

An Inferential Study of Two Kumaraswamy Populations under Joint Ranked Set Sampling

Mahesh K. Bhingikar and D. P. Raykundaliya

*Department of Statistics
Sardar Patel University, Vallabh Vidyanagar, Anand, Gujarat, India*

Received: 24 January 2025; Revised: 02 May 2025; Accepted: 04 May 2025

Abstract

In this study, we have derived the expression for Maximum Likelihood (ML) estimates and likelihood ratio test (LRT) for two Kumaraswamy populations under Joint Ranked Set Sampling (JRSS), Joint Modified Minimum Ranked Set Sampling (JMnRSS), Joint Modified Maximum Ranked Set Sampling (JMxRSS), and Joint Simple Random Sampling (JSRS). The performance of the ML estimates is evaluated using Root Mean Square Error (RMSE) and the bias criterion. LRT is conducted to test the equality of the first shape parameters of two Kumaraswamy populations when the other two shape parameters are known. The power of the LRT is determined to compare the test performance under the mentioned sampling schemes. This simulations for ML and LRT results are obtained using Monte Carlo simulations in R Studio. We found that all joint ranked sampling schemes demonstrate superior performance compared to JSRS in both ML estimation and LRT. However, the JRSS exhibited the best results in LRT when the other two shape parameters are known, while the JMnRSS excelled in ML estimation. Additionally, we provide an illustration using real-life lung cancer data to highlight these findings.

Key words: Kumaraswamy distribution; Joint modified minimum ranked set sampling; Joint modified maximum ranked set sampling; Maximum likelihood estimates.

AMS Subject Classifications: 62D05, 62F07, 62F10, 62F03

The video recording of the paper made under the SSCA's Online Lecture series is available at the Youtube channel URL <https://youtu.be/xp3vRT40tWg>.

1. Introduction

According to World Health Organization (2023), lung cancer is one of the leading causes of cancer-related mortality worldwide, accounting for a significant proportion of all cancer deaths. It's costly to diagnose lung cancer in India: *Rs.*15,000 – *Rs.*1,50,000 and in the USA: \$2,000–\$20,000. To reduce the cost of testing for lung cancer in the laboratory, it is possible to determine the probability of cancer by using modeling based on auxiliary

variables. The probability of cancer can be calculated by logistic regression, Probit Model, Linear Discriminant Analysis (LDA), Support Vector Machine (SVM), Decision Trees (*e.g.*, CART), Random Forests, Neural Networks (SLPs, DNNs), *etc.* We can use the probability (proportion) calculated from modeling to analyze the lung cancer population based on a sample. If the population is finite and the goal is to estimate the population proportion, we can apply sampling theory methods. If our sample size is small relative to the population and we aim to predict the proportion while considering greater variation in the data, we should assume an infinite population. So, to analyze the proportion data in the range of 0 to 1, we may have two choices between Beta-I distribution and Kumaraswamy distribution. According to the available literature, the Kumaraswamy distribution has additional characteristics compared to the Beta-I distribution, such as the fact that Kumaraswamy variables exhibit closeness under exponentiation and linear transformation (Mitnik, 2013). Therefore, to analyze the proportion data, it is advisable to use the Kumaraswamy distribution. If we wish to analyze the proportion of lung cancer by fitting the Kumaraswamy distribution, we can address only the questions related to lung cancer in the overall population, not separately for males and females. To do so, we need a sample of the male population of size m_1 and a sample of the female population of size m_2 for the study. As a result, both the cost and the duration of the study increase. To avoid these issues, we can use joint censoring schemes used in reliability theory (Ashour and Abo-Kasem, 2014; Ding and Gui, 2023). This scheme involves selecting a combined total of m units from the population. To analyze the situation, assume we choose $m_1 = 12$ patients from the male population and $m_2 = 5$ patients from the female population, we may find that the estimates related to the female population are not as efficient as those for the male population. However, by using joint censoring with a total sample size of $m = 17$ (*i.e.*, $m_1 + m_2 = m$), we can still proceed with our analysis. Joint censoring refers to the need for a combined number of sample units rather than individual sample units from each population. This method is applicable when both populations are independent and relatively similar, which is true in this case. To increase efficiency and reduce cost and time, popular schemes are also available in the literature called ranked set sampling (RSS) schemes (McIntyre, 1952; Chen *et al.*, 2004; Wolfe, 2004; Akgul and Senoglu, 2017; Abbasi and Shahd, 2017; Koyuncu, 2018).

Therefore, analysis of lung cancer data, we believe that the various joint ranked set sampling schemes are well-suited compared to JSRS. The aim of any study is to increase efficiency with lesser cost and a smaller sample size. As discussed previously, it is possible to reduce the sample size by using the various joint sampling schemes. Initially various JRSS schemes used to estimate the parameter of the exponential distribution by Raykundaliya and Patel (2022b) and to estimate the parameters of the Rayleigh distribution using JRSS by Patel *et al.* (2022). Raykundaliya and Patel (2022a) used the joint percentile Ranked Set Sampling (JPRSS) scheme to estimate the parameters of the exponential distribution and recently Raykundaliya and Bhingikar (2025) also used this sampling scheme to estimate the parameters for the Weibull population using JRSS. Therefore, we use the JRSS schemes to estimate the parameters by ML method for the Kumaraswamy population. Rest of paper is arranged as follows. Section 2 - 4 derive the expressions required for ML estimates as well as for the Fisher information (I) matrix under JRSS, JMnRSS, JMxRSS, and JSRS. Section 5 discusses the LRT for the JRSS, JMnRSS, JMxRSS, and JSRS. In Section 6, we have conducted the simulation for ML estimates of parameters of two Kumaraswamy populations using the mentioned sampling schemes. Further, we discuss the application of

the Kumaraswamy population for lung cancer data using various joint ranked set sampling. Finally, some concluding remarks are given in Section 7.

2. Joint Ranked Set Sampling (JRSS)

In this Section, we have derived the ML estimates of the parameters of two Kumaraswamy populations under JRSS.

2.1. PDF and CDF of Kumaraswamy populations

Let X_1, X_2, \dots, X_{m_1} be r.s (random sample) drawn from Kumaraswamy (say it population-I) whose pdf and cdf are given as follows,

$$f(x; a_1, b_1) = \begin{cases} a_1 b_1 x^{a_1-1} (1 - x^{a_1})^{b_1-1}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$F(x; a_1, b_1) = 1 - (1 - x^{a_1})^{b_1} \quad (2)$$

where a_1, b_1 are non negative shape parameters

Let Y_1, Y_2, \dots, Y_{m_2} be r.s drawn from Kumaraswamy (say it population-II) whose pdf and cdf are given as follows,

$$g(y; a_2, b_2) = \begin{cases} a_2 b_2 y^{a_2-1} (1 - y^{a_2})^{b_2-1}, & \text{if } 0 < y < 1 \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

$$G(y; a_2, b_2) = 1 - (1 - y^{a_2})^{b_2} \quad (4)$$

where a_2, b_2 are non negative shape parameters

2.2. Algorithm of JRSS

Algorithm 1 The algorithm to select JRSS from two populations

- 1: Select m_1 observations from a population-I and m_2 observations from population-II randomly and combine these observations to create a joint sample of size $m = m_1 + m_2$.
 - 2: Arrange all the observations from the joint sample in ascending order.
 - 3: To get m sets of m sizes, repeat 1 and 2, m times.
 - 4: To get a joint ranked set sample of size m , select i^{th} minimum from i^{th} set. where $i = 1, 2, \dots, m$.
 - 5: Repeat 1 to 4 to increase the size of the joint ranked set sample k times *i.e.* $n = mk$.
-

2.3. Maximum likelihood estimation under JRSS

The likelihood function according to the algorithm 1 based on the observations $w_{ij}, i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, k$ obtained in the joint rank set sample is given as follows,

$$L(a_1, b_1, a_2, b_2 | \underline{w}) = \prod_{j=1}^k \prod_{i=1}^m \frac{m!}{(i-1)!(m-i)!} [F_x(w_{ij})]^{a_{ij}} [1 - F_x(w_{ij})]^{b_{ij}} [f_x(w_{ij})]^{z_{ij}} [G_y(w_{ij})]^{c_{ij}} \\ \times [1 - G_y(w_{ij})]^{d_{ij}} [g_y(w_{ij})]^{1-z_{ij}} \quad (5)$$

where w_{ij} represents the i^{th} order element from the i^{th} joint sample in the j^{th} cycle, where $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, k$. Likelihood under JRSS for Kumaraswamy populations is obtained by substituting equations (1),(2),(3), and (4) in (5).

$$L(a_1, b_1, a_2, b_2 | \underline{w}) = \prod_{j=1}^k \prod_{i=1}^m C \left[1 - (1 - w_{ij}^{a_1})^{b_1} \right]^{a_{ij}} \left[(1 - w_{ij}^{a_1})^{b_1} \right]^{b_{ij}} \left[a_1 b_1 w_{ij}^{a_1-1} (1 - w_{ij}^{a_1})^{b_1-1} \right]^{z_{ij}} \\ \left[1 - (1 - w_{ij}^{a_2})^{b_2} \right]^{c_{ij}} \left[(1 - w_{ij}^{a_2})^{b_2} \right]^{d_{ij}} \left[a_2 b_2 w_{ij}^{a_2-1} (1 - w_{ij}^{a_2})^{b_2-1} \right]^{1-z_{ij}} \quad (6)$$

where,

$$z_{ij} = \begin{cases} 1 & \text{if } w_{ij} \text{ is from first population-I(X)} \\ 0 & \text{if } w_{ij} \text{ is from second population-II(Y),} \end{cases} \quad i = 1, 2, \dots, m; j = 1, 2, \dots, k$$

$$C = \frac{m!}{(i-1)!(m-i)!}$$

$w_{ij} = i^{\text{th}}$ minimum of i^{th} set of combined sample in j^{th} cycle.

