

Transient State Solution of Retrial Queueing System with Catastrophe

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Abstract

In this paper, we consider a single server retrial queueing system with catastrophe. If a primary customer finds the server available on its arrival, it is served immediately, else it retries for service as a secondary customer. Both primary and secondary arrivals follow Poisson processes. The time between two successive arrivals and services follow exponential distribution. Moreover, Catastrophe occurs at the system following a Poisson process. Repair of the failed system starts immediately. The repair times are also distributed exponentially. Time dependent probabilities for exact number of arrivals in the system, departures after taking service from the system and number of customers in orbit when the server is idle or busy are obtained. The probabilities of system being under repair are also obtained. Verification of results is done. Towards the end, numerical results are generated and represented graphically. Sensitivity analysis is performed to study the effect of various parameters.

Key words: Queueing; Retrial; Arrivals; Departures; Catastrophe; Repair.

AMS Subject Classifications: 60K25, 90B15, 90B22

1. Introduction

Queues or waiting lines are the systems where customers wait to get some kind of service and leave after getting service. Queues are a part of our daily lives. Everyday we encounter some form of waiting in lines. People at the checkout counter in supermarket, patients at the doctor's clinic, people to withdraw cash or to check account balance on ATM, *etc* are some examples of queueing systems. In classical queueing models, we observe customers who either wait for their service or leave the system forever. But there exist some systems in which a customer instead of leaving the system forever, retries for service after a random amount of time. Analysis of such systems introduced a new class of queueing systems which is known as retrial queueing systems. In retrial queues, if a customer on arrival finds the server free, it is served immediately else it temporarily leaves the service area and joins the virtual queue known as orbit. Thereafter it retries later from the orbit with a hope of having free server. Such kind of queueing systems play an important role to model many

practical situations like telecommunication systems, restaurants, banks, computer networks, *etc.*

A basic example of retrial queueing systems is a call center where call agent is a server who answers the calls from the customers. When a customer calls, if he is able to connect the call agent immediately, he is answered else he has to repeat the call.

There is extensive literature available on retrial queues. Among these, Cohen (1957) is the earliest work on retrial queues. The book Falin and Templeton (1997) and the survey paper Artalejo and Gómez-Corral (1999) are a great source on retrial queues in which the techniques for the computational analysis of retrial queues are covered. Yang and Templeton (1987) discussed some important retrial queueing models and presented their analytical results and the techniques used. Rajadurai *et al.* (2015) obtained the steady state probability generating function for the system size by using the supplementary variable method. The stationary characteristics of an $M/G/1$ retrial queue are investigated in Kumar *et al.* (2020). Steady state and time dependent solutions for number of customers in the system when server is idle or busy are obtained in Singla and Garg (2022).

As in real situations traffic intensity is constantly varying, so the determination of transient solution is very much essential in analyzing behaviour of the system. Transient solutions deal with behaviour of the queueing system during the initial phase after the system is started. They are useful to study the characteristics of a system on different time points. Therefore, transient analysis of queueing systems is extremely important from theoretical and practical perspective. The classical transient results for the $M/M/1$, $M/M/c$ and $M/G/1$ queue provide little insight into the behaviour of a queueing system through a fixed operation time t . The probability $P_n(t)$ gives the distribution for the number in the system at time t , but practically provides no information on how the system has regulated up until time t .

Pegden and Rosenshine (1982) was the first who introduced the concept of two-state by obtaining a closed form solution for the probability that exactly i arrivals and j services occur over a time interval of length t . Singla and Kalra (2018) used the concept of two-state in retrial queues and analyzed a single server retrial queueing model in which time dependent probabilities for exact number of arrivals and departures in the system when server is free or busy are obtained. An explanation of the retrial queueing system is shown in Figure 1.

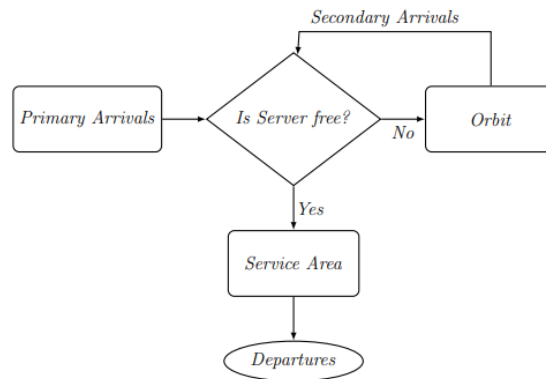


Figure 1: Basic structure of retrial queueing system

Queueing systems are often subject to sudden breakdowns that cause random failures. The sudden, unexpected breakdown of a machine, an electronic device, a computer system or a network at random time intervals is known as catastrophe. When a catastrophe occurs at a system, all the customers present in the system are deleted immediately, which also causes a breakdown of the system. Catastrophe resets the system from its current state to its failed state. The repair of the failed system starts immediately. When the system is repaired, it immediately resumes its functioning to serve new arrivals. Catastrophe models can be found very useful in computer networks and in communication systems. For example: In call centers, the sudden breakdown of power will result in loss of all the calls present at that time.

The loss of customers due to these breakdowns (also referred to as a kind of negative arrivals) was first introduced by Gelenbe (1991). Chao (1995) developed a queueing network model with catastrophes and obtained the steady-state probabilities for the model. Kumar and Arivudainambi (2000) studied ‘Transient solution of an M/M/1 queue with catastrophes’ in which behaviour of the probability of the server being free and mean queue length are discussed. Also steady state probabilities are obtained. Kumar *et al.* (2007) obtained transient solution for the system size in an M/M/1 queueing system with catastrophes, server failures and non-zero repair time. The steady state probabilities and some performance measures are also given. Bura (2022) analyzed infinite server Markovian queue with catastrophe and repairable servers. Figure 2 shows the basic structure of retrial queueing system with catastrophe.

