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Some Improved Separate Estimators of Population Mean in Stratified Ranked Set Sampling

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Abstract

This paper presents improved population mean estimators using auxiliary variable in Stratified Ranked Set Sampling. We have derived the expressions for bias and mean square errors up to the first order of approximation and shown that the proposed estimators under optimum conditions are more efficient than other estimators taken in this paper. In an attempt to verify the efficiencies of proposed estimators, theoretical results are supported by empirical study and simulation study for which we have considered two populations.

Key words: Study variable; Auxiliary variable; Bias; Mean square error; Ranked set sampling.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

In theory of sampling it is evident that suitable use of auxiliary information improves the efficiency of the estimator. These auxiliary information may be used either at the design phase or the estimation phase or at both phases. Cochran (1940) was the first to introduce a ratio estimator of Population Mean using auxiliary information. Shabbir and Gupta (2007), Koyuncu and Kadilar (2009) and Chaudhary *et al.* (2009) have considered the problem of estimating population mean taking into consideration information on auxiliary variable.

When population is heterogenous stratified random sampling (SSRS) is used for better accuracy. Several authors like Kadilar and Cingi (2003), Shabbir and Gupta (2006) and Haq and Shabbir (2013) have proposed estimators in stratified random sampling using information on a single auxiliary variable. Singh and Kumar (2012) have proposed improved estimators of population mean using two auxiliary variables in stratified random sampling. Recently, Muneer *et al.* (2020) have proposed family of chain exponential estimators in SSRS.

Ranked set sampling (RSS) is an improved sampling method over Simple Random Set Sampling (SRS). McIntyre (1952) was the first to explain RSS for estimating the population means. Takahasi and Wakimoto (1968) gave the necessary mathematical theory of RSS.

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Samawi and Muttlak (1996) suggested ratio estimators of population mean in RSS and showed that the RSS estimators gave improved results over their SRS counterparts. Shiva (2006) compared RSS with SRS for estimation of the unknown mean of study variable and the ratio of study variable to auxiliary variable. He concluded that RSS gives a better estimate for both the mean and the ratio. Singh *et al.* (2014) suggested a general procedure for estimating the population mean using RSS. Bouza (2014) and Bouza *et al.* (2018) provided a review of RSS, its modification, and its application.

Stratified ranked set sampling (SRSS) was first introduced by Samawi (1996) for increasing the efficiency of estimator of population mean. Samawi and Siam (2003) have proposed the combined and the separate ratio estimators in SRSS.

2. Sampling methodology

In ranked set sampling (RSS), we rank randomly selected units from the population merely by observation or prior experience after which only a few of these sampled units are measured. In RSS, k independent random sets each of size k are selected from the population and each unit in the set is being selected with equal probability. The members of each random set are ranked with respect to the characteristic of the auxiliary variable. Then the smallest unit is selected from the first ordered set and the second smallest unit is selected from the second ordered set. By this way, this procedure is continued until the largest rank is chosen from the k^{th} set. This cycle may be repeated r times, so rk $(=n)$ units have been measured during this process.

SRSS takes the following steps.

- Step 1: Select k_h^2 bivariate sample units randomly from the h^{th} stratum of the population.
- Step 2: Arrange these selected units randomly into k_h sets, each of size k_h.
- Step 3: The procedure of ranked set sampling (RSS) is then applied, on each of the sets to obtain the k_h sets of ranked set sample units. Here ranking is done with respect to the auxiliary variable X_h.
- Step 4: Repeat the above steps r times for each stratum to get the desired sample of size $n_h = k_h r$.

Consider a finite population $U = (U_1, U_2, ..., U_N)$ based on N identifiable units with a study variable Y and auxiliary variables X associated with each unit U_i, i = 1,2,...,N of the population. Let the population be divided into L disjoint strata with stratum h based on N_h, h = 1,2,...,L units.

Let $(Y_{h[1]j}, X_{h(1)j}), (Y_{h[2]j}, X_{h(2)j}), ..., (Y_{h[k_h]j}, X_{h(k_h)j})$ be the stratified ranked set sample for j^{th}, j=1,2,...,r cycle in h^{th} stratum.

Let $\overline{y}_{[SRSS]} = \sum_{h=1}^{L} W_h \overline{y}_{h[rss]}$ and $\overline{x}_{[SRSS]} = \sum_{h=1}^{L} W_h \overline{x}_{h[rss]}$

respectively be the stratified ranked set sample means corresponding to the population means

$\overline{Y} = \sum_{h=1}^{L} W_h \overline{Y}_h$ and $\overline{X} = \sum_{h=1}^{L} W_h \overline{X}_h$

of variables Y and X, where $W_h = \frac{N_h}{N}$ is the weight in stratum h.

Let $\overline{y}_{h[rss]} = \sum_{i=1}^{k_h} \sum_{j=1}^{r} \frac{Y_{h[i]j}}{k_h r}$ and $\overline{x}_{h[rss]} = \sum_{i=1}^{k_h} \sum_{j=1}^{r} \frac{X_{h(i)j}}{k_h r}$

be the stratified ranked set sample means corresponding to the population means

$\overline{Y}_h = \sum_{j=1}^{N_h} \frac{Y_{h[i]j}}{N_h}$ and $\overline{X}_h = \sum_{j=1}^{N_h} \frac{X_{h(i)j}}{N_h}$

of variables Y and X in stratum h.

Let $s_{yh}^2 = \frac{1}{n_h - 1} \sum_{h=1}^{L} \left(Y_{h[i]} - \overline{y}_{h[rss]}\right)^2$, $s_{xh}^2 = \frac{1}{n_h - 1} \sum_{h=1}^{L} \left(X_{h(i)} - \overline{x}_{h[rss]}\right)^2$ and

$s_{xyh} = \frac{1}{n_h - 1} \sum_{h=1}^{L} (Y_{h[i]} - \overline{y}_{h[rss]})(X_{h(i)} - \overline{x}_{h[rss]})$

respectively be the sample variances and covariances corresponding to the population variances and covariances.