$a_{ij} =$ Number of x's observation less than or equal to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

$b_{ij} =$ Number of x's observation greater than to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

$c_{ij} =$ Number of y's observation less than or equal to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

$d_{ij} =$ Number of y's observation greater than to w_{ij} in i^{th} set of combined samples in j^{th} cycle.

The log-likelihood function for the Kumaraswamy distribution under JRSS can be expressed by taking the logarithm of the equation (6) based on the observations w_{ij} , where $i = 1, 2, 3, \dots, m$ and $j = 1, 2, 3, \dots, k$.

$$l = \log(L(a_1, b_1, a_2, b_2 | \underline{w})) = \sum_{i=1}^m \sum_{j=1}^k a_{ij} \ln \left(1 - (1 - w_{ij}^{a_1})^{b_1} \right) + b_1 \sum_{i=1}^m \sum_{j=1}^k b_{ij} \ln (1 - w_{ij}^{a_1}) \\ + (b_1 - 1) \sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln (1 - w_{ij}^{a_1}) + \sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln (a_1 b_1 w_{ij}^{a_1-1}) \\ + \sum_{i=1}^m \sum_{j=1}^k c_{ij} \ln \left(1 - (1 - w_{ij}^{a_2})^{b_2} \right) + b_2 \sum_{i=1}^m \sum_{j=1}^k d_{ij} \ln (1 - w_{ij}^{a_2}) \\ + (b_2 - 1) \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln (1 - w_{ij}^{a_2}) \\ + \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln (a_2 b_2 w_{ij}^{a_2-1}) \quad (7)$$

$$\frac{\partial l}{\partial a_1} = \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_1 \ln (w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_1} \ln (w_{ij}) ((b_1 - 1) z_{ij} + b_1 b_{ij})}{1 - w_{ij}^{a_1}} \\ + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln (w_{ij}) (1 - w_{ij}^{a_1})^{b_1-1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \quad (8)$$

$$\begin{aligned} \frac{\partial l}{\partial b_1} &= \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_1} + \sum_{i=1}^m \sum_{j=1}^k \ln(1 - w_{ij}^{a_1}) z_{ij} - \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} \ln(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{b_1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k b_{ij} \ln(1 - w_{ij}^{a_1}) \end{aligned} \quad (9)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial a_1^2} &= \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln^2(w_{ij}) + 2b_1 \ln(w_{ij})) - \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k \ln(w_{ij}) z_{ij} (a_1 \ln(w_{ij}) + 1) \\ &\quad - \frac{1}{a_1^2} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) w_{ij}^{a_1} \ln^2(w_{ij}) z_{ij} + b_1 b_{ij} w_{ij}^{a_1} \ln^2(w_{ij})}{1 - w_{ij}^{a_1}} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) w_{ij}^{a_1} \ln(w_{ij}) (w_{ij}^{a_1} \ln(w_{ij}) z_{ij} - b_1 b_{ij} w_{ij}^{a_1} \ln(w_{ij}))}{(1 - w_{ij}^{a_1})^2} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{b_1^2 a_{ij} w_{ij}^{2a_1} \ln^2(w_{ij}) (1 - w_{ij}^{a_1})^{2b_1 - 2}}{(1 - (1 - w_{ij}^{a_1})^{b_1})^2} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) b_1 a_{ij} w_{ij}^{2a_1} \ln^2(w_{ij}) (1 - w_{ij}^{a_1})^{b_1 - 2}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln^2(w_{ij}) (1 - w_{ij}^{a_1})^{b_1 - 1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \end{aligned} \quad (10)$$

$$\frac{\partial^2 l}{\partial b_1^2} = - \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_1^2} - \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} \ln^2(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{b_1}}{((1 - w_{ij}^{a_1})^{b_1} - 1)^2} \quad (11)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial a_1 \partial b_1} &= - \frac{1}{a_1 b_1} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln(w_{ij}) + 1) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln(w_{ij}) \ln(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{b_1 - 1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln(w_{ij}) \ln(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{2b_1 - 1}}{(1 - (1 - w_{ij}^{a_1})^{b_1})^2} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{a_1 w_{ij}^{1 - a_1} (w_{ij}^{a_1 - 1} \ln(w_{ij}) + w_{ij}^{a_1 - 1}) z_{ij}}{a_1 b_1} - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_1} \ln(w_{ij}) z_{ij}}{1 - w_{ij}^{a_1}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} w_{ij}^{a_1} \ln(w_{ij}) (1 - w_{ij}^{a_1})^{b_1 - 1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} - \sum_{i=1}^m \sum_{j=1}^k \frac{b_{ij} w_{ij}^{a_1} \ln(w_{ij})}{1 - w_{ij} z_{ij}^{a_1}} \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{\partial l}{\partial a_2} &= \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_2 \ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) w_{ij}^{a_2} \ln(w_{ij}) z_{ij} - b_2 d_{ij} w_{ij}^{a_2} \ln(w_{ij})}{1 - w_{ij}^{a_2}} \\ &\quad + b_2 \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) (1 - w_{ij}^{a_2})^{b_2 - 1}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\partial l}{\partial b_2} &= \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_2} + \sum_{i=1}^m \sum_{j=1}^k \ln(1 - w_{ij}^{a_2}) z_{ij} - \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} \ln(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{b_2}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k d_{ij} \ln(1 - w_{ij}^{a_2}) \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial a_2^2} &= \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_2 \ln^2(w_{ij}) + 2 \ln(w_{ij})) - \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k \ln(w_{ij}) z_{ij} (a_2 \ln(w_{ij}) + 1) \\ &\quad - \frac{1}{a_2^2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_2 \ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) w_{ij}^{a_2} \ln^2(w_{ij}) z_{ij} - b_2 d_{ij} w_{ij}^{a_2} \ln^2(w_{ij})}{1 - w_{ij}^{a_2}} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_2} \ln(w_{ij}) ((b_2 - 1) w_{ij}^{a_2} \ln(w_{ij}) z_{ij} - b_2 d_{ij} w_{ij}^{a_2} \ln(w_{ij}))}{(1 - w_{ij}^{a_2})^2} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{b_2^2 c_{ij} w_{ij}^{2a_2} \ln^2(w_{ij}) (1 - w_{ij}^{a_2})^{2b_2 - 2}}{(1 - (1 - w_{ij}^{a_2})^{b_2})^2} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) b_2 c_{ij} w_{ij}^{2a_2} \ln^2(w_{ij}) (1 - w_{ij}^{a_2})^{b_2 - 2}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{b_2 c_{ij} w_{ij}^{a_2} \ln^2(w_{ij}) (1 - w_{ij}^{a_2})^{b_2 - 1}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \end{aligned} \quad (15)$$

$$\frac{\partial^2 l}{\partial b_2^2} = - \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_2^2} - \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} \ln^2(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{b_2}}{((1 - w_{ij}^{a_2})^{b_2} - 1)^2} \quad (16)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial a_2 \partial b_2} &= - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{1-a_2} z_{ij} (w_{ij}^{a_2-1} \ln(w_{ij}) a_2 b_2 + w_{ij}^{a_2-1} b_2)}{a_2 b_2^2} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) \ln(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{b_2-1} b_2}{1 - (1 - w_{ij}^{a_2})^{b_2}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) \ln(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{2b_2-1} b_2}{(1 - (1 - w_{ij}^{a_2})^{b_2})^2} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{1-a_2} (w_{ij}^{a_2-1} \ln(w_{ij}) a_2 + w_{ij}^{a_2-1}) z_{ij}}{a_2 b_2} - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_2} \ln(w_{ij}) z_{ij}}{1 - w_{ij}^{a_2}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) (1 - w_{ij}^{a_2})^{b_2-1}}{1 - (1 - w_{ij}^{a_2})^{b_2}} - \sum_{i=1}^m \sum_{j=1}^k \frac{d_{ij} w_{ij}^{a_2} \ln(w_{ij})}{1 - w_{ij}} \end{aligned} \quad (17)$$

To determine the ML estimates for the parameters of the Kumaraswamy distribution under the JRSS, solve the equations (8), (9), (13), and (14) iteratively. Fisher information matrix (I) for the parameters of two Kumaraswamy populations under JRSS can be written as

follows,

$$I(a_1, b_1, a_2, b_2) = \begin{bmatrix} I_{a_1 a_1} & I_{a_1 b_1} & I_{a_1 a_2} & I_{a_1 b_2} \\ I_{b_1 a_1} & I_{b_1 b_1} & I_{b_1 a_2} & I_{b_1 b_2} \\ I_{a_2 a_1} & I_{a_2 b_1} & I_{a_2 a_2} & I_{a_2 b_2} \\ I_{b_2 a_1} & I_{b_2 b_1} & I_{b_2 a_2} & I_{b_2 b_2} \end{bmatrix} = -E \begin{bmatrix} \frac{\partial^2 l}{\partial a_1^2} & \frac{\partial^2 l}{\partial a_1 \partial b_1} & 0 & 0 \\ \frac{\partial^2 l}{\partial b_1 \partial a_1} & \frac{\partial^2 l}{\partial b_1^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 l}{\partial a_2^2} & \frac{\partial^2 l}{\partial a_2 \partial b_2} \\ 0 & 0 & \frac{\partial^2 l}{\partial b_2 \partial a_2} & \frac{\partial^2 l}{\partial b_2^2} \end{bmatrix} \quad (18)$$

3. Joint Modified Minimum Ranked Set Sampling (JMnRSS)

In this Section, we studied the estimation of parameters of the Kumaraswamy population under JMnRSS.