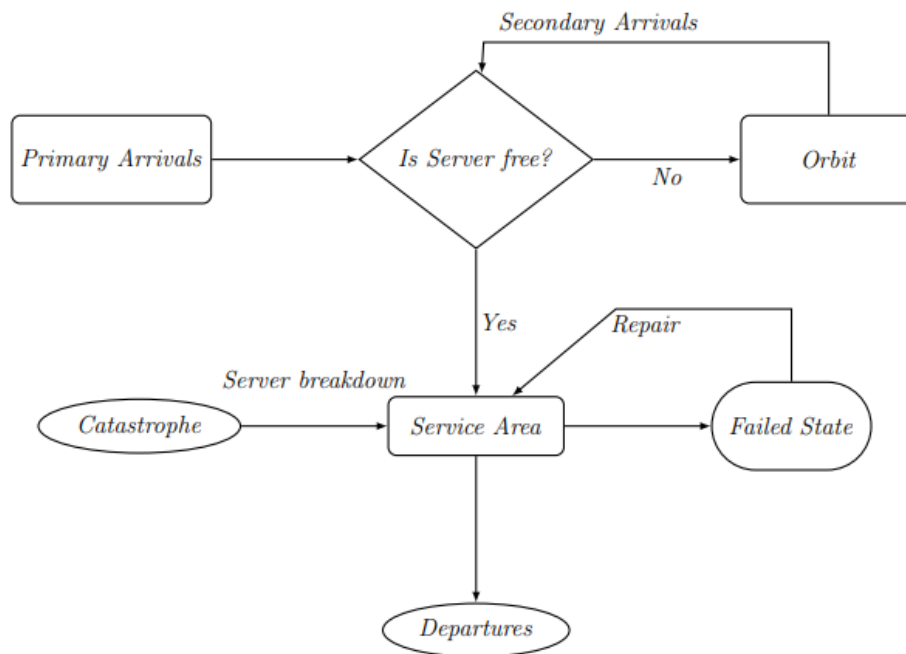


Figure 2: Basic structure of a retrial queue with catastrophe

In this paper, we derive time-dependent probabilities for exact number of arrivals in the system, exact number of departed units after taking service from the system and exact number of customers in the orbit by time t when the server is idle or busy for a single server retrial queueing system. In addition we obtain the probability of server being under repair

when the server fails due to catastrophic events at random time intervals. Besides these theoretical solutions, we present some numerical results graphically to study the behaviour of probabilities with respect to average service times. Sensitivity analysis is performed to study the effect of various parameters.

The paper is organized as follows. In section 2, the complete mathematical description of the model is defined. Also, the difference-differential equations are derived in this section. Mathematical analysis of the model is done in section 3 in which we obtained the recursive probabilities. Transient solutions are obtained in section 4. In section 5 verification of results is done. The numerical results are obtained and represented graphically in section 6. Sensitivity analysis is performed in section 7. Section 8 discusses the conclusion. Finally the references are listed.

2. Model description

We considered a single server retrial queueing system with catastrophe. Assumptions of the model are as follows:

- The primary customers arrive at the system according to Poisson process with mean arrival rate λ .
- If on arrival, a customer finds the server busy, it joins the orbit and retries later from the orbit. These retrials are considered to be secondary arrivals. These customers arrive at the system according to Poisson process with mean retrial rate θ .
- The service times follow exponential distribution with parameter μ .
- Catastrophe occurs at the system following a Poisson process with mean rate ξ .
- The repair times are distributed exponentially with parameter τ . Further it is assumed that arrival will not occur during the repair time of the system.

Also, the primary and secondary arrivals, inter-arrival times, service times, departures and catastrophes are statistically independent.

Laplace transformation $\bar{f}(s)$ of $f(t)$ is given by:

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt; \quad \text{Re}(s) > 0$$

If $L^{-1}\{\bar{f}(s)\} = F(t)$ and $L^{-1}\{\bar{g}(s)\} = G(t)$ then

$$L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t F(u)G(t-u)du = F * G$$

$F * G$ is called the convolution of F and G .

2.1. Notations

$P_{i,j,k}^{(0)}(t)$ = Probability that there are exactly i number of arrivals in the system, j number of units departed from the system after taking the service, k number of units in the orbit by time t and the server is free.

$P_{i,j,k}^{(1)}(t)$ = Probability that there are exactly i number of arrivals in the system, j number of units departed from the system after taking the service, k number of units in the orbit by time t and the server is busy.

$Q_{i,j}(t)$ = Probability that the system is under repair by time t when there are exactly i number of arrivals, j number of units departed from the system after taking service.

$P_{i,j,k}(t)$ = Probability that there are exactly i arrivals in the system, j number of units departed from the system after taking the service, k number of units in the orbit by time t .

$$P_{i,j,k}(t) = P_{i,j,k}^{(0)}(t) + P_{i,j,k}^{(1)}(t) \quad \forall i, j, k \quad i \geq j, k$$

$$\text{and} \quad P_{i,j,k}^{(1)}(t) = 0, \quad i \leq j, k; \quad P_{i,j,k}^{(0)}(t) = 0, \quad i < j, k;$$

Initially

$$P_{0,0,0}^{(0)}(0) = 1; \quad P_{i,j,k}^{(0)}(0) = 0, \quad P_{i,j,k}^{(1)}(0) = 0, \quad i, j, k \neq 0; \quad Q_{i,j}(0) = 0 \quad \forall i, j;$$