$S_{yh}^2 = \frac{1}{N_h - 1} \sum_{h=1}^{L} \left(Y_{h[i]} - \overline{Y}_h\right)^2$, $S_{xh}^2 = \frac{1}{N_h - 1} \sum_{h=1}^{L} \left(X_{h(i)} - \overline{X}_h\right)^2$

and $S_{xyh} = \frac{1}{N_h - 1} \sum_{h=1}^{L} (Y_{h[i]} - \overline{Y}_h)(X_{h(i)} - \overline{X}_h)$ in the stratum h.

Let C_{yh} and C_{xh} respectively be the population coefficient of variation of variables Y and X.

3. Existing estimators

The conventional separate estimator of the population mean $\overline{Y}$ under SRSS is given by

$$t^s = \sum_{h=1}^{L} W_h \overline{y}_{h[rss]} \tag{1}$$

The variance of the estimator t^s is given by

$$Var(t^s) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 U_{20h} \tag{2}$$

The classical separate ratio estimator of the population mean $\overline{Y}$ under SRSS is defined as

$$t_r^s = \sum_{h=1}^{L} W_h \overline{y}_{h[rss]} \frac{\overline{X}}{\overline{x}_{h[rss]}} \tag{3}$$

The Mean Squared Error (MSE) of the estimator t_r^c is given by

$$MSE(t_r^s) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 [U_{20h} + U_{02h} - 2U_{11h}] \tag{4}$$

The classical separate regression estimator of the population mean $\overline{Y}$ under SRSS is given as

$$t^s_{lr} = \sum_{h=1}^{L} W_h \overline{y}_{h[rss]} + \beta(\overline{X} - \overline{x}_{h[rss]}) \tag{5}$$

The Mean Squared Error (MSE) of the estimator t^c_{lr} is given by

$$MSE(t^s_{lr}) = \sum_{h=1}^{L} W_h^2[\overline{Y}_h^2 U_{20h} + \beta_h^2 \overline{X}_h^2 U_{02h} - 2\beta_h \overline{Y}_h \overline{X}_h U_{11h}] \tag{6}$$

where β_h is the regression coefficient of Y_h on X_h.

4. Proposed estimators

Motivated by Bhushan *et al.* (2020), we suggest some estimators of the population mean $\overline{Y}$ using SRSS as

$$t^s_{p1} = \sum_{h=1}^{L} W_h \overline{y}_{h[rss]} \exp\left(\alpha_{1h}\left(\frac{\overline{x}_{h[rss]}}{\overline{X}_h} - 1\right)\right) \tag{7}$$

$$t^s_{p2} = \sum_{h=1}^{L} W_h \overline{y}_{h[rss]} \exp\left(\alpha_{2h} \log \frac{\overline{x}_{h[rss]}}{\overline{X}_h}\right) \tag{8}$$

where α_{1h} and α_{2h} are constants such that MSE of the estimators is minimum.

The biases of the proposed estimators are

$$Bias(t^s_{p1}) = \sum_{h=1}^{L} W_h \overline{Y}_h \left(\frac{\alpha_{1h}^2}{2} U_{02h} + \alpha_{1h} U_{11h}\right) \tag{9}$$

$$Bias(t^s_{p2}) = \sum_{h=1}^{L} W_h \overline{Y}_h \left(\frac{(\alpha_{2h}^2 - \alpha_{2h})}{2} U_{02h} + \alpha_{2h} U_{11h}\right) \tag{10}$$

The mean square errors of the proposed estimators are

$$MSE(t^s_{p1}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} + \alpha_{1h}^2 U_{02h} + 2\alpha_{1h} U_{11h}\right) \tag{11}$$

$$MSE(t^s_{p2}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} + \alpha_{2h}^2 U_{02h} + 2\alpha_{2h} U_{11h}\right) \tag{12}$$

The minimum mean square errors at the optimum values are

$$MinMSE(t^s_{p1}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} - \frac{U_{11h}^2}{U_{02h}}\right) \tag{13}$$

$$MinMSE(t^s_{p2}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} - \frac{U_{11h}^2}{U_{02h}}\right) \tag{14}$$

Outline of the derivations are given in Appendix

5. Some other proposed estimators

We propose modified estimators of population mean by $\overline{Y}$ under SRSS as

$$t_{p3}^{s} = \sum_{h=1}^{L} W_h[(1 + w_{1h})\overline{y}_{h[rss]} + w_{2h}(\overline{X}_h - \overline{x}_{h[rss]})]\frac{\overline{X}_h}{\overline{x}_{h[rss]}} \tag{15}$$

$$t_{p4}^{s} = \sum_{h=1}^{L} W_h[(1 + w_{3h})\overline{y}_{h[rss]} + w_{4h}(\overline{X}_h - \overline{x}_{h[rss]})]\exp\left(\frac{\overline{X}_h - \overline{x}_{h[rss]}}{\overline{X}_h + \overline{x}_{h[rss]}}\right) \tag{16}$$

$$t_{p5}^{s} = \sum_{h=1}^{L} W_h\left[w_{5h}\overline{y}_{h[rss]} + w_{6h}\exp\left(\frac{\overline{X}_h - \overline{x}_{h[rss]}}{\overline{X}_h + \overline{x}_{h[rss]}}\right)\left(1 + log\frac{\overline{x}_{h[rss]}}{\overline{X}_h}\right)\right] \tag{17}$$

$$t_{p6}^{s} = \sum_{h=1}^{L} W_h\left[w_{7h}\overline{y}_{h[rss]} + w_{8h}\left(\frac{\overline{X}_h}{\overline{x}_{h[rss]}}\right)\exp\left(\frac{\overline{X}_h - \overline{x}_{h[rss]}}{\overline{X}_h + \overline{x}_{h[rss]}}\right)\right] \tag{18}$$

The biases of the proposed estimators are

$$bias(t_{p3}^{s}) = \sum_{h=1}^{L} W_h[\overline{Y}_h w_{1h} + \overline{Y}_h(U_{02h} + w_{1h}U_{02h} + w_{2h}\delta U_{02h} - U_{11} - w_{1h}U_{11h})] \tag{19}$$

$$bias(t_{p4}^{s}) = \sum_{h=1}^{L} W_h\left[\overline{Y}_h w_{3h} + \overline{Y}_h\left(\frac{3}{8}U_{02h} + \frac{3}{8}w_{3h}U_{02h} + \frac{1}{2}w_{4h}\delta_h U_{02h} - \frac{1}{2}U_{11h} - \frac{1}{2}w_{3h}U_{11h}\right)\right] \tag{20}$$

$$Bias(t_{p5}^{s}) = \sum_{h=1}^{L} W_h\left[(w_{5h} - 1)\overline{Y}_h + w_{6h}\left(1 - \frac{5}{8}U_{02h}\right)\right] \tag{21}$$

$$Bias(t_{p6}^{s}) = \sum_{h=1}^{L} W_h\left[(w_{7h} - 1)\overline{Y}_h + w_{8h}\left(1 + \frac{15}{8}U_{02h}\right)\right] \tag{22}$$