Algorithm 2 The algorithm to select JMnRSS sample from two populations

- 1: Select m_1 observations from population-I and m_2 observations from population-II randomly and combine these observations to create a joint sample of size $m = m_1 + m_2$.
 - 2: Arrange all these observations of the joint sample in ascending order.
 - 3: To get m joint sets of sizes m , repeat 1 and 2 m times.
 - 4: To get a JMnRSS sample of size m , select the first minimum from all m sets.
 - 5: Repeat 1 to 4 to increase the size of the JMnRSS sample k times *i.e.* $n = mk$.
-

3.1. Maximum likelihood estimation under JMnRSS

The likelihood function according to algorithm 2 based on the observations w_{ij} , $i = 1, 2, 3, \dots, m$; $j = 1, 2, 3, \dots, k$ can be obtained by putting $a_{ij} = c_{ij} = 0$ for all $i = 1, 2, 3, \dots, m$; $j = 1, 2, \dots, k$ in equation (7)

$$\begin{aligned} l &= \log(L(a_1, b_1, a_2, b_2 | \underline{w})) \\ &= b_1 \sum_{i=1}^m \sum_{j=1}^k b_{ij} \ln(1 - w_{ij}^{a_1}) + (b_1 - 1) \sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln(1 - w_{ij}^{a_1}) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln(a_1 b_1 w_{ij}^{a_1 - 1}) + b_2 \sum_{i=1}^m \sum_{j=1}^k d_{ij} \ln(1 - w_{ij}^{a_2}) \\ &\quad + (b_2 - 1) \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln(1 - w_{ij}^{a_2}) + \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln(a_2 b_2 w_{ij}^{a_2 - 1}) \end{aligned} \quad (19)$$

$$\frac{\partial l}{\partial a_1} = \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_1 \ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_1} \ln(w_{ij}) ((b_1 - 1) z_{ij} + b_1 b_{ij})}{1 - w_{ij}^{a_1}} \quad (20)$$

$$\frac{\partial l}{\partial b_1} = \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_1} + \sum_{i=1}^m \sum_{j=1}^k \ln(1 - w_{ij}^{a_1}) z_{ij} + \sum_{i=1}^m \sum_{j=1}^k b_{ij} \ln(1 - w_{ij}^{a_1}) \quad (21)$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial a_1^2} &= \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} \left(\ln^2(w_{ij}) + 2b_1 \ln(w_{ij}) \right) - \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k \ln(w_{ij}) z_{ij} (a_1 \ln(w_{ij}) + 1) \\
&\quad - \frac{1}{a_1^2} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) w_{ij}^{a_1} \ln^2(w_{ij}) z_{ij} + b_1 b_{ij} w_{ij}^{a_1} \ln^2(w_{ij})}{1 - w_{ij}^{a_1}} \\
&\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) w_{ij}^{a_1} \ln(w_{ij}) (w_{ij}^{a_1} \ln(w_{ij}) z_{ij} - b_1 b_{ij} w_{ij}^{a_1} \ln(w_{ij}))}{(1 - w_{ij}^{a_1})^2} \tag{22}
\end{aligned}$$

$$\frac{\partial^2 l}{\partial b_1^2} = - \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_1^2} \tag{23}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial a_1 \partial b_1} &= - \frac{1}{a_1 b_1} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln(w_{ij}) + 1) + \sum_{i=1}^m \sum_{j=1}^k \frac{a_1 w_{ij}^{1-a_1} (w_{ij}^{a_1-1} \ln(w_{ij}) + w_{ij}^{a_1-1}) z_{ij}}{a_1 b_1} \\
&\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_1} \ln(w_{ij}) z_{ij}}{1 - w_{ij}^{a_1}} - \sum_{i=1}^m \sum_{j=1}^k \frac{b_{ij} w_{ij}^{a_1} \ln(w_{ij})}{1 - w_{ij}} \tag{24}
\end{aligned}$$

$$\frac{\partial l}{\partial a_2} = \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_2 \ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) w_{ij}^{a_2} \ln(w_{ij}) z_{ij} - b_2 d_{ij} w_{ij}^{a_2} \ln(w_{ij})}{1 - w_{ij}^{a_2}} \tag{25}$$

$$\frac{\partial l}{\partial b_2} = \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_2} + \sum_{i=1}^m \sum_{j=1}^k \ln(1 - w_{ij}^{a_2}) z_{ij} + \sum_{i=1}^m \sum_{j=1}^k d_{ij} \ln(1 - w_{ij}^{a_2}) \tag{26}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial a_2^2} &= \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} \left(a_2 \ln^2(w_{ij}) + 2 \ln(w_{ij}) \right) - \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k \ln(w_{ij}) z_{ij} (a_2 \ln(w_{ij}) + 1) \\
&\quad - \frac{1}{a_2^2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_2 \ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) w_{ij}^{a_2} \ln^2(w_{ij}) z_{ij} - b_2 d_{ij} w_{ij}^{a_2} \ln^2(w_{ij})}{1 - w_{ij}^{a_2}} \\
&\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_2} \ln(w_{ij}) ((b_2 - 1) w_{ij}^{a_2} \ln(w_{ij}) z_{ij} - b_2 d_{ij} w_{ij}^{a_2} \ln(w_{ij}))}{(1 - w_{ij}^{a_2})^2} \tag{27}
\end{aligned}$$

$$\frac{\partial^2 l}{\partial^2 b_2} = - \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_2^2} \tag{28}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial a_2 \partial b_2} &= - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{1-a_2} z_{ij} (w_{ij}^{a_2-1} \ln(w_{ij}) a_2 b_2 + w_{ij}^{a_2-1} b_2)}{a_2 b_2^2} \\
&\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{1-a_2} (w_{ij}^{a_2-1} \ln(w_{ij}) a_2 + w_{ij}^{a_2-1}) z_{ij}}{a_2 b_2} - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_2} \ln(w_{ij}) z_{ij}}{1 - w_{ij}^{a_2}} \\
&\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{d_{ij} w_{ij}^{a_2} \ln(w_{ij})}{1 - w_{ij}} \tag{29}
\end{aligned}$$

To determine the ML estimates for the parameters of the Kumaraswamy distribution under the JMnRSS, solve the equations (20), (21), (25), and (26) iteratively. Fisher information matrix (I) for the parameters of two Kumaraswamy populations under JMnRSS can be written as follows,

$$I(a_1, b_1, a_2, b_2) = \begin{bmatrix} I_{a_1 a_1} & I_{a_1 b_1} & I_{a_1 a_2} & I_{a_1 b_2} \\ I_{b_1 a_1} & I_{b_1 b_1} & I_{b_1 a_2} & I_{b_1 b_2} \\ I_{a_2 a_1} & I_{a_2 b_1} & I_{a_2 a_2} & I_{a_2 b_2} \\ I_{b_2 a_1} & I_{b_2 b_1} & I_{b_2 a_2} & I_{b_2 b_2} \end{bmatrix} \quad (30)$$

$$= -E \begin{bmatrix} \frac{\partial^2 l}{\partial a_1^2} & \frac{\partial^2 l}{\partial a_1 \partial b_1} & 0 & 0 \\ \frac{\partial^2 l}{\partial b_1 \partial a_1} & \frac{\partial^2 l}{\partial b_1^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 l}{\partial a_2^2} & \frac{\partial^2 l}{\partial a_2 \partial b_2} \\ 0 & 0 & \frac{\partial^2 l}{\partial b_2 \partial a_2} & \frac{\partial^2 l}{\partial b_2^2} \end{bmatrix} \quad (31)$$

4. Joint Modified Maximum Ranked Set Sampling (JMxRSS)

In this Section, we studied the estimation of parameters of the Kumaraswamy population under JMxRSS. The algorithm to select JMxRSS sample from two populations is as follows.