2.2. Difference-differential equations governing the system

$$\frac{d}{dt} P_{i,j,k}^{(0)}(t) = -(\lambda + \xi + k\theta)P_{i,j,k}^{(0)}(t) + \mu(1 - \delta_{0,j})P_{i,j-1,k}^{(1)}(t) + \tau\delta_{0,k}Q_{i,j}(t) \quad i \geq j, k \geq 0 \quad (1)$$

$$\frac{d}{dt} P_{i,0,k}^{(1)}(t) = -(\lambda + \mu + \xi)P_{i,0,k}^{(1)}(t) + \lambda\delta_{0,k}P_{i-1,0,k}^{(0)}(t) + \lambda(1 - \delta_{0,k})P_{i-1,0,k-1}^{(1)}(t) \quad i \geq 1, k \geq 0 \quad (2)$$

$$\begin{aligned} \frac{d}{dt} P_{i,j,k}^{(1)}(t) = & -(\lambda + \mu + \xi)P_{i,j,k}^{(1)}(t) + \lambda P_{i-1,j,k}^{(0)}(t) + \lambda(1 - \delta_{0,k})P_{i-1,j,k-1}^{(1)}(t) \\ & + (k+1)\theta(1 - \delta_{0,j})P_{i,j,k+1}^{(0)}(t) \quad i \geq 2, j > 0, k \geq 0 \end{aligned} \quad (3)$$

$$\frac{d}{dt} Q_{i,j}(t) = -\tau Q_{i,j}(t) + \xi \left(P_{i,j,0}^{(0)}(t) + \sum_{k=1}^{i-j} (1 - \delta_{0,j})P_{i,j,k}^{(0)}(t) + \sum_{k=0}^{i-j-1} P_{i,j,k}^{(1)}(t) \right) \quad i \geq j \geq 0 \quad (4)$$

where

$$\delta_{0,j} = \begin{cases} 1 & \text{when } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta_{0,k} = \begin{cases} 1 & \text{when } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

3. Mathematical analysis of the model

Using the Laplace transformation $\bar{f}(s)$ of $f(t)$ given by

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt, \quad \operatorname{Re}(s) > 0$$

in the equations (1)-(4) along with the initial conditions, we have

$$(s + \lambda + \xi + k\theta) \bar{P}_{i,j,k}^{(0)}(s) = \mu(1 - \delta_{0,j}) \bar{P}_{i,j-1,k}^{(1)}(s) + \tau \delta_{0,k} \bar{Q}_{i,j}(s) \quad i \geq j, k \geq 0 \quad (5)$$

$$(s + \lambda + \mu + \xi) \bar{P}_{i,0,k}^{(1)}(s) = \lambda \delta_{0,k} \bar{P}_{i-1,0,k}^{(0)}(s) + \lambda(1 - \delta_{0,k}) \bar{P}_{i-1,0,k-1}^{(1)}(s) \quad i \geq 1, k \geq 0 \quad (6)$$

$$(s + \lambda + \mu + \xi) \bar{P}_{i,j,k}^{(1)}(s) = \lambda \bar{P}_{i-1,j,k}^{(0)}(s) + \lambda(1 - \delta_{0,k}) \bar{P}_{i-1,j,k-1}^{(1)}(s) \\ + (k+1)\theta(1 - \delta_{0,j}) \bar{P}_{i,j,k+1}^{(0)}(s) \quad i \geq 2, j > 0, k \geq 0 \quad (7)$$

$$(s + \tau) \bar{Q}_{i,j}(s) = \xi \left(\bar{P}_{i,j,0}^{(0)}(s) + \sum_{k=1}^{i-j} (1 - \delta_{0,j}) \bar{P}_{i,j,k}^{(0)}(s) + \sum_{k=0}^{i-j-1} \bar{P}_{i,j,k}^{(1)}(s) \right) \\ i \geq j \geq 0 \quad (8)$$

Solving equations (5)-(8) recursively, we obtained the following recursive solutions.

$$\bar{Q}_{0,0}(s) = \frac{\xi}{(s + \tau)(s + \lambda + \xi) - \xi\tau} \quad (9)$$

$$\bar{P}_{0,0,0}^{(0)}(s) = \frac{1}{s + \lambda + \xi} + \frac{\tau}{s + \lambda + \xi} \bar{Q}_{0,0}(s) \quad (10)$$

$$\bar{P}_{i,0,0}^{(0)}(s) = \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i,0}(s) \quad i \geq 1 \quad (11)$$

$$\bar{P}_{i,0,k}^{(1)}(s) = \left(\frac{\lambda}{s + \lambda + \mu + \xi} \right)^{k+1} \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i-k-1,0}(s) \quad i \geq k+2, k \geq 1 \quad (12)$$

$$\bar{Q}_{i,0}(s) = \frac{\xi}{s + \tau} \left[\bar{P}_{i,0,0}^{(0)}(s) + \sum_{k=0}^{i-1} \bar{P}_{i,0,k}^{(1)}(s) \right] \quad i \geq 1 \quad (13)$$

$$\bar{P}_{i,0,0}^{(1)}(s) = \frac{\lambda}{s + \lambda + \mu + \xi} \bar{P}_{i-1,0,0}^{(0)}(s) \quad i \geq 1 \quad (14)$$

$$\bar{P}_{i,0,i-1}^{(1)}(s) = \left(\frac{\lambda}{s + \lambda + \mu + \xi} \right)^i \bar{P}_{0,0,0}^{(0)}(s) \quad i \geq 2 \quad (15)$$

$$\bar{P}_{i,1,k}^{(0)}(s) = \frac{\mu}{s + \lambda + \xi + k\theta} \left[\left(\frac{\lambda}{s + \lambda + \mu + \xi} \right)^{k+1} \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i-k-1,0}(s) \right] \\ i \geq k+2, k \geq 1 \quad (16)$$

$$\bar{P}_{i,1,i-1}^{(0)}(s) = \frac{\mu}{s + \lambda + \xi + (i-1)\theta} \left[\left(\frac{\lambda}{s + \lambda + \mu + \xi} \right)^i \bar{P}_{0,0,0}^{(0)}(s) \right] \quad i \geq 2 \quad (17)$$