The mean square errors of the proposed estimators are

$$MSE(t_{p3}^{s}) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2(A_{1h} + w_{1h}^2 B_{1h} + w_{2h}^2 C_{1h} + 2w_{1h}D_{1h} - 2w_{2h}E_{1h} - 2w_{1h}w_{2h}F_{1h}) \tag{23}$$

$$MSE(t_{p4}^{s}) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2(A_{2h} + w_{3h}^2 B_{2h} + w_{4h}^2 C_{2h} + 2w_{3h}D_{2h} - 2w_{4h}E_{2h} - 2w_{3h}w_{4h}F_{2h}) \tag{24}$$

The minimum mean square errors at the optimum values are

$$MinMSE(t_{p3}^{s}) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left(A_{1h} + \frac{C_{1h}D_{1h}^2 + B_{1h}E_{1h}^2 - 2D_{1h}E_{1h}F_{1h}}{F_{1h}^2 - B_{1h}C_{1h}}\right) \tag{25}$$

$$MinMSE(t_{p4}^{s}) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left(A_{2h} + \frac{C_{2h}D_{2h}^2 + B_{2h}E_{2h}^2 - 2D_{2h}E_{2h}F_{2h}}{F_{2h}^2 - B_{2h}C_{2h}}\right) \tag{26}$$

Outline of the derivations are given in Appendix

5.1. Case 1: Sum of weights is unity ($w_5 + w_6 = 1$ and $w_7 + w_8 = 1$)

The mean square errors of the proposed estimators are

$$MSE(t^s_{p5}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 (U_{20h} + w_{6h}^2 U_{02h} - 2w_{6h} V_{11h}) \tag{27}$$

$$MSE(t^c_{p6}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 (U_{20h} + w_{8h}^2 U_{02h} - 2w_{8h} U_{11h}) \tag{28}$$

The minimum mean square errors at the optimum values are

$$MinMSE(t^s_{p5}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} - \frac{U_{11h}^2}{U_{02h}} \right) \tag{29}$$

$$MinMSE(t^s_{p6}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} - \frac{U_{11h}^2}{U_{02h}} \right) \tag{30}$$

Outline of the derivations are given in Appendix

5.2. Case 2: Sum of weights is flexible ($w_5 + w_6 \neq 1$ and $w_7 + w_8 \neq 1$)

The mean square errors of the proposed estimators are

$$MSE(t^s_{p5}) = \sum_{h=1}^{L} W_h^2 [C_{3h} + w_{5h}^2 A_{3h} + w_{6h}^2 B_{3h} - 2w_{5h} C_{3h} - 2w_{6h} D_{3h} + 2w_{5h} w_{6h} E_{3h}] \tag{31}$$

$$MSE(t^s_{p6}) = \sum_{h=1}^{L} W_h^2 [C_{4h} + w_{7h}^2 A_{4h} + w_{8h}^2 B_{4h} - 2w_{7h} C_{4h} - 2w_{8h} D_{4h} + 2w_{7h} w_{8h} E_{4h}] \tag{32}$$

The minimum mean square errors at the optimum values are

$$MinMSE(t^s_{p5}) = \sum_{h=1}^{L} W_h^2 \left[C_{3h} + \frac{B_{3h} C_{3h}^2 + A_{3h} D_{3h}^2 - 2C_{3h} D_{3h} E_{3h}}{E_{3h}^2 - A_{3h} B_{3h}} \right] \tag{33}$$

$$MinMSE(t^s_{p6}) = \sum_{h=1}^{L} W_h^2 \left[C_{4h} + \frac{B_{4h} C_{4h}^2 + A_{4h} D_{4h}^2 - 2C_{4h} D_{4h} E_{4h}}{E_{4h}^2 - A_{4h} B_{4h}} \right] \tag{34}$$

Outline of the derivations are given in Appendix

6. Empirical study

In this section, we compare the performance of the proposed estimators with the other estimators considered in this paper. For comparison, we have taken a stratified population with 3 strata of sizes 20, 30, 17 respectively from the Singh (2003) (page no. 1119 (Appendix)). Where y is production (study variable) in metric tons and x is area (auxiliary variable) in hectares. For the above population, the parameters are given as below: For total population, N=67, $\overline{Y}$=72247.6, $\overline{X}$=26438

Table 1

Stratum 1	Stratum 2	Stratum 3
N_1=20	N_2=30	N_3=17
n_1=12	n_2=18	n_3=9
W_1=0.29851	W_2=0.44776	W_3=0.25373
$\overline{X}_1$=6801.25	$\overline{X}_2$=11025.3	$\overline{X}_3$=82464.1
$\overline{Y}_1$=17511.7	$\overline{Y}_2$=18937.4	$\overline{Y}_3$=377960.5
S^2_{x1}=175539558	S^2_{x2}=595679198.4	S^2_{x3}=20255478994
S^2_{y1}=1366895911	S^2_{y2}=2421559069	S^2_{y3}=687956456787
S_{y1x1}=489224338	S_{y2x2}=1174423304	S_{y3x3}=46735680920
C_{x1}=1.94804	C_{x2}=2.21368	C_{x3}=1.72586
$D^2_{yh1[i]}$=0.322701311	$D^2_{yh2[i]}$= 0.284750439	$D^2_{yh3[i]}$= 0.352112122
$D^2_{xh1[i]}$=0.277106302	$D^2_{xh2[i]}$= 0.191404888	$D^2_{xh3[i]}$= 0.201142044
$D_{yxh1[i]}$=0.298636371	$D_{yxh2[i]}$= 0.227030958	$D_{yxh3[i]}$= 0.01248969
R_1=2.57477	R_2=1.71763	R_3=4.58333

From this population we took ranked set samples of sizes k_1=4 , k_2=6 and k_3=3 from the stratum 1^{st} , 2^{nd} and 3^{rd} respectively. Further each ranked set sample from each stratum were repeated with number of cycles r=3. Hence sample size of stratifird ranked set sample is equivalent to $n_h = k_h r$.