Algorithm 3 The algorithm to select JMxRSS sample from two populations

- 1: Select m_1 observation from a population-I and m_2 observation from a population-II randomly and combine these observations to create a joint sample of size $m = m_1 + m_2$.
 - 2: Arrange all these observations of the joint sample in ascending order.
 - 3: To get m joint sets of sizes m , repeat the 1 and 2 m times.
 - 4: To have a JMxRSS sample of size m , select the maximum from all m sets.
 - 5: Repeat 1 to 4 to increase the size of the JMxRSS sample k times *i.e.* $n = mk$.
-

4.1. Maximum likelihood estimation under JMxRSS

The likelihood function according to the algorithm 3 discuss above based on the observations $w_{ij}, i = 1, 2, 3, \dots, m; j = 1, 2, 3, \dots, k$ can be obtained by putting $b_{ij} = d_{ij} = 0$ for all $i = 1, 2, 3, \dots, m; j = 1, 2, \dots, k$ in equation (6)

$$\begin{aligned} l = \log(L(a_1, b_1, a_2, b_2 | \underline{w})) &= \sum_{i=1}^m \sum_{j=1}^k a_{ij} \ln \left(1 - (1 - w_{ij}^{a_1})^{b_1} \right) + (b_1 - 1) \\ &\sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln (1 - w_{ij}^{a_1}) + \sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln (a_1 b_1 w_{ij}^{a_1 - 1}) + \sum_{i=1}^m \sum_{j=1}^k c_{ij} \ln \left(1 - (1 - w_{ij}^{a_2})^{b_2} \right) \\ &+ (b_2 - 1) \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln (1 - w_{ij}^{a_2}) + \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln (a_2 b_2 w_{ij}^{a_2 - 1}) \end{aligned} \quad (32)$$

$$\frac{\partial l}{\partial a_1} = \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_1 \ln(w_{ij}) + 1) + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln(w_{ij}) (1 - w_{ij}^{a_1})^{b_1 - 1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \quad (33)$$

$$\frac{\partial l}{\partial b_1} = \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_1} + \sum_{i=1}^m \sum_{j=1}^k \ln(1 - w_{ij}^{a_1}) z_{ij} - \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} \ln(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{b_1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \quad (34)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial a_1^2} &= \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln^2(w_{ij}) + 2b_1 \ln(w_{ij})) - \frac{1}{a_1} \sum_{i=1}^m \sum_{j=1}^k \ln(w_{ij}) z_{ij} (a_1 \ln(w_{ij}) + 1) \\ &\quad - \frac{1}{a_1^2} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) w_{ij}^{a_1} \ln^2(w_{ij}) z_{ij}}{1 - w_{ij}^{a_1}} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) w_{ij}^{2a_1} \ln^2(w_{ij}) z_{ij}}{(1 - w_{ij}^{a_1})^2} - \sum_{i=1}^m \sum_{j=1}^k \frac{b_1^2 a_{ij} w_{ij}^{2a_1} \ln^2(w_{ij}) (1 - w_{ij}^{a_1})^{2b_1 - 2}}{(1 - (1 - w_{ij}^{a_1})^{b_1})^2} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_1 - 1) b_1 a_{ij} w_{ij}^{2a_1} \ln^2(w_{ij}) (1 - w_{ij}^{a_1})^{b_1 - 2}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln^2(w_{ij}) (1 - w_{ij}^{a_1})^{b_1 - 1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \end{aligned} \quad (35)$$

$$\frac{\partial^2 l}{\partial b_1^2} = - \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_1^2} - \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} \ln^2(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{b_1}}{((1 - w_{ij}^{a_1})^{b_1} - 1)^2} \quad (36)$$

$$\begin{aligned} \frac{\partial^2 l}{\partial a_1 \partial b_1} &= - \frac{1}{a_1 b_1} \sum_{i=1}^m \sum_{j=1}^k a_1 z_{ij} (\ln(w_{ij}) + 1) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln(w_{ij}) \ln(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{b_1 - 1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{b_1 a_{ij} w_{ij}^{a_1} \ln(w_{ij}) \ln(1 - w_{ij}^{a_1}) (1 - w_{ij}^{a_1})^{2b_1 - 1}}{(1 - (1 - w_{ij}^{a_1})^{b_1})^2} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{a_1 w_{ij}^{1-a_1} (w_{ij}^{a_1 - 1} \ln(w_{ij}) + w_{ij}^{a_1 - 1}) z_{ij}}{a_1 b_1} - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_1} \ln(w_{ij}) z_{ij}}{1 - w_{ij}^{a_1}} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} w_{ij}^{a_1} \ln(w_{ij}) (1 - w_{ij}^{a_1})^{b_1 - 1}}{1 - (1 - w_{ij}^{a_1})^{b_1}} \end{aligned} \quad (37)$$

$$\begin{aligned} \frac{\partial l}{\partial a_2} &= \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_2 \ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) w_{ij}^{a_2} \ln(w_{ij}) z_{ij}}{1 - w_{ij}^{a_2}} \\ &\quad + b_2 \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) (1 - w_{ij}^{a_2})^{b_2 - 1}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \end{aligned} \quad (38)$$

$$\frac{\partial l}{\partial b_2} = \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_2} + \sum_{i=1}^m \sum_{j=1}^k \ln(1 - w_{ij}^{a_2}) z_{ij} - \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} \ln(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{b_2}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \quad (39)$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial a_2^2} &= \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} \left(a_2 \ln^2(w_{ij}) + 2 \ln(w_{ij}) \right) - \frac{1}{a_2} \sum_{i=1}^m \sum_{j=1}^k \ln(w_{ij}) z_{ij} (a_2 \ln(w_{ij}) + 1) \\
&- \frac{1}{a_2^2} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a_2 \ln(w_{ij}) + 1) - \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) w_{ij}^{a_2} \ln^2(w_{ij}) z_{ij}}{1 - w_{ij}^{a_2}} \\
&- \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) w_{ij}^{2a_2} \ln^2(w_{ij}) z_{ij}}{(1 - w_{ij}^{a_2})^2} - \sum_{i=1}^m \sum_{j=1}^k \frac{b_2^2 c_{ij} w_{ij}^{2a_2} \ln^2(w_{ij}) (1 - w_{ij}^{a_2})^{2b_2 - 2}}{(1 - (1 - w_{ij}^{a_2})^{b_2})^2} \\
&- \sum_{i=1}^m \sum_{j=1}^k \frac{(b_2 - 1) b_2 c_{ij} w_{ij}^{2a_2} \ln^2(w_{ij}) (1 - w_{ij}^{a_2})^{b_2 - 2}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \\
&+ \sum_{i=1}^m \sum_{j=1}^k \frac{b_2 c_{ij} w_{ij}^{a_2} \ln^2(w_{ij}) (1 - w_{ij}^{a_2})^{b_2 - 1}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \tag{40}
\end{aligned}$$

$$\frac{\partial^2 l}{\partial b_2^2} = - \sum_{i=1}^m \sum_{j=1}^k \frac{z_{ij}}{b_2^2} - \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} \ln^2(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{b_2}}{\left((1 - w_{ij}^{a_2})^{b_2} - 1 \right)^2} \tag{41}$$

$$\begin{aligned}
\frac{\partial^2 l}{\partial a_2 \partial b_2} &= - \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{1-a_2} z_{ij} (w_{ij}^{a_2-1} \ln(w_{ij}) a_2 b_2 + w_{ij}^{a_2-1} b_2)}{a_2 b_2^2} \\
&+ \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) \ln(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{b_2-1} b_2}{1 - (1 - w_{ij}^{a_2})^{b_2}} \\
&+ \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) \ln(1 - w_{ij}^{a_2}) (1 - w_{ij}^{a_2})^{2b_2-1} b_2}{\left(1 - (1 - w_{ij}^{a_2})^{b_2} \right)^2} \\
&+ \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{1-a_2} (w_{ij}^{a_2-1} \ln(w_{ij}) a_2 + w_{ij}^{a_2-1}) z_{ij}}{a_2 b_2} \\
&- \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^{a_2} \ln(w_{ij}) z_{ij}}{1 - w_{ij}^{a_2}} + \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^{a_2} \ln(w_{ij}) (1 - w_{ij}^{a_2})^{b_2-1}}{1 - (1 - w_{ij}^{a_2})^{b_2}} \tag{42}
\end{aligned}$$

To determine the ML estimates for the parameters of the Kumaraswamy distribution under the JMxRSS, solve the equations (33), (34), (38), and (39) iteratively. Fisher information matrix (I) for the parameters of two Kumaraswamy populations under JMxRSS can be written as follows,

$$I(a_1, b_1, a_2, b_2) = \begin{bmatrix} I_{a_1 a_1} & I_{a_1 b_1} & I_{a_1 a_2} & I_{a_1 b_2} \\ I_{b_1 a_1} & I_{b_1 b_1} & I_{b_1 a_2} & I_{b_1 b_2} \\ I_{a_2 a_1} & I_{a_2 b_1} & I_{a_2 a_2} & I_{a_2 b_2} \\ I_{b_2 a_1} & I_{b_2 b_1} & I_{b_2 a_2} & I_{b_2 b_2} \end{bmatrix} = -E \begin{bmatrix} \frac{\partial^2 l}{\partial a_1^2} & \frac{\partial^2 l}{\partial a_1 \partial b_1} & 0 & 0 \\ \frac{\partial^2 l}{\partial b_1 \partial a_1} & \frac{\partial^2 l}{\partial b_1^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 l}{\partial a_2^2} & \frac{\partial^2 l}{\partial a_2 \partial b_2} \\ 0 & 0 & \frac{\partial^2 l}{\partial b_2 \partial a_2} & \frac{\partial^2 l}{\partial b_2^2} \end{bmatrix} \tag{43}$$