$$\bar{P}_{i,1,0}^{(0)}(s) = \frac{\mu}{s + \lambda + \xi} \bar{P}_{i,0,0}^{(1)}(s) + \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i,1}(s) \quad i \geq 1 \quad (18)$$

$$\begin{aligned} \bar{P}_{i,j,0}^{(0)}(s) &= \frac{\mu}{s + \lambda + \xi} \left[\frac{\lambda}{s + \lambda + \mu + \xi} \bar{P}_{i-1,j-1,0}^{(0)}(s) + \frac{\theta}{s + \lambda + \mu + \xi} \bar{P}_{i,j-1,1}^{(0)}(s) \right] \\ &\quad + \frac{\tau}{s + \lambda + \xi} \bar{Q}_{i,j}(s) \quad i \geq j \geq 2 \end{aligned} \quad (19)$$

$$\bar{Q}_{i,j}(s) = \frac{\xi}{s + \tau} \left[\sum_{k=0}^{i-j} \bar{P}_{i,j,k}^{(0)}(s) + \sum_{k=0}^{i-j-1} \bar{P}_{i,j,k}^{(1)}(s) \right] \quad i \geq j \geq 1 \quad (20)$$

$$\bar{P}_{i,j,0}^{(1)}(s) = \frac{\lambda}{s + \lambda + \mu + \xi} \bar{P}_{i-1,j,0}^{(0)}(s) + \frac{\theta}{s + \lambda + \mu + \xi} \bar{P}_{i,j,1}^{(0)}(s) \quad i > j \geq 1 \quad (21)$$

$$\bar{P}_{i,j,k}^{(1)}(s) = \sum_{p=0}^{k+1} \left(\frac{\lambda}{s + \lambda + \mu + \xi} \right)^{k+1-p} \eta_p \bar{P}_{i-k-1+p,j,p}^{(0)}(s) \quad i \geq k + j + 1, \quad k, j \geq 1 \quad (22)$$

where

$$\eta_p = \begin{cases} 1 & \text{if } p = 0 \\ 1 + \frac{p\theta}{s + \lambda + \mu + \xi} & \text{if } p = 1 \text{ to } i - k - 1 \\ \frac{p\theta}{s + \lambda + \mu + \xi} & \text{if } p = i - k \end{cases}$$

$$\bar{P}_{i,j,k}^{(0)}(s) = \frac{\mu}{s + \lambda + \xi + k\theta} \left[\sum_{p=0}^{k+1} \left(\frac{\lambda}{s + \lambda + \mu + \xi} \right)^{k+1-p} \eta_p \bar{P}_{i-k-1+p,j-1,p}^{(0)}(s) \right] \quad (23)$$

$$i \geq k + j, j \geq 2, k \geq 1 \quad (24)$$

where

$$\eta_p = \begin{cases} 1 & \text{if } p = 0 \\ 1 + \frac{p\theta}{s + \lambda + \mu + \xi} & \text{if } p = 1 \text{ to } i - k \\ \frac{p\theta}{s + \lambda + \mu + \xi} & \text{if } p = i - k + 1 \end{cases}$$

4. Transient solution of the model

Taking the Inverse Laplace transform of equations (9)–(23), we obtained the following transient solutions.

$$Q_{0,0}(t) = 2\xi \left[\left(\lambda^2 + \tau^2 + \xi^2 + 2 \left(\tau\xi + \lambda\xi - \lambda\tau \right) \right)^{1/2} e^{-1/2(\lambda+\xi+\tau)t} \sinh \left(\frac{1}{2} t \left\{ \lambda^2 + 2\lambda(\xi - \tau) + (\xi + \tau)^2 \right\}^{1/2} \right) \right]^{-1} \quad (25)$$

$$P_{0,0,0}^{(0)}(t) = e^{-(\lambda+\xi)t} + \tau e^{-(\lambda+\xi)t} * Q_{0,0}(t) \quad (26)$$

$$P_{i,0,i-1}^{(1)}(t) = \lambda^i e^{-(\lambda+\mu+\xi)t} \frac{t^{(i-1)}}{(i-1)!} * P_{0,0,0}^{(0)}(t) \quad i \geq 2 \quad (27)$$

$$P_{i,0,0}^{(0)}(t) = \tau e^{-(\lambda+\xi)t} * Q_{i,0}(t) \quad i \geq 1 \quad (28)$$

$$P_{i,0,0}^{(1)}(t) = \lambda e^{-(\lambda+\mu+\xi)t} * P_{i-1,0,0}^{(0)}(t) \quad i \geq 1 \quad (29)$$

$$P_{i,j,0}^{(0)}(t) = \mu \lambda e^{-(\lambda+\xi)t} \left[\frac{1}{\mu + \xi} - \frac{e^{-(\mu+\xi)t}}{\mu + \xi} \right] * P_{i-1,j-1,0}^{(0)}(t) + \mu \theta e^{-(\lambda+\xi)t} \left[\frac{1}{\mu + \xi} - \frac{e^{-(\mu+\xi)t}}{\mu + \xi} \right] * P_{i,j-1,1}^{(0)}(t) + \tau e^{-(\lambda+\xi)t} * Q_{i,j}(t) \quad i \geq j \geq 2 \quad (30)$$

$$P_{i,j,0}^{(1)}(t) = \lambda e^{-(\lambda+\mu+\xi)t} * P_{i-1,j,0}^{(0)}(t) + \theta e^{-(\lambda+\mu+\xi)t} * P_{i,j,1}^{(0)}(t) \quad i > j \geq 1 \quad (31)$$

$$P_{i,0,k}^{(1)}(t) = \lambda^{k+1} \tau e^{-(\lambda+\xi)t} \left[\frac{1}{(\mu + \xi)^{k+1}} - e^{-(\mu+\xi)t} \sum_{r=0}^k \frac{t^r}{r!} \frac{1}{(\mu + \xi)^{k+1-r}} \right] * Q_{i-k-1,0}(t) \quad i \geq k+2, k \geq 1 \quad (32)$$

