language without using cadl laws to imitate classification theory of finite (or infinite) fields. In this connection it may be interesting to observe that Albert studied division rings in some what similar manner Albert (1961). He defined a "generalized twisting" of a field to get such division rings and conjectured that all division rings are obtained from a field in this manner. The conjecture is known to be true for dimension 3 or 4.

The language developed in above two papers though very general does not still include all PTR's. For example two isotopic PTR may have very different structure. In this connection it may be interesting to look at structures more general than Hall's PTR. Grari studies such general structures (see Grari (2004)).

3. t-designs

In this section we will review some basic construction methods and results on t-designs which evolved over last few decades and describe a generalization of t-designs to t-list designs, given in a joint paper of the author with Raychaudhuri, Singhi and Raychaudhuri (2012).

Let $D = (X, f)$ be a design. Thus $f \in N(X)$. Define a function $\boldsymbol{\partial_t}$: $N(X) \to N_t(X)$ as follows. Let $\partial_t(f)(T) =$ the number of blocks containing T, $T \in \mathbb{P}_t(X)$. Thus

$$\partial_t(f)(T) = \textstyle\sum_{T \in B} f(B), \text{ where the sum is over all } B \in \mathbb{P}(X)$$

The number of blocks in $D = \partial_0(f)(\emptyset)$, will be denoted by $\boldsymbol{b}$ or $\boldsymbol{b(D)}$.

If $\partial_1(f)(\{x\}) = \partial_1(f)(\{y\})$ for $x, y \in X$, we will denote the common value of $\partial_1(f)(\{x\})$, $x \in X$ by $\boldsymbol{r}$ or $\boldsymbol{r(D)}$.

We will denote by $\boldsymbol{\partial_{t,k}}$, the restriction of ∂_t on $N_k(X)$. Thus $\partial_{t,k} : N_k(X) \to N_t(X)$, is defined by $\partial_{t,k}(f)(T) = \sum_{T \in B} f(B)$, where the sum is over all $B \in \mathbb{P}_k(X)$.

A design $D = (x, f)$ is said to be a $\boldsymbol{t}$**-**$\boldsymbol{(v, k, \lambda)}$**-design** if D has, v points, *i.e.* $|X| = v$, block size in D is k and $\partial_t(f)(T) = \lambda$ for every t-subset T of X.

If $D = (X, f)$ is a signed design (or rational design), all the above terms signed t-(v, k, λ)-designs and rational t-(v, k, λ)-designs *etc.* are similarly defined in those cases too. In particular $\boldsymbol{\partial_t}$ and $\boldsymbol{\partial_{t,k}}$ are similarly defined over $M(X)$ and $M_k(X)$ (or for $V(x)$ and $V_k(X)$) also.

Remark 1: It is easy to see that if $D = (x, f)$ is a t-(v, k, λ)-design (or signed design or rational design) then

(A). $\partial_t(f)(W) = \lambda \frac{\binom{v-w}{t-w}}{\binom{k-w}{t-w}}$ for every w-subset W of X, $0 \leq w \leq t$.

(B). Thus in particular $b = \lambda \frac{\binom{v}{t}}{\binom{k}{t}}$ and $r = \lambda \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}$

(C). Thus a t-(v, k, λ)-design is also a w-(v, k, λ_w)-design with $\lambda_w = \lambda \frac{\binom{v-w}{t-w}}{\binom{k-w}{t-w}}$, $0 \leq w \leq t$.

When $t = 2$, a 2-(v, k, λ)-design is also called a **BIBD** (Balanced Incomplete Block Design). A BIBD is also called **BIBD$(\boldsymbol{v, b, r, k, \lambda})$** or **$(\boldsymbol{v, b, r, k, \lambda})$-design**. It is difficult to construct t-designs with $t \geq 3$, specially when parameters are small. Very few such designs with small parameters are known. And yet there are interesting results which show that all such designs, with v sufficiently large, exist. We will discuss these results, how they evolved over the decades.