4.2. Maximum likelihood estimation under Joint Simple Random Sampling (JSRS)

Likelihood under JSRS for the sample size $n_1 = m_1k$ of population-I and sample size $n_2 = m_2k$ is written as follows,

$$L(a_1, b_1, a_2, b_2|x, y) = \prod_{i=1}^{n_1} \prod_{i=1}^{n_2} f(x_i)g(y_i) = \prod_{i=1}^{n_1} f(x_i) \prod_{i=1}^{n_2} g(y_i) \quad (44)$$

Substituting equations (1) and (3) into equation (44) and applying the logarithm, we obtain the following equation.

$$\begin{aligned} l &= \log(L(a_1, b_1, a_2, b_2|x, y)) = \sum_{i=1}^{n_1} \log(f(x_i; a_1, b_1)) + \sum_{i=1}^{n_2} \log(g(y_i; a_2, b_2)) \\ l &= n_1 \log(a_1 b_1) + (a_1 - 1) \sum_{i=1}^{n_1} \log(x_i) + (b_1 - 1) \sum_{i=1}^{n_1} \log(1 - x_i^{a_1}) \\ &+ n_2 \log(a_2 b_2) + (a_2 - 1) \sum_{i=1}^{n_2} \log(y_i) + (b_2 - 1) \sum_{i=1}^{n_2} \log(1 - y_i^{a_2}) \end{aligned} \quad (45)$$

$$\frac{\partial l}{\partial a_1} = \frac{n_1}{a_1} - (b_1 - 1) \sum_{i=1}^{n_1} \frac{x_i^{a_1} \ln(x_i)}{1 - x_i^{a_1}} + \sum_{i=1}^{n_1} \ln(x_i) \quad (46)$$

$$\frac{\partial^2 l}{\partial a_1^2} = -\frac{n_1}{a_1^2} - (b_1 - 1) \sum_{i=1}^{n_1} \frac{x_i^{a_1} \ln^2(x_i)}{1 - x_i^{a_1}} - (b_1 - 1) \sum_{i=1}^{n_1} \frac{x_i^{2a_1} \ln^2(x_i)}{(1 - x_i^{a_1})^2} \quad (47)$$

$$\frac{\partial^2 l}{\partial a_1 \partial b_1} = \sum_{i=1}^{n_1} \frac{x_i^{a_1} \ln(x_i)}{x_i^{a_1} - 1} \quad (48)$$

$$\frac{\partial l}{\partial b_1} = \frac{n_1}{b_1} + \sum_{i=1}^{n_1} \ln(1 - x_i^{a_1}) \quad (49)$$

$$\frac{\partial^2 l}{\partial^2 b_1} = -\frac{n_1}{b_1^2} \quad (50)$$

$$\frac{\partial l}{\partial a_2} = \frac{n_2}{a_2} - (b_2 - 1) \sum_{i=1}^{n_2} \frac{y_i^{a_2} \ln(y_i)}{1 - y_i^{a_2}} + \sum_{i=1}^{n_2} \ln(y_i) \quad (51)$$

$$\frac{\partial^2 l}{\partial a_2^2} = -\frac{n_2}{a_2^2} - (b_2 - 1) \sum_{i=1}^{n_2} \frac{y_i^{a_2} \ln^2(y_i)}{1 - y_i^{a_2}} - (b_2 - 1) \sum_{i=1}^{n_2} \frac{y_i^{2a_2} \ln^2(y_i)}{(1 - y_i^{a_2})^2} \quad (52)$$

$$\frac{\partial^2 l}{\partial a_2 \partial b_2} = \sum_{i=1}^{n_2} \frac{y_i^{a_2} \ln(y_i)}{y_i^{a_2} - 1} \quad (53)$$

$$\frac{\partial l}{\partial b_2} = \frac{n_2}{b_2} + \sum_{i=1}^{n_2} \ln(1 - y_i^{a_2}) \quad (54)$$

$$\frac{\partial^2 l}{\partial b_2^2} = -\frac{n_2}{b_2^2} \quad (55)$$

To determine the ML estimates for the parameters of the Kumaraswamy distribution under the JSRS, solve the equations (46), (49), (51), and (54) iteratively. Fisher information matrix (I) for the parameters of two Kumaraswamy populations under JSRS can be written

as follows,

$$I(a_1, b_1, a_2, b_2) = \begin{bmatrix} I_{a_1 a_1} & I_{a_1 b_1} & I_{a_1 a_2} & I_{a_1 b_2} \\ I_{b_1 a_1} & I_{b_1 b_1} & I_{b_1 a_2} & I_{b_1 b_2} \\ I_{a_2 a_1} & I_{a_2 b_1} & I_{a_2 a_2} & I_{a_2 b_2} \\ I_{b_2 a_1} & I_{b_2 b_1} & I_{b_2 a_2} & I_{b_2 b_2} \end{bmatrix} = -E \begin{bmatrix} \frac{\partial^2 l}{\partial a_1^2} & \frac{\partial^2 l}{\partial a_1 \partial b_1} & 0 & 0 \\ \frac{\partial^2 l}{\partial b_1 \partial a_1} & \frac{\partial^2 l}{\partial b_1^2} & 0 & 0 \\ 0 & 0 & \frac{\partial^2 l}{\partial a_2^2} & \frac{\partial^2 l}{\partial a_2 \partial b_2} \\ 0 & 0 & \frac{\partial^2 l}{\partial b_2 \partial a_2} & \frac{\partial^2 l}{\partial b_2^2} \end{bmatrix} \quad (56)$$

5. Likelihood ratio test

In this Section, we derive the likelihood ratio test to compare the two Kumaraswamy distributions for equality of the first shape parameter when the second shape parameter is known, considering various joint ranked set sampling methods.

5.1. Likelihood ratio test for JRSS

$H_0 : a_1 = a_2 = a$ Vs $H_1 : a_1 \neq a_2$, when b_1 and b_2 known.

The log-likelihood under H_0 for JRSS is expressed as follows:

$$\begin{aligned} l = \log(L(a_1, b_1, a_2, b_2 | \underline{w})) &= \sum_{i=1}^m \sum_{j=1}^k a_{ij} \ln(1 - (1 - w_{ij}^{a_1})^{b_1}) + b_1 \sum_{i=1}^m \sum_{j=1}^k b_{ij} \ln(1 - w_{ij}^{a_1}) \\ &+ (b_1 - 1) \sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln(1 - w_{ij}^{a_1}) + \sum_{i=1}^m \sum_{j=1}^k z_{ij} \ln(a_1 b_1 w_{ij}^{a_1 - 1}) \\ &+ \sum_{i=1}^m \sum_{j=1}^k c_{ij} \ln(1 - (1 - w_{ij}^{a_2})^{b_2}) + b_2 \sum_{i=1}^m \sum_{j=1}^k d_{ij} \ln(1 - w_{ij}^{a_2}) \\ &+ (b_2 - 1) \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln(1 - w_{ij}^{a_2}) \\ &+ \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) \ln(a_2 b_2 w_{ij}^{a_2 - 1}) \end{aligned} \quad (57)$$

Differentiate (57) with respect to a and equate to zero.

$$\begin{aligned} \frac{\partial l}{\partial a} &= \frac{1}{a} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a \ln(w_{ij}) + 1) + \frac{1}{a} \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) (a \ln(w_{ij}) + 1) \\ &- (b_1 - 1) \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^a \ln(w_{ij}) z_{ij}}{1 - w_{ij}^a} - (b_2 - 1) \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^a \ln(w_{ij}) (1 - z_{ij})}{1 - w_{ij}^a} \\ &+ b_1 \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} w_{ij}^a \ln(w_{ij}) (1 - w_{ij}^a)^{b_1 - 1}}{1 - (1 - w_{ij}^a)^{b_1}} - b_1 \sum_{i=1}^m \sum_{j=1}^k \frac{b_{ij} w_{ij}^a \ln(w_{ij})}{1 - w_{ij}^a} \\ &+ b_2 \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^a \ln(w_{ij}) (1 - w_{ij}^a)^{b_2 - 1}}{1 - (1 - w_{ij}^a)^{b_2}} - b_2 \sum_{i=1}^m \sum_{j=1}^k \frac{d_{ij} w_{ij}^a \ln(w_{ij})}{1 - w_{ij}^a} \end{aligned} \quad (58)$$

To get MLE of a under H_0 solve the (58) numerically. MLEs of a_1 and a_2 obtained by numerically solving the equations (8), (9), (13), and (14). LRT test statistic is determined by putting the MLEs of a_1, a_2 , and a into following equation.

$$\lambda_{JRSS}(\underline{w}) = \frac{\sup_a L(a|\underline{w})}{\sup_{a_1, a_2} L(a_1, a_2|\underline{w})}$$

LRT reject H_0 when $\lambda_{JRSS}(w) > \chi_{1, \alpha}^2$, where α is the level of significance.

5.2. Likelihood ratio test for JMnRSS

$H_0 : a_1 = a_2 = a \quad Vs \quad H_1 : a_1 \neq a_2$, when b_1 and b_2 known.