$$Q_{i,j}(t) = \xi e^{-\tau t} * \left[\sum_{k=0}^{i-j} P_{i,j,k}^{(0)}(t) + \sum_{k=0}^{i-j-1} P_{i,j,k}^{(1)}(t) \right] \quad i \geq j \geq 1 \quad (33)$$

$$P_{i,j,k}^{(1)}(t) = \lambda^{k+1} e^{-(\lambda+\mu+\xi)t} \frac{t^k}{k!} * P_{i-k-1,j,0}^{(0)}(t) + e^{-(\lambda+\mu+\xi)t} \sum_{p=0}^{k-1} \lambda^{k-p} \frac{t^{k-p-1}}{(k-p-1)!} * P_{i-k+p,j,p+1}^{(0)}(t) + e^{-(\lambda+\mu+\xi)t} \sum_{p=0}^{k-1} \lambda^{k-p} (p+1) \theta \frac{t^{k-p}}{(k-p)!} * P_{i-k+p,j,p+1}^{(0)}(t) + (k+1) \theta e^{-(\lambda+\mu+\xi)t} * P_{i,j,k+1}^{(0)}(t) \quad i \geq j+k+1, j, k \geq 1 \quad (34)$$

$$P_{i,1,i-1}^{(0)}(t) = \mu \lambda^i e^{-(\lambda+(i-1)\theta+\xi)t} \left[\frac{1}{(\mu + \xi)^i} - e^{-(\mu+\xi)t} \sum_{r=0}^{i-1} \frac{t^r}{r!} \frac{1}{(\mu + \xi)^{i-r}} \right] * P_{0,0,0}^{(0)}(t) \quad i \geq 2 \quad (35)$$

$$P_{i,1,0}^{(0)}(t) = \mu e^{-(\lambda+\xi)t} * P_{i,0,0}^{(1)}(t) + \tau e^{-(\lambda+\xi)t} * Q_{i,1}(t) \quad i \geq 1 \quad (36)$$

$$Q_{i,0}(t) = \xi e^{-\tau t} * \left[P_{i,0,0}^{(0)}(t) + \sum_{k=0}^{i-1} P_{i,0,k}^{(1)}(t) \right] \quad i \geq 1 \quad (37)$$

$$\begin{aligned}
P_{i,1,k}^{(0)}(t) &= \mu \lambda^{k+1} \tau \left[\frac{e^{-(\lambda+\xi+k\theta)t}}{(k\theta - \mu)^{m+k}(k\theta)} \right. \\
&\quad + \frac{\sum_{l=1}^{k+1} \sum_{m=1}^l e^{-(\lambda+\mu+\xi)t} t^{k+1-l} (-1)^{m+1} \binom{l-1}{m-1} \left(\prod_{g_1=0}^{l-m-1} (1+g_1) \right) \left(\prod_{g_2=0}^{m-2} (1+g_2) \right)}{(k+1-l)!(m-1)!(k\theta - \mu)^m (-\mu)^{1+l-m}} \\
&\quad \left. + \frac{e^{-(\lambda+\xi)t}}{(k\theta)^m (\mu)^{k+1+l-m}} \right] * Q_{i-k-1,0}(t) \quad i \geq k+2, k \geq 1 \quad (38)
\end{aligned}$$

$$\begin{aligned}
P_{i,j,k}^{(0)}(t) &= \mu \lambda^{k+1} e^{-(\lambda+\xi+k\theta)t} \left[\frac{1}{(\mu + \xi)^{k+1}} - e^{-(\mu+\xi)t} \sum_{r=0}^k \frac{t^r}{r!} \frac{1}{(\mu + \xi)^{k+1-r}} \right] * P_{i-k-1,j-1,0}^{(0)}(t) \\
&\quad + \mu e^{-(\lambda+\xi+k\theta)t} \sum_{p=0}^{k-1} \lambda^{k-p} \left[\frac{1}{(\mu + \xi)^k} - e^{-(\mu+\xi)t} \sum_{r=0}^{k-1} \frac{t^r}{r!} \frac{1}{(\mu + \xi)^{k-r}} \right] * P_{i-k+p,j-1,p,0}(t) \\
&\quad + \mu e^{-(\lambda+\xi+k\theta)t} \sum_{p=0}^{k-1} \lambda^{k-p} (p+1) \theta \left[\frac{1}{(\mu + \xi)^{k+1}} - e^{-(\mu+\xi)t} \sum_{r=0}^k \frac{t^r}{r!} \frac{1}{(\mu + \xi)^{k+1-r}} \right] * \quad (39)
\end{aligned}$$

$$\begin{aligned}
&P_{i-k+p,j-1,p}^{(0)}(t) + \mu e^{-(\lambda+\xi+k\theta)t} (k+1) \theta \left[\frac{1}{\mu + \xi} - \frac{e^{-(\mu+\xi)t}}{\mu + \xi} \right] * P_{i,j-1,k+1}^{(0)}(t) \\
&i \geq k+j, j \geq 2, k \geq 1 \quad (40)
\end{aligned}$$

5. Verification of results

Taking $\xi = 0$, $\tau = 0$ and

$P_{i,j,k}^{(0)}(t) = P_{i,j}^{(0)}(t)$ where $k = \text{number of arrivals} - \text{number of departures}$.