A t-design with $\lambda = 1$ is also called a **Steiner system**, named after Swiss Mathematician Steiner, who studied them almost 200 years back. Most of the focus is on BIBD's, specially because they are very useful in Statistics, in Designs of Experiments. Perhaps Fisher and Yates were the ones, who formalized using such designs for designs of experiments. Main current interest arose with the 1939 paper of R.C.Bose , who was perhaps the first one to methodically study them by using algebra, geometry and number theory (see Bose (1939)). He used affine and projective planes, finite fields, difference sets *etc.* to construct them. Some good reference books are Raghavarao's book on designs and their applications in designs of experiments Raghavarao (1971), Colbourn and Dinitz Handbook of Combinatorial Designs Colbourn and Dinitz (2006), Beth Jugnickel and Lenz book on Design Theory Beth *et al.* (2000) and Peter Dembowski's book on finite geometries Dembowski (1968).

Example 5: (A). Suppose $D = (x, f)$ is a projective plane of order n, then it can be easily seen that D is a 2-$(n^2+n+1, n+1, 1)$ design, *i.e.* BIBD$(n^2+n+1, n^2+n+1, n+1, n+1, 1)$. Interestingly, thus in this design, the number of blocks, $b = v$, the number of treatments. A 2-(v, k, λ)-design, in which $b = v$, is called a **SBIBD$(\boldsymbol{v, k, \lambda})$, (symmetric BIBD).** Conversely every SBIBD with $\lambda = 1$ is essentially a projective plane.

One can similarly construct an SBIBD with higher lambda using projective spaces of higher dimensions. Blocks in these designs are the hyperplanes. In an SBIBD(v, k, λ), any two blocks intersect in exactly λ treatments. SBIBD's are very challenging objects of study. There are many unsolved problems about them. We briefly mentioned them in the Section 2 also, while discussing nets. The above example from projective planes shows that there are infinitely many SBIBD's with $\lambda = 1$. SBIBD's with $\lambda = 2$ are called biplanes.

Interestingly only finitely many SBIBD with a given λ are known for any $\lambda \geq 2$. There is a well-known conjecture formulated by Marashal Hall Jr.

Conjecture. For any given integer $m \geq 2$, there are only finitely many SBIBD with $\lambda = m$.

(B). Now suppose $D = (X, f)$ is an affine plane of order n. Again it can be easily seen that D is a 2-$(n^2, n, 1)$-design, a BIBD$(n^2, n^2 + n, n + 1, n, 1)$. As we had noted in Section 2, blocks of this design can be partitioned into parallel classes.

A design $D = (X, f)$ is said to be **resolvable**, if blocks of D can be partitioned into parallel classes. Affine planes and nets are examples of resolvable designs.

A resolvable design is called an **affine design**, if any two blocks from different parallel classes intersect in exactly the same number of treatments. In an affine plane clearly they intersect in exactly one treatment. Thus an affine plane is also an affine design. Conversely every BIBD with $\lambda = 1$, which is an affine design, actually is obtained form the affine plane in this manner.

Affine spaces of higher dimension can also be used to form similarly affine designs. The hyperplanes of affine spaces form the blocks of such designs.

Thus in affine designs intersection number of two blocks takes only two possible values, one of which is 0. BIBD's in which blocks intersect in only two possible values are called **quasi-symmetric BIBD's**. These are sort of designs next best to symmetric BIBD's. As remarked in (A), blocks in a symmetric BIBD intersect in a unique value. qausi-symmetric BIBD's have been extensively studied. It has become a subject by itself. A good source for results and theory on quasi-symmetric designs is the book by M.S. Shrikhande and S.S. Sane (Shrikhande and Sane (1991)).

(C). In High school geometry we learn that there is a unique circle through any 3 noncollinear points in a real affine plane. Suppose we take all "‘circles" in an affine plane and an extra point say ∞, added to every line of the plane, it is not hard to see these new extended lines together with circles considered as blocks, give us a 3-design, in the sense that any 3 points are on a unique block.