The MLE of a under H_0 for JMnRSS can be obtained by setting $a_{ij} = c_{ij} = 0$ in equation (57) and differentiating with respect to a .

$$\begin{aligned} \frac{\partial l}{\partial a} &= \frac{1}{a} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a \ln(w_{ij}) + 1) + \frac{1}{a} \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) (a \ln(w_{ij}) + 1) \\ &\quad - (b_1 - 1) \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^a \ln(w_{ij}) z_{ij}}{1 - w_{ij}^a} - (b_2 - 1) \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^a \ln(w_{ij}) (1 - z_{ij})}{1 - w_{ij}^a} \\ &\quad - b_1 \sum_{i=1}^m \sum_{j=1}^k \frac{b_{ij} w_{ij}^a \ln(w_{ij})}{1 - w_{ij}^a} - b_2 \sum_{i=1}^m \sum_{j=1}^k \frac{d_{ij} w_{ij}^a \ln(w_{ij})}{1 - w_{ij}^a} \end{aligned} \quad (59)$$

To get MLE of a under H_0 , solve the (59) numerically. MLEs of a_1 and a_2 obtained by numerically solving the equations (20), (21), (25), and (26). LRT test statistic is determined by putting the MLEs of a_1, a_2 , and a into following equation.

$$\lambda_{JMnRSS}(W) = \frac{\sup_a L(a|\underline{w})}{\sup_{a_1, a_2} L(a_1, a_2|\underline{w})}$$

LRT reject H_0 when $\lambda_{JMnRSS}(w) > \chi_{1, \alpha}^2$ where α is the level of significance.

5.3. Likelihood ratio test for JMxRSS

$H_0 : a_1 = a_2 = a \quad Vs \quad H_1 : a_1 \neq a_2$, when b_1 and b_2 known.

The MLE of a under H_0 for JMxRRS can be obtained by setting $b_{ij} = d_{ij} = 0$ in equation (57) and differentiating with respect to a .

$$\begin{aligned} \frac{\partial l}{\partial a} &= \frac{1}{a} \sum_{i=1}^m \sum_{j=1}^k z_{ij} (a \ln(w_{ij}) + 1) + \frac{1}{a} \sum_{i=1}^m \sum_{j=1}^k (1 - z_{ij}) (a \ln(w_{ij}) + 1) \\ &\quad - (b_1 - 1) \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^a \ln(w_{ij}) z_{ij}}{1 - w_{ij}^a} - (b_2 - 1) \sum_{i=1}^m \sum_{j=1}^k \frac{w_{ij}^a \ln(w_{ij}) (1 - z_{ij})}{1 - w_{ij}^a} \\ &\quad + b_1 \sum_{i=1}^m \sum_{j=1}^k \frac{a_{ij} w_{ij}^a \ln(w_{ij}) (1 - w_{ij}^a)^{b_1 - 1}}{1 - (1 - w_{ij}^a)^{b_1}} + b_2 \sum_{i=1}^m \sum_{j=1}^k \frac{c_{ij} w_{ij}^a \ln(w_{ij}) (1 - w_{ij}^a)^{b_2 - 1}}{1 - (1 - w_{ij}^a)^{b_2}} \end{aligned} \quad (60)$$

To get MLE of a under H_0 Solve the (60) numerically. MLEs of a_1 and a_2 also obtained by solving the equations (33), (34), (38), and (39) numerically. LRT test statistic is determined by putting the MLEs of a_1, a_2 , and a into following equation.

$$\lambda_{JMxRSS}(\underline{w}) = \frac{\sup_a L(a|\underline{w})}{\sup_{a_1, a_2} L(a_1, a_2|\underline{w})}$$

LRT reject H_0 when $\lambda_{JMxRSS}(\underline{w}) > \chi_{1, \alpha}^2$ where α is the level of significance.

5.4. Likelihood ratio test for JSRS

$$H_0 : a_1 = a_2 = a \quad \text{Vs} \quad H_1 : a_1 \neq a_2, \quad \text{when } b_1 \text{ and } b_2 \text{ known.}$$

The MLE under H_0 can be obtained by substituting $a_1 = a_2 = a$ in equation (45) and differentiate with respect to a .

$$\frac{\partial l}{\partial a} = \frac{n_1 + n_2}{a} - (b_1 - 1) \sum_{i=1}^{n_1} \frac{x_i^a \ln(x_i)}{1 - x_i^a} - (b_2 - 1) \sum_{i=1}^{n_2} \frac{y_i^a \ln(y_i)}{1 - y_i^a} + \sum_{i=1}^{n_2} \ln(y) + \sum_{i=1}^{n_2} \ln(x_i) \quad (61)$$

MLEs of a_1 and a_2 can be found by solving the equations (46), (49), (51), and (54). LRT test statistic is determined by putting the MLEs of a_1, a_2 , and a into following equation.

$$\lambda_{JSRS}(x, y) = \frac{\sup_a L(a|x, y)}{\sup_{a_1, a_2} L(a_1, a_2|x, y)}$$

LRT reject H_0 when $\lambda_{JSRS}(x, y) > \chi_{1, \alpha}^2$ where α is the level of significance.

6. Simulation results and applications

In this Section, we present theoretical simulation results and also analyze a real data set.

6.1. Simulation results

In this Section, we evaluate the efficiency of ML estimates of the Kumaraswamy distribution parameters under JRSS, JMnRSS, JMxRSS, and JSRS. To evaluate the efficiency of our ML estimates, we employed Monte Carlo simulation techniques utilizing R Studio. For this specifically, we used the "mle" function for optimization and the Conjugate Gradient (CG) method given in R Studio. We compared these ML estimates using RMSE and bias calculated by 100 repetitions of various sample sizes of the first Kumaraswamy population $m_1 = (3, 4, 5)$, and from the second Kumaraswamy population $m_2 = (4, 5, 6)$, for different cycles $k = 2, 3, 4$. Additionally, we simulate the power of the LRT for the mentioned joint sampling schemes based on the algorithm 5. These results were also performed for various values of $m_1 = (3, 4, 5)$, $m_2 = (4, 5, 6)$, and $k = (2, 3, 4)$ with 1000 repetitions to compare the power of the test, ensuring at least three digits after the decimal point.

From Table (1) (see Annexure), we observe that with an increase in the total sample size ($m = m_1 + m_2$) and the number of cycles k , the RMSE and bias of the ML estimates of the parameters for the Kumaraswamy population decrease for all joint ranked sampling schemes. We observed that all joint ranked sampling schemes performed better than the

Algorithm 4 MLE and MSE Calculation for JRSS / JMnRSS/ JMxRSS

- 1: **Initialization:** Set parameters for the Kumaraswamy population-I: (a_1, b_1) and sample size m_1 .
 - 2: Set parameters for the Kumaraswamy population-II: (a_2, b_2) and sample size m_2 .
 - 3: Ensure the total sample size is $m = m_1 + m_2$.
 - 4: Specify the number of cycles k and repetitions B .
 - 5: **for** each sample size m_1 **do**
 - 6: **for** each sample size m_2 **do**
 - 7: **for** each cycle size k **do**
 - 8: **while** $i \leq 1$ to B **do**
 - 9: Assign (a_1, b_1, a_2, b_2) to vector $Para$.
 - 10: Generate $m_1 \times k$ random observation from the Kumaraswamy population-I and $m_2 \times k$ observation from the Kumaraswamy population-II.
 - 11: Select a sample using the JRSS / JMnRSS / JMxRSS algorithms 1, 2, 3
 - 12: Construct the likelihood function $\mathcal{L}(Para|W)$.
 - 13: Maximize $\mathcal{L}(Para)$ to estimate \hat{para}_i . using maxLik function in R and initial values of the parameters are found by the method of moments Dey *et al.* (2018).
 - 14: Compute RMSE: $RMSE_i = \sqrt{(P\hat{a}ra_i - Para)^2}$.
 - 15: **end while**
 - 16: ML estimates: $P\hat{a}ra = \frac{1}{B} \sum_{i=1}^B P\hat{a}ra_i$
 - 17: bias= $P\hat{a}ra - Para$
 - 18: RMSE = $\frac{1}{B} \sum_{i=1}^B RMSE_i$.
 - 19: **end for**
 - 20: **end for**
 - 21: **end for**
 - 22: **Output:** Average MLE and RMSE for each n .
-

Algorithm 5 Algorithm for power calculation of LRT using JRSS/ JMxRSS/ JMnRSS/ JSRS

- 1: **Step 1:** Determine LRT test Statistic $\lambda(w)$ for $(m = m_1 + m_2, k)$ using one of the joint sampling schemes JRSS/ JMxRSS/ JMnRSS/ JSRS (for JSRS $n_1 = m_1k, n_2 = m_2k$).
 - 2: **Step 2:** Choose level of significance ($\alpha = 0.05$) and determine type-II error ($p_value \geq 0.05$ when H_1 is True).
 - 3: **Step 3:** Repeat step.1 and Step.2, $B=1000$ times and calculate $p(\text{type-II error}) = \frac{\text{No. of times } p_value \geq 0.05}{B}$.
 - 4: **Step 4:** Determine the power of the test, power= $1 - p(\text{type-II error})$.
 - 5: **Step 5:** Repeat above procedure for different combination of (m_1, m_2, k) .
-

JSRS scheme. Additionally, the simulation study for ML estimation shows that JMnRSS outperforms both JRSS and JMxRSS. In general, Table (2) (see Annexure) shows that as the difference between the parameters of the Kumaraswamy distribution increases, the RMSE and bias also increase. Table (3) (see Annexure) gives the power of the LRT for testing the equality of parameters (*i.e.*, $a_1 = a_2 = a$) when b_1 and b_2 are known, and the result indicates that JRSS has better power compared to other joint sampling schemes.