Table 2: The MSE and PRE of the estimators

Estimators	MSE	Bias	PRE
t^s	1759632517	0.0000	100.0000
t^s_r	1204001473	17677.2090	146.1400
t^s_{lr}	11702271788	0.0000	150.3600
t^s_{p1}	11702271788	-2020.0767	150.3600
t^s_{p2}	11702271788	321.8933	150.3600
t^s_{p3}	811711525	-18442.3400	216.7800
t^s_{p4}	545563651	-27933.6290	281.5500
t^s_{p5}	425689034	11761.6920	413.3600
t^s_{p6}	315596791	-8835.3558	557.5500

The formula for Percent Relative Efficiency (PRE) is
PRE (estimators) = $\frac{MSE(t^s)}{MSE(estimator)} \times 100$

From Table 2, it is observed that

- The estimators t^s_{p1} and t^s_{p2} are almost equally efficient estimators as separate linear regression estimators under SRSS as these estimators show the MSE almost equal to the MSE of the combined linear regression estimator (t^s_{lr}). These two estimators t^s_{p1} and t^s_{p2} are more efficient estimators than that the other competitive estimators.

- t^s_{p3} ,t^s_{p4} , t^s_{p5} and t^s_{p6} are more efficient than other estimators used in this paper. It is observed that t^s_{p3} ,t^s_{p4} , t^s_{p5} and t^s_{p6} are more efficient than convention, ratio estimator and linear regression estimator under SRSS.

- From Table 2, we can conclude that the proposed estimators perform better than existing estimators as our proposed estimators have greater PRE.

7. Simulation study

To generalize the results of the numerical study, we have conducted simulation study over two hypothetically generated normal populations. The simulation procedure is explained in the following points:

- We generated bivariate random observations of size N=600 units from a bivariate normal distribution with parameters μ_y =20, σ_y =15, and μ_x=15, σ_x=10 and possibly chosen values of ρ_{yx}= 0.6, 0.7, 0.8, 0.9.
- Similarly, generate the population-2 with the parameters μ_y =120, σ_y =25, and μ_x=100, σ_x=20 .
- The population generated above is divided into 3 equal strata and a stratified ranked set sample of size 12 units with number of cycles 4 and set size 3 is drawn from each stratum.
- Compute the required statistics.
- Iterate the above steps 10,000 times to calculate the MSE and PRE of various combined estimators using the following expression.

$$MSE(T) = \frac{1}{10000}\sum_{i=1}^{10000}(T_i - \overline{Y})^2 \tag{35}$$

$$PRE = \frac{Var(t^c)}{MSE(T)} \times 100 \tag{36}$$

The MSE and PRE of the separate estimators are calculated using (35) and (36) and the results are reported for various values of correlation coefficients in Table 3.

Table 3 also shows that our proposed estimators perform better than the existing estimators. The MSE of the estimators decreases when the correlation and sample size increases for the population 1 and 2.

8. Conclusions

In this article we have proposed estimators for the population mean in stratified Ranked set sampling using the information of auxiliary variable. The expressions for Bias and MSE of the suggested estimators have been derived up to the first order of approximation. Empirical approach and simulation study for comparing the efficiency of the proposed estimators with other estimators have been used. The results have been shown the Tables 2 and 3. The Tables show that the proposed estimators turn out to be more efficient as compared to the other estimators for both populations. The proposed estimators are found to be rather improved in terms of lesser MSE and greater PRE as compared to the existing

Table 3: The MSE and PRE of the estimators

ρ_{yx}	Estimators	Population1			Population2		
		MSE	Bias	PRE	MSE	Bias	PRE
0.9	t^s	0.007284	0.000000	100.000000	0.066100	0.000000	100.000000
	t_r^s	0.006384	-0.000207	114.095495	0.043496	0.002922	151.969606
	t_{lr}^s	0.004961	0.000000	146.827826	0.042869	0.000000	154.189069
	t_{p1}^s	0.004945	-0.000239	147.285914	0.042656	-0.001315	154.960149
	t_{p2}^s	0.004943	-0.000279	147.352653	0.042767	-0.002903	154.558070
	t_{p3}^s	0.003387	-0.000311	215.050896	0.034651	-0.001423	190.761305
	t_{p4}^s	0.003339	0.000190	218.094473	0.024678	0.001060	267.848401
	t_{p5}^s	0.003245	-0.000178	224.414416	0.020015	-0.001060	330.255808
	t_{p6}^s	0.003090	-0.000426	235.710841	0.019309	-0.002208	342.320400
0.8	t^s	0.006984	0.000000	100.000000	0.090219	0.000000	100.000000
	t_r^s	0.006809	0.000672	102.560246	0.070926	0.001426	127.201411
	t_{lr}^s	0.004670	0.000000	149.540410	0.059455	0.000000	151.741804
	t_{p1}^s	0.004634	-0.000181	150.699420	0.059354	-0.004426	152.000269
	t_{p2}^s	0.004687	-0.000102	148.992987	0.059456	0.004611	151.739251
	t_{p3}^s	0.004174	-0.000314	167.327264	0.043999	-0.008287	205.046909
	t_{p4}^s	0.003587	0.000145	194.697844	0.030173	0.001628	299.004742
	t_{p5}^s	0.002991	-0.000314	233.454669	0.026272	-0.001067	343.397088
	t_{p6}^s	0.002657	-0.000537	262.768911	0.024469	-0.002098	368.707344
0.7	t^s	0.009693	0.000000	100.000000	0.074859	0.000000	100.000000
	t_r^s	0.006455	0.000136	150.158004	0.061038	0.002001	122.642758
	t_{lr}^s	0.005928	0.000000	163.507324	0.058294	0.000000	128.416597
	t_{p1}^s	0.005934	-0.000124	163.345039	0.058345	0.002856	128.303578
	t_{p2}^s	0.005976	0.000362	162.191669	0.058568	0.001072	127.814362
	t_{p3}^s	0.005562	-0.000330	174.267715	0.040771	-0.007203	183.608732
	t_{p4}^s	0.005109	0.000123	189.710173	0.038156	0.001765	196.189650
	t_{p5}^s	0.003267	-0.000429	296.645336	0.036670	-0.001165	204.142055
	t_{p6}^s	0.002752	-0.000625	352.113929	0.020043	-0.002084	373.482308
0.6	t^s	0.008782	0.000000	100.000000	0.091577	0.000000	100.000000
	t_r^s	0.008134	0.000191	107.954650	0.086847	0.002652	105.447165
	t_{lr}^s	0.007273	0.000000	120.745030	0.078933	0.000000	116.018800
	t_{p1}^s	0.007145	-0.000832	122.898593	0.078557	0.001270	116.573953
	t_{p2}^s	0.007108	0.000521	123.537736	0.078345	0.001939	116.889576
	t_{p3}^s	0.005597	-0.000356	156.880083	0.030133	-0.004797	303.904598
	t_{p4}^s	0.003695	0.000945	237.630218	0.023471	0.002048	390.162195
	t_{p5}^s	0.002241	-0.000552	391.795672	0.016980	-0.009593	539.310974
	t_{p6}^s	0.001342	-0.000719	654.023741	0.013250	-0.001763	691.136804