$P_{i,j,k}^{(1)}(t) = P_{i,j}^{(1)}(t)$ where $k = \text{number of arrivals} - \text{number of departures} - 1$.

in equations (1 – 4) we get,

$$\frac{d}{dt} P_{i,j}^{(0)}(t) = -(\lambda + (i-j)\theta) P_{i,j}^{(0)}(t) + \mu(1 - \delta_{0,j}) P_{i,j-1}^{(1)}(t) \quad i \geq j \geq 0 \quad (41)$$

$$\frac{d}{dt} P_{1,0}^{(1)}(t) = -(\lambda + \mu) P_{1,0}^{(1)}(t) + \lambda P_{0,0}^{(0)}(t) \quad (42)$$

$$\begin{aligned}
\frac{d}{dt} P_{i,j}^{(1)}(t) &= -(\lambda + \mu) P_{i,j}^{(1)}(t) + \lambda(1 - \delta_{0,j}) P_{i-1,j}^{(0)}(t) + \lambda(1 - \delta_{i-1,j}) P_{i-1,j}^{(1)}(t) \\
&\quad + (i-j)\theta(1 - \delta_{0,j}) P_{i,j}^{(0)}(t) \quad i > 1, i > j \geq 0 \quad (43)
\end{aligned}$$

which coincides with the results (1 – 3) of Singla and Kalra (2018). Furthermore, the model given in equations (39) – (41) can be converted into one-dimensional model by defining the probability $U_n^{(m)}(t)$ as:

$U_n^{(m)}(t)$ = Probability that there are n customers in the orbit at time t and the server is free or busy according as $m = 0$ or 1 .

When server is free, it is defined as probability:

$$U_n^{(0)}(t) = \sum_{j=0}^{\infty} P_{j+n,j}^{(0)}(t)$$

where n = number of arrivals - number of departures.
when server is busy, it is defined as probability:

$$U_n^{(1)}(t) = \sum_{j=0}^{\infty} P_{j+n+1,j}^{(1)}(t)$$

where n = number of arrivals - number of departures - 1.

By using above definitions in equations (39 – 41) the equations in statistical equilibrium are:

$$\begin{aligned} (\lambda + n\theta)U_n^{(0)} &= \mu U_n^{(1)} & n \geq 0 \\ (\lambda + \mu)U_n^{(1)} &= \lambda(U_n^{(0)} + U_{n-1}^{(1)}) + (n+1)\theta U_{n+1}^{(0)} & n \geq 1 \end{aligned}$$

which coincides with the results (1.5) – (1.6) of Falin and Templeton (1997)

6. Numerical solution and graphical representation

The Numerical results are generated using MATLAB programming and following Bunday (1986) for the case $\rho = \left(\frac{\lambda}{\mu}\right) = 0.7$, $\eta = \left(\frac{\theta}{\mu}\right) = 0.6$, $\tau' = \left(\frac{\tau}{\mu}\right) = 0.4$, $\xi' = \left(\frac{\xi}{\mu}\right) = 0.3$. In following tables, we observe some probabilities at various time instants whose sum approaches to 1.

Table 1: At time $t = 1$

t	$P_{0,0,0}^{(0)}$	$P_{1,1,0}^{(0)}$	$P_{2,1,1}^{(0)}$	$P_{1,0,0}^{(1)}$	$P_{2,0,1}^{(1)}$	$P_{2,1,0}^{(1)}$	$P_{3,0,2}^{(1)}$	$P_{3,1,1}^{(1)}$	$Q_{0,0}$	$Q_{1,0}$	$Q_{1,1}$
1	0.3954	0.0972	0.0161	0.1676	0.0483	0.0212	0.0102	0.0064	0.1542	0.0365	0.0122

$Q_{2,0}$	$Q_{2,1}$	Sum
0.0068	0.0035	0.9756

Table 2: At time $t = 25$

t	$P_{5,0,0}^{(0)}$	$P_{4,3,0}^{(0)}$	$P_{3,2,0}^{(0)}$	$P_{5,1,0}^{(0)}$	$P_{5,5,0}^{(0)}$	$P_{5,4,0}^{(0)}$	$P_{5,3,0}^{(0)}$	$P_{5,2,0}^{(0)}$	$Q_{5,0}$	$Q_{5,1}$	$Q_{5,2}$
25	0.0076	0.0039	0.003	0.0408	0.0898	0.1383	0.1467	0.1001	0.006	0.0315	0.0762

$Q_{5,3}$	$Q_{5,4}$	$Q_{5,5}$	Sum
0.1097	0.1014	0.0649	0.9199

Table 3: At time $t = 40$

t	$P_{5,0,0}^{(0)}$	$P_{5,1,0}^{(0)}$	$P_{5,5,0}^{(0)}$	$P_{5,4,0}^{(0)}$	$P_{5,3,0}^{(0)}$	$P_{5,2,0}^{(0)}$	$Q_{5,0}$	$Q_{5,1}$	$Q_{5,2}$	$Q_{5,3}$	$Q_{5,4}$
40	0.0082	0.0442	0.0942	0.1503	0.1611	0.1097	0.0062	0.0333	0.0823	0.1207	0.1125

$Q_{5,5}$	Sum
0.0705	0.9932

The probabilities against time are represented graphically in the following figures.

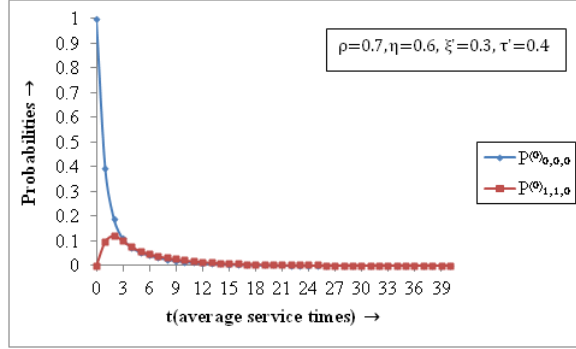


Figure 3: Probabilities $P_{0,0,0}^{(0)}$ and $P_{1,1,0}^{(0)}$ against average service times

In Figure 3, the probabilities $P_{0,0,0}^{(0)}$ and $P_{1,1,0}^{(0)}$ are plotted against average service times t for the given case. Here we observe that the probability $P_{0,0,0}^{(0)}$ decreases rapidly from its initial value 1 (at time $t = 0$) whereas the probability $P_{1,1,0}^{(0)}$ increases from its initial value 0 (at $t = 0$) in the beginning and then decreases.