6.2. Application

In this Section, we delve into a practical example that illustrates the application of the Kumaraswamy distribution in estimating the probability of lung cancer within a specific population.

In this study, we have taken a lung cancer data set ([click here](#)) from Kaggle, which includes the following variables: Gender, Age, Smoking, Yellow fingers, Anxiety, Peer pressure, Chronic disease, Fatigue, Allergy, Wheezing, Alcohol consuming, Coughing of breath, Swallowing difficulty, Chest pain, lung cancer. Lung cancer is a binary (yes/no) variable. Hence, we used the Probit model for binary regression to classify individuals based on the explanatory variable as either having or not having lung cancer. Further, we checked the significance of the variable and found that Smoking, Yellow fingers, Peer pressure, Chronic disease, Fatigue, Allergy, Coughing of breath, and Swallowing difficulty, all these variables contribute significantly. For the given lung cancer data, we found that the Probit link function is suitable and also assessed the adequacy of the model using the `hoslem.test`, with a *p-value* of 0.9391, which is greater than $\alpha = 0.05$, indicating that the model fits well. Based on the Probit model, we predict the probability of lung cancer using the data. Then, we classify the predicted probabilities into two groups, male and female, and check the fit of the Kumaraswamy distribution using the goodness-of-fit test (Anderson-Darling test). All these results are provided in Table (4) (see Annexure) for the whole population and the separated male and female populations. For this fitting, we estimate the parameters under SRS using the ML method, with initial values obtained by the method of moments. So, in Table (4), the first column represents the groups (combined, male, and female), and the second column represents the initial estimates. The third column represents the ML estimates used for fitting the distribution using SRS, along with the corresponding test statistic value of the Anderson-Darling test and its *p-value*. In the fourth, fifth, and sixth columns, we estimate the parameters under JRSS, JMnRSS, and JMxRSS (which means considering the male and female populations as a joint sample) respectively. So, estimates for the combined sample under JRSS, JMnRSS, and JMxRSS are obtained by averaging the estimates from the male and female populations. The last three columns show the error in the estimates, or the difference between the estimates under SRS and JRSS, JMnRSS, and JMxRSS. So, the last three columns represent the difference in the estimates when we estimate parameters separately for the male and female populations, as well as based on a joint sample. The purpose of Table (4) is to compare the error in the estimates for the combined sample based on the male and female populations, with those estimated based on the joint sample, considering the average. From Figure (1) and (2), we can easily observe the distributional structure for the male, female, and combined populations. In Table (5) (see Annexure), we also provide the values of ML estimates, RMSE, and bias using methods such as JRSS, JSRS, JMnRSS, and JMxRSS. We do this by following the steps of Algorithm 4. From Table (5), we observe

that JMnRSS performs better than JRSS, JMxRSS, and JSRS, which also aligns with the theoretical simulation study.

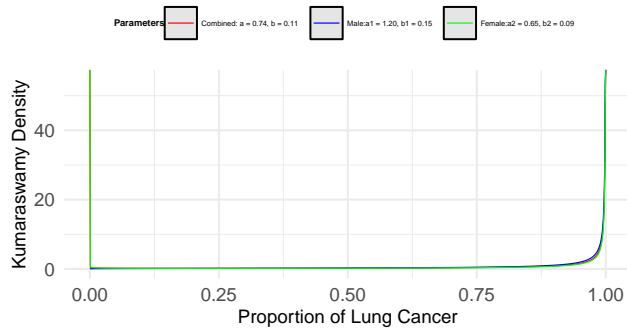


Figure 1: Density Plot for Lung Cancer

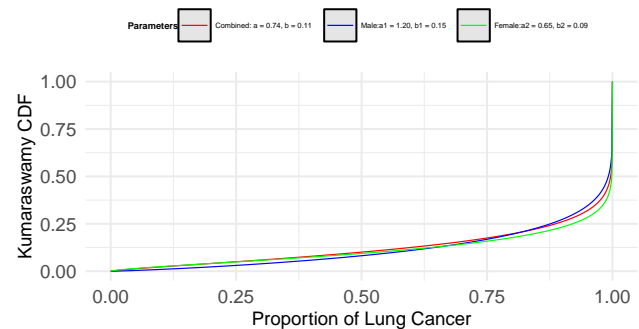


Figure 2: CDF Plot for Lung Cancer

7. Conclusion

Our study indicates that the JMnRSS scheme outperforms the JRSS, JMxRSS, and JSRS methods in terms of RMSE for ML estimation of the Kumaraswamy population. We also observed the behaviour of ML estimates of the Kumaraswamy population under all joint sampling schemes, and according to the simulation tables, as the total sample size $m = m_1 + m_2$ and cycle k increase, RMSE and bias decrease. As the difference (d) in the parameters of the two Kumaraswamy populations increases, the RMSE and bias also increase. We recommend using JMnRSS schemes for the estimation of parameters of two Kumaraswamy distributions. In the real scenario, we observed that JMnRSS is more suitable for predicting the probability of lung cancer compared to JRSS, JMxRSS, and JSRS. More research is needed in the area of joint ranked set sampling, including methods like joint percentile sampling. Additionally, Bayesian estimation can be applied to estimate the parameters of the Kumaraswamy distribution using various joint sampling.

Acknowledgements

The first author is thankful to SHODH, Gujarat for providing financial support under the state government of Gujarat (2022016452). The authors are grateful to the Chief Editor and anonymous referees for their valuable suggestions to improve our research paper.

Conflict of interest

The authors declare no conflict of interest.

References

- Abbasi, A. M. and Shahd, M. Y. (2017). Estimation of population mean and median using double robust truncation based ranked set sampling. *Pakistan Journal of Statistics and Operation Research*, **13**, 379–394.

- Akgul, F. G. and Senoglu, B. (2017). Estimation of using modifications of ranked set sampling for Weibull distribution. *Pakistan Journal of Statistics and Operation Research*, **13**, 931–958.
- Ashour, S. and Abo-Kasem, O. (2014). Parameter estimation for two Weibull populations under joint type II censored scheme. *International Journal of Engineering*, **5**, 31–36.
- Chen, Z., Bai, Z., and Sinha, B. K. (2004). *Ranked Set Sampling: Theory and Applications*, volume 176. Springer.
- Dey, S., Mazucheli, J., and Nadarajah, S. (2018). Kumaraswamy distribution: different methods of estimation. *Computational and Applied Mathematics*, **37**, 2094–2111.
- Ding, L. and Gui, W. (2023). Statistical inference of two gamma distributions under the joint type-II censoring scheme. *Mathematics*, **11**, 2003.
- Koyuncu, N. (2018). Regression estimators in ranked set, median ranked set and neoteric ranked set sampling. *Pakistan Journal of Statistics and Operation Research*, **14**, 89–94.
- McIntyre, G. (1952). A method for unbiased selective sampling, using ranked sets. *Australian Journal of Agricultural Research*, **3**, 385–390.
- Mitnik, P. A. (2013). New properties of the Kumaraswamy distribution. *Communications in Statistics-Theory and Methods*, **42**, 741–755.
- Patel, M. N., Muralidharan, K., and Bavagosai, P. (2022). Comparing two Rayleigh populations using joint ranked set sampling. *International Journal of Mathematics and Statistics*, **23**, 385–390.
- Raykundaliya, D. and Bhingikar, M. (2025). Estimation of parameters of two Weibull populations under the joint ranked set Sampling . *Revista Investigación Operacional*, **46**, 439–472.
- Raykundaliya, D. and Patel, M. (2022a). Estimation for two exponential populations based on joint percentile rank set sampling. *International Journal of Statistics and Reliability Engineering*, **9**, 282–292.
- Raykundaliya, D. and Patel, M. (2022b). Estimation for two exponential populations based on joint rank set sampling. *Revista Investigación Operacional*, **43**, 544–559.
- Wolfe, D. A. (2004). Ranked set sampling: an approach to more efficient data collection. *Statistical Science*, **19**, 636–643.
- World Health Organization (2023). Lung cancer. <https://www.who.int/news-room/fact-sheets/detail/lung-cancer>. Accessed: 2025-01-16.