estimators in both real and simulated data sets. It is also observed from the simulation that the MSE of the proposed estimators decreases as the values of the correlation coefficient increase whereas the PRE of the suggested estimators increases as the values of the correlation coefficients increase. Based on our empirical study and simulation study, we can conclude that our proposed estimators can be preferred over the other estimators taken in this paper in several real situations.

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APPENDIX

This section consider the proof of the Theorems of Section 4 & 5.
To derive the MSE of the proposed estimators, the following notations will be used throughout the paper.

$$\overline{y}_{h[srss]} = \overline{Y_h}(1+\epsilon_{0h})$$
$$\overline{x}_{h[srss]} = \overline{X_h}(1+\epsilon_{1h})$$

such that E(ϵ_{0h})= E(ϵ_{1h})= 0

$$E(\epsilon_{0h}^2) = (\eta_h C_{yh}^2 - D_{yh[i]}^2) = U_{20h}$$
$$E(\epsilon_{1h}^2) = (\eta_h C_{xh}^2 - D_{xh[i]}^2) = U_{02h}$$
$$E(\epsilon_{0h}\epsilon_{1h}) = (\eta_h C_{xyh} - D_{xyh[i]}) = U_{11h}$$

where $\eta_h = \frac{1}{k_h r}$, $C_{xh} = \frac{S_{xh}}{\overline{X}}$,$C_{yh} = \frac{S_{yh}}{\overline{Y}}$, $D_{xh[i]}^2 = \frac{1}{k_h^2 r \overline{X}^2}\sum_{i=1}^{k_h}(\overline{X}_{h(i)} - \overline{X}_h)^2$,

$D_{yh[i]}^2 = \frac{1}{k_h^2 r \overline{Y}^2}\sum_{i=1}^{k_h}(\overline{Y}_{h(i)} - \overline{Y}_h)^2$ and $D_{xyh[i]} = \frac{1}{k_h^2 r \overline{Y}\overline{X}}\sum_{i=1}^{k_h}(\overline{Y}_{h(i)} - \overline{Y}_h)(\overline{X}_{h(i)} - \overline{X}_h)$

where $\overline{Y}_{h[i]}$ and $\overline{X}_{h(i)}$ are the means of the i^{th} is ranked set and are given by

$$\overline{Y}_{h[i]} = \frac{1}{r}\sum_{j=1}^{r} Y_{h[i]j}, \overline{X}_{h(i)} = \frac{1}{r}\sum_{j=1}^{r} X_{h(i)j}$$

Now, consider the estimator

$$t_{p1}^s = \sum_{h=1}^{L} W_h \overline{y}_{h[rss]} \exp\left(\alpha_{1h}\left(\frac{\overline{x}_{h[rss]}}{\overline{X}_h} - 1\right)\right)$$

Using the above notations we have

$$t_{p1}^s = \sum_{h=1}^{L} W_h \overline{Y}_h (1+\epsilon_{0h}) \exp\left(\alpha_{1h}\left(\frac{\overline{X}_h(1+\epsilon_1)}{\overline{X}_h} - 1\right)\right) \tag{37}$$

The bias of the estimator t_{p1}^s is given by

$$Bias(t_{p1}^s) = \sum_{h=1}^{L} W_h \overline{Y}_h \left(\frac{\alpha_{1h}^2}{2} U_{02h} + \alpha_{1h} U_{11h}\right) \tag{38}$$

The MSE of the estimator t^s_{p1} is given by

$$MSE(t^s_{p1}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} + \alpha_{1h}^2 U_{02h} + 2\alpha_{1h} U_{11h} \right) \tag{39}$$

To find out the minimum MSE for t^s_{p1}, we partially differentiate equation (39) *w.r.t.* α_{1h} and equating to zero we get

$$\alpha^*_{1h} = -\frac{U_{11h}}{U_{02h}} \tag{40}$$

Putting the optimum value of α_{1h} in the equation (39), we get a minimum MSE of t^s_{p1} as

$$MinMSE(t^s_{p1}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} - \frac{U_{11h}^2}{U_{02h}} \right) \tag{41}$$

Similarly, we can obtain the optimum values of constants and minimum MSEs of other proposed estimators which are given as

$$t^s_{p2} = \sum_{h=1}^{L} W_h \overline{Y}_h (1 + \epsilon_{0h}) \exp \left(\alpha_{2h} \log \frac{\overline{X}_h (1 + \epsilon_{1h})}{\overline{X}_h} \right) \tag{42}$$

The bias of the estimator t^s_{p2} is given by

$$Bias(t^s_{p2}) = \sum_{h=1}^{L} W_h \overline{Y}_h \left(\frac{(\alpha_{2h}^2 - \alpha_{2h})}{2} U_{02h} + \alpha_{2h} U_{11h} \right) \tag{43}$$

The MSE of the estimator t^s_{p2} is given by

$$MSE(t^s_{p2}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} + \alpha_{2h}^2 U_{02h} + 2\alpha_{2h} U_{11h} \right) \tag{44}$$