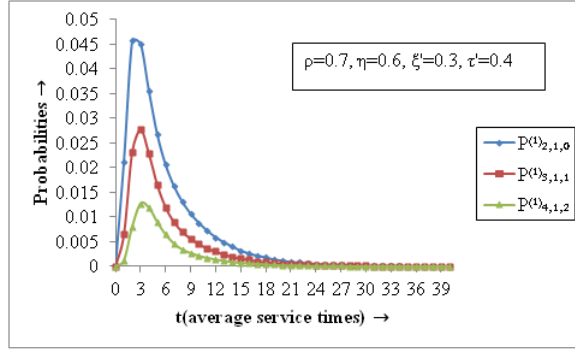


Figure 4: Comparison among the probabilities $P_{2,1,0}^{(1)}(t)$, $P_{3,1,1}^{(1)}(t)$, and $P_{4,1,2}^{(1)}(t)$

Figure 4 shows the comparison between the probabilities $P_{2,1,0}^{(1)}$, $P_{3,1,1}^{(1)}$ and $P_{4,1,2}^{(1)}$ against times t . From the graph it is observed that all the probabilities increase in the beginning and then decrease. Furthermore, the above graph shows that the probabilities achieve lower values with the increase in arrivals for the same number of departures.

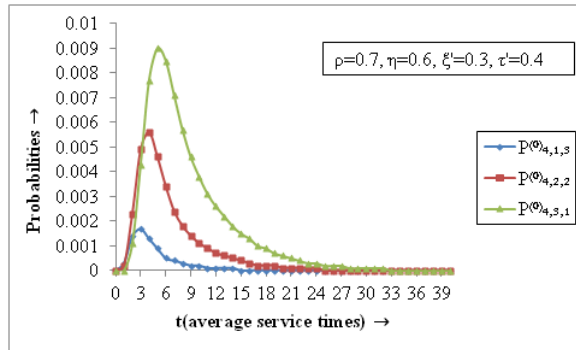


Figure 5: Comparison among the probabilities $P_{4,1,3}^{(0)}(t)$, $P_{4,2,2}^{(0)}(t)$, and $P_{4,3,1}^{(0)}(t)$

In Figure 5, the probabilities $P_{4,1,3}^{(0)}$, $P_{4,2,2}^{(0)}$ and $P_{4,3,1}^{(0)}$ are compared against average service times t for the given case. All the probabilities increase initially and then start decreasing after reaching their peak. It is also interpreted from the graph that the considered probabilities are higher when the number of departures is larger for the same number of arrivals.

7. Sensitivity analysis of the model

In this section, the effect of change in parameters of the system on the various probabilities is studied. The data is summarized in the following tables. In Table 4, the probability $P_{0,0,0}^{(0)}(t)$ is given for different values of ρ . It can be seen that in general whenever the arrival rate per unit service time increases the probability $P_{0,0,0}^{(0)}(t)$ decreases against average service time t . In Table 5, the effect of catastrophe rate per unit service time on the probability of system being under repair is studied. It is observed that the probability of system being under repair increases with the increase in ξ' . Similarly the effect of repair rate per unit service time on the probability of system being under repair is studied in Table 6. Here we notice that the probability decreases with the increase in τ' .

Table 4: Effect of arrival rate per unit service time on probability $P_{0,0,0}^{(0)}(t)$

	$\rho = 0.5$	$\rho = 0.7$	$\rho = 0.9$
t	$P_{0,0,0}^{(0)}$	$P_{0,0,0}^{(0)}$	$P_{0,0,0}^{(0)}$
0	1	1	1
3	0.1745	0.1092	0.0704
6	0.0772	0.0413	0.0243
9	0.0414	0.0196	0.0107
12	0.0227	0.0095	0.0047
15	0.0124	0.0046	0.0021
18	0.0068	0.0022	0.0009
21	0.0037	0.0011	0.0004
24	0.0021	0.0005	0.0002
27	0.0011	0.0003	0.0001
30	0.0006	0.0001	0
33	0.0003	0.0001	0
36	0.0002	0	0
39	0.0001	0	0

Table 5: Effect of catastrophe rate per unit service time on probability of system being under repair

	$\xi' = 0.3$			$\xi' = 0.6$			$\xi' = 0.9$		
t	$Q_{2,1}$	$Q_{3,1}$	$Q_{4,1}$	$Q_{2,1}$	$Q_{3,1}$	$Q_{4,1}$	$Q_{2,1}$	$Q_{3,1}$	$Q_{4,1}$
1	0.0035	0.0007	0.0001	0.0057	0.001	0.0001	0.0069	0.0012	0.0002
3	0.0317	0.0133	0.0046	0.0409	0.0165	0.0054	0.041	0.0158	0.005
6	0.0409	0.0242	0.0121	0.0572	0.0325	0.0153	0.0609	0.0329	0.0145
9	0.0328	0.0233	0.0138	0.0559	0.0388	0.0219	0.0663	0.0439	0.0234
12	0.0233	0.019	0.0128	0.0481	0.0388	0.0251	0.0636	0.0491	0.0301
15	0.0154	0.0141	0.0105	0.0385	0.0351	0.0253	0.0565	0.0495	0.034
18	0.0097	0.0098	0.008	0.0293	0.0296	0.0235	0.0477	0.0466	0.0354
21	0.0059	0.0065	0.0057	0.0215	0.0238	0.0206	0.0388	0.0418	0.0346
24	0.0035	0.0042	0.0039	0.0153	0.0185	0.0172	0.0307	0.036	0.0322
27	0.0021	0.0026	0.0026	0.0107	0.0139	0.0139	0.0237	0.0301	0.029
30	0.0012	0.0016	0.0017	0.0073	0.0102	0.0109	0.018	0.0246	0.0253
33	0.0007	0.001	0.0011	0.0049	0.0074	0.0083	0.0135	0.0197	0.0215
36	0.0004	0.0006	0.0007	0.0033	0.0052	0.0062	0.0099	0.0155	0.0179
39	0.0002	0.0003	0.0004	0.0022	0.0037	0.0046	0.0073	0.012	0.0146