Some what similar construction can be carried out with an affine plane over a finite field. What we get 3-$(n^2 + 1, n + 1, 1)$-design as an extension of an affine plane of order n.

Formally such a design is constructed from an ovoid in a projective space of dimension 3 over a field of order n. An **ovoid** in this 3-dimensional projective space is a set of $n^2 + 1$ points, no three of which are collinear. It can be shown that every hyperplane of this projective space (it is actually a plane since we have taken projective space of dimension 3), intersects this ovoid in 0 or $n + 1$ points.

When we take all these sets of intersections with the ovoid of size $n + 1$ as blocks, we get a 3-$(n^2 + 1, n + 1, 1)$- design.

Any 3-$(n^2 + 1, n + 1, 1)$-design is called an **inversive plane**. There are some interesting unsolved problems associated with study of inversive planes. For more details see Beth *et al.* (2000) or Dembowski (1968).

Remark 2: Necessary conditions for existence of t-designs.
From Remark 1. It is clear that a necessary conditions for existence of t-designs is that

$$\lambda\binom{v-w}{t-w} = 0 \ (mod \ \binom{k-w}{t-w}), \ 0 \le w \le t$$

Remark 3: Basic problem in the theory of t-(v,k,λ)-designs.

A. **Existence Problem:**. Characterize all quadruples t-(v,k,λ) satisfying the necessary conditions of Remark 2. for which there exists a t-(v,k,λ) designs.

B. **Classifying problem:**. For a given t-(v,k,λ) satisfying necessary conditions, construct all non-isomorphic t-(v,k,λ)-designs.

Main effort in the subject has been to solve the Existence Problem. This general problem is quite hard. Even for very small parameters designs are not known, nor one can prove that they do not exist. Some examples of such parameters are 2-$(22,8,4)$-design, (BIBD(22,33,12,84)), 2-(157,13,1)-design (projective plane of order 12) or 6-designs with $\lambda = 1$ for small v. Even with best computers one can not do much in such cases. May be AI and simulations could be used to study such problems properly. There are 1000's of papers on this topic, still many designs in the useful range for Statistical studies are not known.

On the other hand, Bose's 1939 paper, Bose (1939), started a spurt in research activity of studies of t-designs, specially BIBD's, which still continues. Constructing new families of BIBD's whose existence is not known or which are not isomorphic to already known designs, still creates a lot of new interest in the subject.

Remark 4: Constructing t-designs.
Two types of types of methods are used generally to construct BIBD's or t designs.

(A). **Direct construction methods:**.
One constructs a new design or a new family of design directly by using some algebraic objects like difference sets, transitive permutation groups *etc.* Or one constructs them from geometric objects like projective spaces, affine space or ovals *etc.* Some examples we have described in the above Example 5. Bose himself gave some examples of such constructions in his paper. Among many others, who have given very interesting such constructions, included are S.S. Shrikhande, Marshal Hall, Wilson, Ray-Chaudhuri, Hanani (see Wilson (1972a), Ray-Chaudhuri and Wilson (1971), Wilson (1973),Ray-Chaudhuri and Singhi (1988), Colbourn and Dinitz (2006), Raghavarao (1971)).

(B). **Composition Techniques:**.
Smaller Designs are used to paste together a bigger design by using a base design. We will discuss some composition methods evolved, later in this section.

Let us just note here first that these studies had led to a conjecture, the so called the existence conjecture. It stated that if v is sufficiently large compared k and λ then necessary conditions of existence for a t-(v, k, λ)-design are sufficient. Conjecture was proved by Wilson in 1975 for the $t = 2$ case, *i.e.*, BIBD's, We will describe some details of his method later. Though many of Wilson's ideas were generalized for all t- designs. But the conjecture for $t \geq 2$, remained unsolved until 2014. The conjecture was proved in the general case by Keevash in 2014 by very different methods. He used probabilistic arguments to prove the conjecture. His method may be considered as a modification of the famous Rodl's nibble method (see Rodl (1985)). Though Keevash is able to get exactness of a very different order, which was needed to construct such exact designs. Keevash calls his method randomized algebraic construction (see Keevash (2014), Keevash (2015), See also an interesting lecture by Kalai, explaining Keevash's papers, Kalai (2015)). It is a bit amazing to see that probabilistic methods can give such exact geometric objects, even though v is large for such objects. Perhaps Kim and Vu were among the first ones to show such a potential of probabilistic methods in finding such exact constructions. They showed existence of small complete arcs in projective planes with high probability (Kim and Vu (2003)).