To find out the minimum MSE for t^s_{p2}, we partially differentiate equation (44) *w.r.t.* α_{2h} and equating to zero we get

$$\alpha^*_{2h} = -\frac{U_{11h}}{U_{02h}} \tag{45}$$

Putting the optimum value of α_{2h} in the equation (44), we get a minimum MSE of t^s_{p2} as

$$MinMSE(t^s_{p2}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(U_{20h} - \frac{U_{11h}^2}{U_{02h}} \right) \tag{46}$$

$$t^s_{p3} = \sum_{h=1}^{L} W_h [(1 + w_{1h}) \overline{Y}_h (1 + \epsilon_{0h}) + w_{2h} \epsilon_{1h}](1 - \epsilon_{1h} + \epsilon_{1h}^2) \tag{47}$$

$$t^s_{p3} - \overline{Y} = \sum_{h=1}^{L} W_h \overline{Y}_h [(\epsilon_{0h} + w_{1h} + \epsilon_{0h} w_{1h} - \epsilon_{1h} - \epsilon_{1h} w_{1h} - \epsilon_{0h}\epsilon_{1h} - \epsilon_{0h}\epsilon_{1h} w_{1h} + \epsilon_{1h}^2 + w_{1h}\epsilon_{1h}^2) - w_{2h}\delta(\epsilon_{1h} - \epsilon_{1h}^2)] \tag{48}$$

The bias of the estimator t^s_{p3} is given by

$$Bias(t^s_{p3}) = \sum_{h=1}^{L} W_h[\overline{Y}_h w_{1h} + \overline{Y}_h(U_{02h} + w_{1h}U_{02h} + w_{2h}\delta U_{02h} - U_{11} - w_{1h}U_{11h})] \tag{49}$$

The MSE of the estimator t^s_{p3} is given by

$$MSE(t^s_{p3}) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2[U_{20h} + U_{02h} - 2U_{11h} + w_{1h}^2(1 + U_{20h} + 3U_{02h} - 4U_{11h}) + w_{2h}^2\delta_h^2 U_{02h} + 2w_{1h}(U_{20h} + 2U_{02h} - 3U_{11h}) - 2w_{2h}\delta(U_{11h} - U_{02h}) - 2w_{1h}w_{2h}\delta(U_{11h} - 2U_{02h})] \tag{50}$$

$$MSE(t^s_{p3}) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2(A_{1h} + w_{1h}^2 B_{1h} + w_{2h}^2 C_{1h} + 2w_{1h}D_{1h} - 2w_{2h}E_{1h} - 2w_{1h}w_{2h}F_{1h}) \tag{51}$$

where

$$A_{1h} = U_{20h} + U_{02h} - 2U_{11h}$$
$$B_{1h} = 1 + U_{20h} + 3U_{02h} - 4U_{11h}$$
$$C_{1h} = \delta^2 U_{02h}, \delta_h = \frac{\overline{X}_h}{\overline{Y}_h}$$
$$D_{1h} = U_{20h} + 2U_{02h} - 3U_{11h}$$
$$E_{1h} = \delta_h(U_{02h} - U_{11h})$$
$$F_{1h} = \delta_h(U_{02h} - 2U_{11h})$$

To find out the minimum MSE for t^s_{p3}, we partially differentiate equation (51) *w.r.t.* w_{1h} and w_{2h} and equating to zero we get

$$w_{1h}^* = \frac{C_{1h}D_{1h} - E_{1h}F_{1h}}{F_{1h}^2 - B_{1h}C_{1h}} \tag{52}$$

$$w_{2h}^* = \frac{D_{1h}F_{1h} - B_{1h}C_{1h}}{F_{1h}^2 - B_{1h}C_{1h}} \tag{53}$$

Putting the optimum values of w_{1h} and w_{2h} in the equation (51), we get a minimum MSE of t^s_{p3} as

$$MinMSE(t^s_{p3}) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left(A_{1h} + \frac{C_{1h}D_{1h}^2 + B_{1h}E_{1h}^2 - 2D_{1h}E_{1h}F_{1h}}{F_{1h}^2 - B_{1h}C_{1h}}\right) \tag{54}$$

$$t^s_{p4} = \sum_{h=1}^{L} W_h[(1 + w_{3h})\overline{Y}_h(1 + \epsilon_{0h}) + w_{4h}\epsilon_{1h}]\left(1 - \frac{3}{2}\epsilon_{1h} + \frac{15}{8}\epsilon_{1h}^2\right) \tag{55}$$

$$t^s_{p4} - \overline{Y} = \sum_{h=1}^{L} W_h\overline{Y}_h[(\epsilon_{0h} + W_{3h} + \epsilon_{0h}w_{3h} - \frac{1}{2}\epsilon_{1h} - \frac{1}{2}\epsilon_{1h}w_{3h} - \frac{1}{2}\epsilon_{0h}\epsilon_{1h} - \frac{1}{2}\epsilon_{0h}\epsilon_{1h}w_{3h} + \frac{3}{8}\epsilon_{1h}^2 + \frac{3}{8}w_{3h}\epsilon_1^2) - w_{4h}\delta_h(\epsilon_{1h} - \epsilon_{1h}^2)] \tag{56}$$

The bias of the estimator t^s_{p4} is given by

$$Bias(t^s_{p4}) = \sum_{h=1}^{L} W_h \left[\overline{Y}_h w_{3h} + \overline{Y}_h \left(\frac{3}{8}U_{02h} + \frac{3}{8}w_{3h}U_{02h} + \frac{1}{2}w_{4h}\delta_h U_{02h} - \frac{1}{2}U_{11h} - \frac{1}{2}w_{3h}U_{11h}\right)\right] \tag{57}$$