Table 6: Effect of repair rate per unit service time on probability of system being under repair

	$\tau' = 0.2$			$\tau' = 0.5$			$\tau' = 0.9$		
t	$Q_{4,2}$	$Q_{4,3}$	$Q_{4,4}$	$Q_{4,2}$	$Q_{4,3}$	$Q_{4,4}$	$Q_{4,2}$	$Q_{4,3}$	$Q_{4,4}$
1	0	0	0	0	0	0	0	0	0
3	0.0038	0.0013	0.0002	0.0035	0.0012	0.0002	0.0031	0.0011	0.0001
6	0.017	0.0116	0.0035	0.0152	0.0103	0.003	0.0128	0.0087	0.0025
9	0.0228	0.0185	0.0074	0.0193	0.0156	0.0061	0.0143	0.0119	0.0046
12	0.0247	0.0207	0.0087	0.0178	0.0153	0.0063	0.0106	0.0097	0.0041
15	0.0246	0.0207	0.0086	0.014	0.0123	0.0051	0.0064	0.0061	0.0027
18	0.0232	0.0195	0.0079	0.0099	0.0089	0.0037	0.0034	0.0033	0.0015
21	0.021	0.0177	0.0071	0.0066	0.006	0.0024	0.0016	0.0016	0.0007
24	0.0185	0.0156	0.0061	0.0041	0.0038	0.0015	0.0007	0.0007	0.0003
27	0.016	0.0134	0.0052	0.0025	0.0023	0.0009	0.0003	0.0003	0.0001
30	0.0135	0.0113	0.0043	0.0015	0.0013	0.0005	0.0001	0.0001	0.0001
33	0.0112	0.0094	0.0035	0.0008	0.0008	0.0003	0	0.0001	0
36	0.0092	0.0077	0.0029	0.0005	0.0004	0.0002	0	0	0
39	0.0075	0.0063	0.0023	0.0002	0.0002	0.0001	0	0	0

This variation can also be shown graphically in the figures (6) – (8).

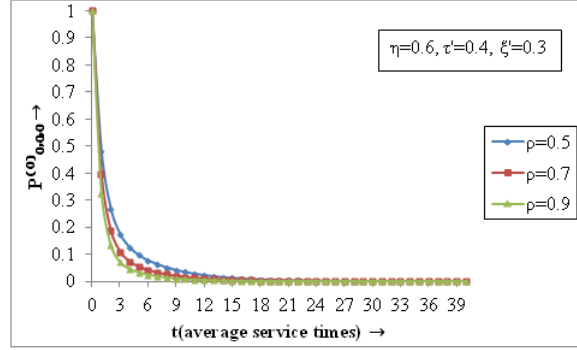


Figure 6: Probability $P_{0,0,0}^{(0)}$ for different values of ρ

Figure 6 shows the variation in probability $P_{0,0,0}^{(0)}$ by taking the different values of arrival rate per unit service time (ρ). The probability decreases with the increasing values of ρ .

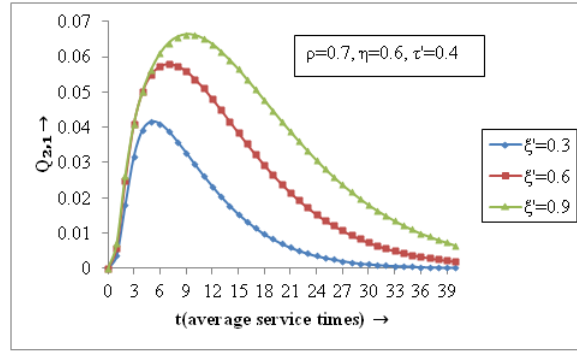


Figure 7: Probability $Q_{2,1}$ for different values of ξ'

In Figure 7, the variation is shown graphically by taking the probability of system being under repair when the number of arrivals are two and the departed units are one for different values of ξ' .

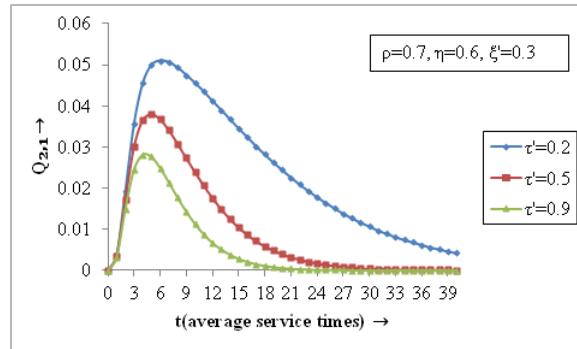


Figure 8: Probability $Q_{2,1}$ for different values of τ'

Similarly in Figure 8, the effect of τ' (change of repair rate per unit service time) on the probability $Q_{2,1}$ is studied. It is observed from the graph that the probability of system being under repair decreases with the increasing values of repair rate per unit service time which is as desired.

8. Conclusion

In this paper, we studied a single server retrial queueing system with catastrophe. It is very crucial to identify and mitigate the factors that lead to catastrophic events in designing efficient queueing systems. The proposed model is highly applicable to represent many practical situations like submitting any application online, ticket booking services using telephone facility, withdrawing cash at an ATM, manufacturing sectors, call centers, *etc.* Results are well-quantified as the obtained transient probabilities consider arrivals and departures separately rather than total number of customers in the system. The generated numerical results for the developed model and their graphical representation successfully depicts the effect of various parameters on various probabilities. Future research will aim to develop queueing models that incorporate other influencing factors, such as server heterogeneity and customer impatience.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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