Though Keevash's theorem implies that all t-designs with sufficiently large v exist, still the existence problem in many of the general useful practical cases remains unsolved. There is a possibility that Wilson's method and composition techniques could be modified to get existence problems solved for practical cases. In fact in three interesting papers Blanchard proved some thing similar to existence conjecture for transversal designs or orthogonal arrays, using such methods (see Blanchard (1995b), Blanchard (1995a), Blanchard (1997)). We now describe in short how one of such basic composition technique evolved and many similar composition techniques were developed. Wilson also developed some of them. Such techniques formed the main core of his proof of the existence conjecture in the BIBD case.

A design $D = (X, F)$ is called a **PBD** (pairwise balanced design) with index λ if for all $x, y \in X$, the number of blocks of D containing x, y is λ. Thus $\sum_{x,y \in B} f(B) = \lambda$ for all $x, y \in X$. A PBD is similar to a 2-(v, k, λ)-design, only now the block size is not constant. Let us define for a PBD, $D = (X, f)$, of index λ, the set K (or $K(D)$) to be the set of all block sizes of D, *i.e.*, $K = \{k \in \mathbb{N}|$ there exists $B \in \mathbb{P}(X)$ such that $f(B) \neq 0$ and $|b| = k\}$. We will say that D is a $\mathbf{PBD}(\boldsymbol{v}, \boldsymbol{K}, \boldsymbol{\lambda})$ or a $(\boldsymbol{v}, \boldsymbol{K}, \boldsymbol{\lambda})$**-design**.

PBD's were first defined by Bose and Shrikhande. They were interested in a famous problem on nets, the so called Euler's conjecture, posed almost 200 years back. The conjecture stated that there is no Net$(n, 4)$ when ever $n = 2(mod\ 4)$. Note that a net with 4 parallel classes corresponds to two mutually orthogonal latin squares. Thus Euler's conjecture was that there are no mutually orthogonal latin squares of order n, if $n = 2(mod\ 4)$. Euler became interested in this problem because of some arrangement of army regiments and ranks, Russian Czar had asked him to arrange. It corresponded to creating 2 mutually orthogonal squares of order 6. Euler could prove that no such mutually orthogonal squares of order 6 exist. He then conjectured the same for orders $n = 2(mod\ 4)$. Bose and Shrikhande proved that the conjecture is false.

Their method was to use smaller nets or planes and paste them together by using a PBD as a base. Crucial aspect in the construction was to use these designs with unequal block sizes to get designs with equal block sizes. Before them Parker was also trying to study the same problem. He had also come up with similar construction but he was using projective or affine planes which have blocks with the same sizes. He came up with interesting results but could not prove falsity of Euler's conjecture. Later all three of them together proved that Euler's conjecture was only true for 2 and 6, it was false in all other cases (Bose and Shrikhande (1959) and Bose *et al.* (1960)). Later, Chowla Erdos and Strauss proved using similar compositions that if n is large and $r \leq n^{1/91}$ then a net $\mathrm{N}(n, r)$ exists. Wilson later improved this bound to $n^{1/17}$. Thus largest r for which $\mathrm{N}(n, r)$ exist, does not depend on prime power decomposition of n (see Chowla *et al.* (1960) and Wilson (1974)).