The MSE of the estimator t^s_{p4} is given by

$$MSE(t^s_{p4}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 (U_{20h} + \frac{1}{4}U_{02h} - U_{11} + w_{3h}^2(1 + U_{20h} + U_{02h} - 2U_{11h}) + w_{4h}^2\delta_h^2 U_{02h} + 2w_{3h} (U_{20h} + \frac{5}{4}U_{02h} - \frac{3}{2}U_{11h}) - 2w_{4h}\delta_h(U_{11h} - \frac{1}{2}U_{02h}) - 2w_{3h}w_{4h}\delta(U_{11h} - U_{02h})) \tag{58}$$

$$MSE(t^s_{p4}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 (A_{2h} + w_{3h}^2 B_{2h} + w_{4h}^2 C_{2h} + 2w_{3h}D_{2h} - 2w_{4h}E_{2h} - 2w_{3h}w_{4h}F_{2h}) \tag{59}$$

where

$$A_{2h} = U_{20h} + \frac{1}{4}U_{02h} - U_{11h}$$
$$B_{2h} = 1 + U_{20h} + U_{02h} - 2U_{11h}$$
$$C_{2h} = \delta^2 U_{02h}, \delta_h = \frac{\overline{X}_h}{\overline{Y}_h}$$
$$D_{2h} = U_{20h} + \frac{5}{4}U_{02h} - \frac{3}{2}U_{11h}$$
$$E_{2h} = \delta\left(U_{02h} - \frac{1}{2}U_{11h}\right)$$
$$F_{2h} = \delta(U_{02h} - U_{11h})$$

To find out the minimum MSE for t^s_{p4}, we partially differentiate equation (59) *w.r.t.* w_{3h} and w_{4h} and equating to zero we get

$$w_{3h}^* = \frac{C_{2h}D_{2h} - E_{2h}F_{2h}}{F_{2h}^2 - B_{2h}C_{2h}} \tag{60}$$

$$w_{4h}^* = \frac{D_{2h}F_{2h} - B_{2h}C_{2h}}{F_{2h}^2 - B_{2h}C_{2h}} \tag{61}$$

Putting the optimum values of w_{3h} and w_{4h} in the equation (59), we get a minimum MSE of t^s_{p4} as

$$MinMSE(t^s_{p4}) = \sum_{h=1}^{L} W_h^2 \overline{Y}_h^2 \left(A_{2h} + \frac{C_{2h}D_{2h}^2 + B_{2h}E_{2h}^2 - 2D_{2h}E_{2h}F_{2h}}{F_{2h}^2 - B_{2h}C_{2h}}\right) \tag{62}$$

$$t^s_{p5} = \sum_{h=1}^{L} W_h \left[w_{5h}\overline{Y}_h(1 + \epsilon_{0h}) + w_{6h}\exp\left(\frac{-\epsilon_{1h}}{2 + \epsilon_{1h}}\right)(1 + log(1 + \epsilon_{1h}))\right] \tag{63}$$

$$t^s_{p5} - \overline{Y} = \sum_{h=1}^{L} W_h \left[(w_{5h} - 1)\overline{Y}_h + w_{5h}\overline{Y}_h\epsilon_{0h} + w_{6h}\left(1 + \frac{\epsilon_{1h}}{2} - \frac{5}{8}\epsilon_{1h}^2\right)\right] \tag{64}$$

$$Bias(t_{p5}^s) = \sum_{h=1}^{L} W_h \left[(w_{5h}-1)\overline{Y}_h + w_{6h}\left(1-\frac{5}{8}U_{02h}\right)\right] \tag{65}$$

$$t_{p6}^s = \sum_{h=1}^{L} W_h \left[w_{7h}\overline{Y}_h(1+\epsilon_{0h}) + w_{8h}\exp\left(\frac{-\epsilon_{1h}}{2+\epsilon_{1h}}\right)(1+\epsilon_{1h})^{-1}\right] \tag{66}$$

$$t_{p6}^s - \overline{Y} = \sum_{h=1}^{L} W_h \left[(w_{7h}-1)\overline{Y}_h + w_{7h}\overline{Y}_h\epsilon_{0h} + w_{8h}\left(1-\frac{3}{2}\epsilon_{1h}-\frac{15}{8}\epsilon_{1h}^2\right)\right] \tag{67}$$

$$Bias(t_{p6}^s) = \sum_{h=1}^{L} W_h \left[(w_{7h}-1)\overline{Y}_h + w_{8h}\left(1+\frac{15}{8}U_{02h}\right)\right] \tag{68}$$

CASE 1: SUM OF WEIGHTS IS UNITY ($w_5 + w_6 = 1$ and $w_7 + w_8 = 1$)

The MSE of the estimator t_{p5}^s is given by

$$MSE(t_{p5}^s) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2(U_{20h} + w_{6h}^2U_{02h} - 2w_{6h}V_{11h}) \tag{69}$$

To find out the minimum MSE for t_{p5}^s , we partially differentiate equation (69) *w.r.t.* w_{6h}, and equating to zero we get

$$w_{6h}^* = \frac{V_{11h}}{V_{02h}} \tag{70}$$

Putting the optimum value of w_{6h} in the equation (69), we get a minimum MSE of t_{p5}^s as

$$MinMSE(t_{p5}^s) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left(U_{20h} - \frac{U_{11h}^2}{U_{02h}}\right) \tag{71}$$

The MSE of the estimator t_{p6}^s is given by

$$MSE(t_{p6}^c) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2(U_{20h} + w_{8h}^2U_{02h} - 2w_{8h}U_{11h}) \tag{72}$$

To find out the minimum MSE for t_{p6}^s , we partially differentiate equation (72) *w.r.t.* w_{8h}, and equating to zero we get

$$w_{8h}^* = \frac{U_{11h}}{U_{02h}} \tag{73}$$

Putting the optimum value of w_{8h} in the equation (72), we get a minimum MSE of t_{p6}^s as

$$MinMSE(t_{p6}^s) = \sum_{h=1}^{L} W_h^2\overline{Y}_h^2\left(U_{20h} - \frac{U_{11h}^2}{U_{02h}}\right) \tag{74}$$

CASE 2: SUM OF WEIGHTS IS FLEXIBLE ($w_5 + w_6 \neq 1$ and $w_7 + w_8 \neq 1$)

$$t_{p5}^s - \overline{Y} = \sum_{h=1}^{L} W_h \left[(w_{5h}-1)\overline{Y}_h + w_{5h}\overline{Y}_h\epsilon_{0h} + w_{6h}\left(1+\frac{\epsilon_{1h}}{2}-\frac{5}{8}\epsilon_{1h}^2\right)\right] \tag{75}$$