The composing bigger designs from smaller designs with nonconstant block sizes became an important technique to study different type of designs and arrays. In particular it helped in construction of many specialized designs and arrays, PBIBD (partially balanced incomplete block designs), Orthogonal arrays, association schemes, resolvable designs *etc.* Wilson and Ray-Chaudhari developed several such methods to solve the famous Kirkman's School Girl Problem, posed by Kirkman, almost 200 years back (see Ray-Chaudhuri and Wilson (1971), Beth *et al.* (2000)).

Later Wilson used all this development, to unify most of such work by then. He defined a very interesting closure operation **PBD closure** on any subset of $\mathbb{N}$, in the following manner. A set $K \subseteq \mathbb{N}$ is said to be PBD-closed if the existence of a $\mathrm{PBD}(v, K, 1)$ implies that $v \in K$. Let $K \subseteq \mathbb{N}$ and let $B(K) = \{v|$ there exists a $PBD(v, K, 1)\}$. Then $B(K)$ is a PBD-closed set, called the **closure** of K. Given any set K define $\beta(K)$ to be the $\gcd\{k(k-1)|k \in K\}$. Using this closure operation, Wilson proved the following interesting result in 1972. Every closed set K is eventually periodic with period $\beta(K)$. That is, there exists a constant C such that, for every $k \in K$, $\{v|v \geq C, v = k(mod\ \beta(K))\} \subseteq K$. What this theorem implies, for example, is that if $v = k\ (mod\ k(k-1))$ and is sufficiently large, then a 2-(v, k, λ)- design exists. In fact the result implied for many such congruent classes, the existence of BIBD's for all large v (see for more details Wilson (1972b), Wilson (1972c)). Ultimately by 1975, he proved the existence conjecture for BIBD case completely, (Wilson (1975)). We will describe basic steps in his proof.

Another tool which helped in construction of BIBD's, and more generally t-(v, k, λ)-designs was studying the structure of the module which is kernel of the mapping $\partial_{t,k}$: $M_k(X) \to M_t(X)$. Note that if $f \in ker(\partial_{t,k})$, $\partial_{t,k}(f)(T) = 0$ for all $T \in \mathbb{P}_t(X)$. Thus we can think of such an f as a signed t-(v, k, λ) design with $\lambda = 0$. Such signed t-$(v, k, 0)$ designs are called **null *t*-designs**. Thus $\ker(\partial_{t,k})$ is a $\mathbb{Z}$-module of all null t-designs. Its rank is clearly $\binom{v}{k} - \binom{v}{t}$. Constructing a natural basis $\ker(\partial_{t,k})$ acting on $M_k(X)$ or vector-space $V_k(X)$ helps a lot in developing a proper understanding of the signed designs. Graver and Jurkat and Wilson constructed such a basis. While Graver and Jukart studied it for module $M_k(X)$, Wilson studied over the vector space $V_k(X)$. Wilson actually proved that all t-$(v, k.\lambda)$-designs exist if λ is sufficiently large (see Graver and Jurkat (1973), Wilson (1973)).

These results were used by them to show that signed t-$(v, k.\lambda)$-designs always exist. Thus interestingly rational or signed t-(v, k, λ)-designs can be directly constructed by using such algebraic methods. Could a more careful study and better base or generating set for $\ker\partial_{t,k}$ or $M_k(X)$ itself, help in direct construction of t-designs? The method was used by

Ray-Chaudhri and Singhi to construct t-(v, k, λ) designs, for large λ and v, in which no block is repeated more than 2 times (see Ray-Chaudhuri and Singhi (1988)).

We describe an interesting natural set of generators of the $\mathbb{Z}$-submodule $\ker\partial_{t,k}$ of $M_k(X)$. This interesting generating set was first described by Graham Li Li, (see Graham *et al.* (1980)).

Let $X = \{x_1, x_2, \ldots, x_v\}$. Let $A = \{y_1, y_2, \ldots y_{2t+2}, w_1, w_2 \cdots w_{k-t-1}\}$ be a $(k+t+1)$-subset of X. Define a polynomial P_A by

$$P_A = (y_1 - y_2)(y_3 - y_4) \ldots (y_{2t+1} - y_{2t+2}) w_1 w_2 \ldots w_{k-t-1}$$

Now define a function $f_A \in M_k(X)$ as follows. For a set $B \in \mathbb{P}_k(X)$, $B = \{q_1, q_2 \ldots q_k\}$, define $f_A(B)$ to be the coefficient of the monomial $q_1 q_1 \ldots q_k$ in P_A. Thus $f_A(B)$ is ± 1 or 0. Using the fact that there are $t+1$ brackets in the above expression of P_A, it can be easily seen that $\partial_{t,k}(f_A)(T) = 0$ for all $T \in \mathbb{P}_t(X)$. Thus $f_A \in \ker\partial_{t,k}$ is a null t-design. Graham Li and Li showed that such singed designs f_A generate the submodule $\ker\partial_{t,k}$ of $M_k(X)$. Chahal and Singhi, using these ideas, constructed a natural basis of the module $M_k(X)$ by using lexicographic ordering. Elements of this basis they called **tags** (see for more details Chahal and Singhi (2001), Singhi (2006)). Tags can be used to study many other problems too.

Wilson's proof of existence conjecture for BIBD can be summarized in a 3-step process.

(i). Existence theorem for signed designs:. Step 1 is to show that the necessary conditions are sufficient for signed t-designs or more general similar structures. This was done, as described above, first by Graver and Jurkar and Wilson.

(ii) $\boldsymbol{\lambda}$ large theorem: Step 2 is to prove that given v, t and k for all sufficiently large λ the necessary conditions are sufficient. This was proved by Wilson by studying his famous $W_{t,k}$ matrices and corresponding vector spaces, which he also used for solving many other interesting problems Wilson (1973).

(iii) Block spreading: Step 3 is to replace a set X in designs constructed in Step 2 with $X \times V$ for a large set V to reduce repetitions and create 2-designs on the set $X \times V$ with much smaller λ, for example a Steiner system. This was done by Wilson by taking V to be a vector space. The method is now known as Wilson's block spreading technique (see Wilson (1980) Wilson (1975) Wilson (1990)). As we already remarked, the method was later generalized for transversal designs for any t by Blanchard.

Finally we describe the generalization of t-designs to t designs for multisets (or lists, as we remarked in Section 1, two concepts lists or multisets are the same). These generalized designs should be useful in Statistics too. Also, another possibility is that Wilson's ideas of block spreading could be applied to them too, to get construction of actual t-designs.

We first define designs on multisets. A **list Design** is an ordered pair $D = (X, f)$, where X is finite set and f is a list on $L(X)$. Thus for each multiset ℓ of X, $f(\ell) \in \mathbb{N}$. We may also consider f as a multiset $f = [\ell_i | i \in I_{|f|}]$. Each element of f is called a block of the list design f. Thus $\ell \in L(X)$ is a block if and only if $f(\ell)) \neq 0$. Elements of X are called, as in the case of sets, points or treatments. D is said to be of **block size *k***, if all blocks are multisets of size k, i.e $f(\ell) \neq 0$ implies that $\ell \in L_k(X)$. We define ∂_t and $\partial_{t,k}$, for lists in quite similar manner, as we defined them for $N(X)$ and $N_k(X)$, only now they will be defined over $L(X)$ and $L_k(X)$ respectively. Thus, for example, if f is a list on $L_k(X)$, $\partial_{t,k}(f)$ is a list on $L_t(X)$, defined by $\partial_{t,k}(f)(s) = \sum_{s \subseteq \ell} c(\ell, s) f(\ell)$, for all $s \in L_t(X)$. Thus $\partial_{t,k}(f)(s)$, essentially gives the number of ways in which s occurs as a submultiset in the blocks of f.