Squaring on both sides we get

$$(t^s_{p5}-\overline{Y})^2=\sum_{h=1}^{L}W_h^2\Big[\overline{Y}_h^2+\overline{Y}_h^2w_{5h}^2(1+\epsilon_{0h}^2)+w_{6h}^2(1-\epsilon_{1h}^2)-2w_{5h}\overline{Y}_h^2-2w_{6h}\overline{Y}_h\Big(1-\frac{5}{8}\epsilon_{1h}^2\Big)$$
$$+2w_{5h}w_{6h}\Big(1-\frac{5}{8}\epsilon_{1h}^2+\frac{1}{2}\epsilon_{0h}\epsilon_{1h}\Big)\Big] \quad (76)$$

Taking expectations on both sides we get

$$MSE(t^s_{p5})=\sum_{h=1}^{L}W_h^2\Big[\overline{Y}_h^2+\overline{Y}_h^2w_{5h}^2(1+U_{20h})+w_{6h}^2(1-U_{02h})-2w_{5h}\overline{Y}_h^2-2w_{6h}\overline{Y}_h\Big(1-\frac{5}{8}U_{02h}\Big)$$
$$+2w_{5h}w_{6h}\Big(1-\frac{5}{8}U_{02h}+\frac{1}{2}U_{11h}\Big)\Big] \quad (77)$$

$$MSE(t^s_{p5})=\sum_{h=1}^{L}W_h^2[C_{3h}+w_{5h}^2A_{3h}+w_{6h}^2B_{3h}-2w_{5h}C_{3h}-2w_{6h}D_{3h}+2w_{5h}w_{6h}E_{3h}] \quad (78)$$

where

$$A_{3h}=\overline{Y}_h^2(1+U_{20h})$$
$$B_{3h}=1-U_{02h}$$
$$C_{3h}=\overline{Y}_h^2$$
$$D_{3h}=\overline{Y}_h\left(1-\frac{5}{8}U_{02h}\right)$$
$$E_{3h}=\overline{Y}_h\left(1-\frac{5}{8}U_{02h}+\frac{1}{2}U_{11h}\right)$$

To find out the minimum MSE for the estimator t^s_{p5} , we partially differentiate equation (78) *w.r.t.* w_{5h} and w_{6h} and equating to zero we get

$$w_{5h}^*=\frac{B_{3h}C_{3h}-D_{3h}E_{3h}}{A_{3h}B_{3h}-E_{3h}^2} \quad (79)$$

$$w_{6h}^*=\frac{A_{3h}D_{3h}-C_{3h}E_{3h}}{A_{3h}B_{3h}-E_{3h}^2} \quad (80)$$

Putting the optimum values of w_{5h} and w_{6h} in the equation (78), we get a minimum MSE of t^s_{p5} as

$$MinMSE(t^s_{p5})=\sum_{h=1}^{L}W_h^2\left[C_{3h}+\frac{B_{3h}C_{3h}^2+A_{3h}D_{3h}^2-2C_{3h}D_{3h}E_{3h}}{E_{3h}^2-A_{3h}B_{3h}}\right] \quad (81)$$

$$t^c_{p6}-\overline{Y}=\sum_{h=1}^{L}W_h\left[(w_{7h}-1)\overline{Y}_h+w_{7h}\overline{Y}_h\epsilon_{0h}+w_{8h}\left(1-\frac{3}{2}\epsilon_{1h}+\frac{15}{8}\epsilon_{1h}^2\right)\right] \quad (82)$$

Squaring on both sides we get

$$(t^s_{p6}-\overline{Y})^2=\sum_{h=1}^{L}W_h^2\Big[\overline{Y}_h^2+\overline{Y}_h^2w_{7h}^2(1+\epsilon_{0h}^2)+w_{8h}^2(1+6\epsilon_{1h}^2)-2w_{7h}\overline{Y}_h^2-2w_{8h}\overline{Y}_h\Big(1-\frac{15}{8}\epsilon_{1h}^2\Big)$$
$$+2w_{7h}w_{8h}\Big(1+\frac{15}{8}\epsilon_{1h}^2-\frac{3}{2}\epsilon_{0h}\epsilon_{1h}\Big)\Big] \quad (83)$$

Taking expectations on both sides we get

$$MSE(t^s_{p6})=\sum_{h=1}^{L}W_h^2\Big[\overline{Y}_h^2+\overline{Y}_h^2w_{7h}^2(1+U_{20h})+w_{8h}^2(1+6U_{02h})-2w_{7h}\overline{Y}_h^2-2w_{8h}\overline{Y}_h\Big(1+\frac{15}{8}U_{02h}\Big)$$
$$+2w_{7h}w_{8h}\Big(1+\frac{15}{8}U_{02h}-\frac{3}{2}U_{11h}\Big)\Big] \quad (84)$$

$$MSE(t^s_{p6})=\sum_{h=1}^{L}W_h^2[C_{4h}+w_{7h}^2A_{4h}+w_{8h}^2B_{4h}-2w_{7h}C_{4h}-2w_{8h}D_{4h}+2w_{7h}w_{8h}E_{4h}] \quad (85)$$

where

$$A_{4h}=\overline{Y}_h^2(1+U_{20h})$$
$$B_{4h}=1+6U_{02h}$$
$$C_{4h}=\overline{Y}_h^2$$
$$D_{4h}=\overline{Y}_h\left(1+\frac{15}{8}U_{02h}\right)$$
$$E_{4h}=\overline{Y}_h\left(1+\frac{15}{8}U_{02h}-\frac{3}{2}U_{11h}\right)$$

To find out the minimum MSE for the estimator t^s_{p6}, we partially differentiate equation (85) *w.r.t.* w_{7h} and w_{8h} and equating to zero we get

$$w_{7h}^*=\frac{B_{4h}C_{4h}-D_{4h}E_{4h}}{A_{4h}B_{4h}-E_{4h}^2} \quad (86)$$

$$w_{8h}^*=\frac{A_{4h}D_{4h}-C_{4h}E_{4h}}{A_{4h}B_{4h}-E_{4h}^2} \quad (87)$$

Putting the optimum values of w_{7h} and w_{8h} in the equation (85), we get a minimum MSE of t^s_{p6} as

$$MinMSE(t^s_{p6})=\sum_{h=1}^{L}W_h^2\left[C_{4h}+\frac{B_{4h}C_{4h}^2+A_{4h}D_{4h}^2-2C_{4h}D_{4h}E_{4h}}{E_{4h}^2-A_{4h}B_{4h}}\right] \quad (88)$$