For a finite set X we will denote by $\boldsymbol{S(X)}$ the symmetric group of all permutations of X. We note that $S(X)$ also acts as permutation group on the set of all k-subsets $\mathbb{P}_k(X)$ as well as on the set of all k-multisets $L_k(X)$. We also note that $S(X)$ acts transitively on $\mathbb{P}_k(X)$, *i.e.*, given any two k-subsets A_1, A_2 of X, we can always find an element $\sigma \in S(X)$, such that $\sigma(A_1) = A_2$. But this is not true with k-multisets.

Example 6: Let X be finite set. $x, y \in X$, $x \neq y$. Consider two 5-multisets $A_1 = [x, x, y, y, y]$ and $A_2 = [x, x, x, y, y]$. Define a permutation $\sigma : X \to X$ by $\sigma(x) = y$, $\sigma(y) = x$ and $\sigma(z) = z$, if $z \neq x$ or y. Then clearly $\sigma(A_1) = A_2$. Now consider the 5-multiset $A_3 = [x, y, y, y, y]$. It can be easily seen that there is no $\tau \in S(X)$ such that $\tau(A_1) = A_3$. Thus in general $S(X)$ is not transitive on $L_k(X)$.

Let $\ell \in L_t(X)$ we will denote $\boldsymbol{orb_t(\ell)}$, the orbit of $\ell \in L_t(X)$ under $S(X)$. Thus $orb_t(\ell) = \{\ell_1 | \sigma(\ell) = \sigma(\ell_1)$ for some $\sigma \in S(X)\}$. Let $\boldsymbol{ORB_t(X)}$ be the set $\{orb_t(\ell) | \ell \in L_t(X)\}$ of all orbits of elements of $L_t(X)$.

Let $m \in \mathbb{N}$. a **partition** π of m is a list on the set $I_m = \{1, 2, \ldots m\}$ such that $\sum i\pi(i) = m$, where the sum is over all $i \in I_n$.

Example 7: Consider the list π on the set I_{13} defined by $\pi(1) = 3$, $\pi(2) = \pi(3) = 2$ and $\pi(g) = 0$ if $g \neq 1, 2, 3$. π corresponds to the multiset $[1, 1, 1, 2, 2, 3, 3]$. Clearly π is a partition of 13.

Now suppose $\ell \in L(X)$. Define a partition of $\boldsymbol{\pi(\ell)}$ of integer $|\ell|$ by $\pi(\ell)(i) = |\{x \in supp(\ell) | \ell(x) = i\}|$. Thus $\pi(\ell) = [\ell(x)\ x \in supp(\ell)]$.

Remark 5: Suppose $\ell, \ell_1 \in L_t(X)$. Then, it can be easily seen that $orb_t(\ell) = orb_t(\ell_1)$, if and only if $\pi(\ell) = \pi(\ell_1)$.

We can now define t-list designs. Let $0 \leq t \leq k$, $|X| = v$. A ***t*-list design** on X with block size k is a list design $D = (X, f)$, with block size k such that for all $s_1, s_2 \in L_t(X)$ with $\pi(s_1) = \pi(s_2)$, $(\partial_{t,k}(f))(s_1) = (\partial_{t,k}(f))(s_2)$. Thus t-list design with block size k is a list design with block size k on the set X such that if any two t-lists s_1, s_2 on X are in the same orbit under the action of $S(X)$, then they occur the same number of times in blocks of D. We define t-list designs also in terms of parameters, only note that now λ is not a constant, it is a function on $ORB_t(X)$.

Let $\lambda : ORB_t(X) \to \mathbb{N}$ be a list on $ORB_t(X)$. A list design $D = (X, f)$ on a set X of size v and block size k is said to be a t-(v, k, λ)-list design if for all $s \in L_t(X)$, $\partial_{t,k}(f)(s) = \lambda(orb_t(s))$.

In the paper Singhi and Raychaudhuri (2012), list designs, signed list designs, rational list designs are studied, the concept of tags is extended to list designs. Signed list designs are constructed for all parameters. And similarly second step in Wilson's three step process described above, of creating list designs when λ is large for all orbits, is carried out. some ideas of block spreading are also discussed.

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Conflict of interest

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