ANNEXURE

Table 1: Simulation Result of MLE of Kumaraswamy parameters $a_1 = 1.5, b_1 = 8$ and $a_2 = 1.65, b_2 = 8.8$

m_1	m_2	k	JRSS						JMhRSS						JRSRS																			
			\hat{a}_1		\hat{a}_2		\hat{b}_1		\hat{b}_2		\hat{a}_1		\hat{a}_2		\hat{b}_1		\hat{b}_2		\hat{a}_1		\hat{a}_2		\hat{b}_1		\hat{b}_2									
			RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias						
4	3	2	1.52042	8.74694	1.67808	9.4451	1.5071	8.3475	1.6543	8.8928	1.6105	9.4153	1.725	10.2453	1.70427	11.48768	1.82056	11.82987	1.52042	8.74694	1.67808	9.4451	1.5071	8.3475	1.6543	8.8928	1.6105	9.4153	1.725	10.2453	1.70427	11.48768	1.82056	11.82987
			0.20423	2.53662	0.17952	2.16132	0.1801	2.3415	0.1776	2.3886	0.3301	2.823	0.3112	2.9758	0.37004	4.69534	0.34346	4.32441	0.20423	2.53662	0.17952	2.16132	0.1801	2.3415	0.1776	2.3886	0.3301	2.823	0.3112	2.9758	0.37004	4.69534	0.34346	4.32441
			0.02402	0.74694	0.02808	0.6451	0.0071	0.3475	0.0043	0.0928	0.1105	1.4153	0.075	1.4453	0.20427	3.48768	0.17056	3.02987	0.02402	0.74694	0.02808	0.6451	0.0071	0.3475	0.0043	0.0928	0.1105	1.4153	0.075	1.4453	0.20427	3.48768	0.17056	3.02987
			1.48903	8.08997	1.65839	9.17293	1.4781	8.1915	1.6839	9.3758	1.535	8.7852	1.6761	9.4917	1.59852	10.16494	1.72934	10.64433	1.48903	8.08997	1.65839	9.17293	1.4781	8.1915	1.6839	9.3758	1.535	8.7852	1.6761	9.4917	1.59852	10.16494	1.72934	10.64433
4	3	2	0.15559	1.61661	0.15126	1.83125	0.1503	2.4149	0.1275	1.8764	0.2918	2.535	0.2893	2.4968	0.32773	3.52876	0.29465	3.55104	0.15559	1.61661	0.15126	1.83125	0.1503	2.4149	0.1275	1.8764	0.2918	2.535	0.2893	2.4968	0.32773	3.52876	0.29465	3.55104
			-0.01097	0.08997	0.00839	0.37293	-0.0219	0.1915	0.0339	0.5758	0.035	0.7852	0.0261	0.6917	0.09852	2.16494	0.07934	1.84433	-0.01097	0.08997	0.00839	0.37293	-0.0219	0.1915	0.0339	0.5758	0.035	0.7852	0.0261	0.6917	0.09852	2.16494	0.07934	1.84433
			1.52597	8.46756	1.62412	8.74326	1.4869	7.9214	1.6448	9.0335	1.569	9.0162	1.6414	9.0979	1.55196	9.22715	1.71098	10.45493	1.52597	8.46756	1.62412	8.74326	1.4869	7.9214	1.6448	9.0335	1.569	9.0162	1.6414	9.0979	1.55196	9.22715	1.71098	10.45493
			0.15298	1.6016	0.12414	1.38213	0.1444	1.4674	0.1245	1.3401	0.2402	2.1269	0.2198	1.9701	0.22626	2.82601	0.25781	3.27956	0.15298	1.6016	0.12414	1.38213	0.1444	1.4674	0.1245	1.3401	0.2402	2.1269	0.2198	1.9701	0.22626	2.82601	0.25781	3.27956
3	5	2	0.02597	0.46756	-0.02588	-0.05674	-0.0131	-0.0786	-0.0052	0.2335	0.069	1.0162	-0.0086	0.2979	0.05196	0.06098	1.65493	0.02597	0.46756	-0.02588	-0.05674	-0.0131	-0.0786	-0.0052	0.2335	0.069	1.0162	-0.0086	0.2979	0.05196	0.06098	1.65493		
			1.47201	8.14049	1.61921	8.88119	1.5433	8.6529	1.634	8.8785	1.5259	9.0356	1.6417	9.3497	1.62578	10.32224	1.69332	10.33858	1.47201	8.14049	1.61921	8.88119	1.5433	8.6529	1.634	8.8785	1.5259	9.0356	1.6417	9.3497	1.62578	10.32224	1.69332	10.33858
			0.18974	2.13464	0.16257	1.79604	0.1723	1.9977	0.1638	2.594	0.3601	2.9143	0.3254	2.8226	0.36054	3.95396	0.29393	3.84344	0.18974	2.13464	0.16257	1.79604	0.1723	1.9977	0.1638	2.594	0.3601	2.9143	0.3254	2.8226	0.36054	3.95396	0.29393	3.84344
			-0.02799	0.14049	-0.03079	0.08119	0.0433	0.6529	-0.016	0.0785	0.0259	1.0356	-0.0083	0.5497	0.12578	2.32224	0.04332	1.53858	-0.02799	0.14049	-0.03079	0.08119	0.0433	0.6529	-0.016	0.0785	0.0259	1.0356	-0.0083	0.5497	0.12578	2.32224	0.04332	1.53858
3	5	3	1.48583	8.25521	1.64063	8.85063	1.5005	8.313	1.656	9.226	1.4668	8.3746	1.7127	9.5877	1.62752	10.10057	1.73163	10.22245	1.48583	8.25521	1.64063	8.85063	1.5005	8.313	1.656	9.226	1.4668	8.3746	1.7127	9.5877	1.62752	10.10057	1.73163	10.22245
			0.13487	1.54567	0.14191	1.72893	0.127	1.7582	0.1526	1.995	0.3294	2.4214	0.2326	2.2042	0.29952	3.56592	0.28102	3.34221	0.13487	1.54567	0.14191	1.72893	0.127	1.7582	0.1526	1.995	0.3294	2.4214	0.2326	2.2042	0.29952	3.56592	0.28102	3.34221
			-0.01417	0.25521	-0.00937	0.05063	0.0005	0.313	0.006	0.426	-0.0332	0.3746	0.0627	0.7877	0.12752	2.10057	0.08163	1.42245	-0.01417	0.25521	-0.00937	0.05063	0.0005	0.313	0.006	0.426	-0.0332	0.3746	0.0627	0.7877	0.12752	2.10057	0.08163	1.42245
			1.49401	8.14764	1.6395	9.04624	1.4864	8.127	1.6355	9.1881	1.5185	8.3664	1.659	8.9808	1.60941	9.50905	1.69059	10.23561	1.49401	8.14764	1.6395	9.04624	1.4864	8.127	1.6355	9.1881	1.5185	8.3664	1.659	8.9808	1.60941	9.50905	1.69059	10.23561
3	5	4	0.13804	1.31315	0.11549	1.40191	0.1122	1.4002	0.1027	1.6387	0.1928	1.2865	0.1691	1.33	0.25071	3.11481	0.20955	2.90981	0.13804	1.31315	0.11549	1.40191	0.1122	1.4002	0.1027	1.6387	0.1928	1.2865	0.1691	1.33	0.25071	3.11481	0.20955	2.90981
			-0.00599	0.14764	-0.0105	0.24624	-0.0136	0.127	-0.0145	0.3881	0.0185	0.3664	0.009	0.1808	0.10941	1.50905	0.04059	1.43561	-0.00599	0.14764	-0.0105	0.24624	-0.0136	0.127	-0.0145	0.3881	0.0185	0.3664	0.009	0.1808	0.10941	1.50905	0.04059	1.43561
			1.53153	8.35245	1.65106	9.24652	1.5197	8.4047	1.6463	9.2382	1.4556	8.2586	1.7266	10.0508	1.55013	10.02691	1.68086	10.37209	1.53153	8.35245	1.65106	9.24652	1.5197	8.4047	1.6463	9.2382	1.4556	8.2586	1.7266	10.0508	1.55013	10.02691	1.68086	10.37209
			0.1506	1.25993	0.11308	1.63126	0.1698	2.1377	0.1496	1.9254	0.3666	2.6342	0.2819	2.6538	0.2737	3.28596	0.2671	3.53977	0.1506	1.25993	0.11308	1.63126	0.1698	2.1377	0.1496	1.9254	0.3666	2.6342	0.2819	2.6538	0.2737	3.28596	0.2671	3.53977
3	6	2	0.03153	0.35245	0.00106	0.44652	0.0197	0.4047	-0.0037	0.4382	-0.0444	0.2586	0.0766	1.2508	0.05013	0.03686	1.57209	0.03153	0.35245	0.00106	0.44652	0.0197	0.4047	-0.0037	0.4382	-0.0444	0.2586	0.0766	1.2508	0.05013	0.03686	1.57209		
			1.49639	8.33889	1.63636	8.90609	1.5043	8.261	1.6567	8.9116	1.4939	8.342	1.6673	9.1017	1.50812	9.01897	1.74196	10.75446	1.49639	8.33889	1.63636	8.90609	1.5043	8.261	1.6567	8.9116	1.4939	8.342	1.6673	9.1017	1.50812	9.01897	1.74196	10.75446
			0.14747	1.58555	0.11467	1.34916	0.1231	1.3795	0.1006	1.355	0.1855	1.2142	0.1839	1.4168	0.31534	3.16649	0.22777	3.07345	0.14747	1.58555	0.11467	1.34916	0.1231	1.3795	0.1006	1.355	0.1855	1.2142	0.1839	1.4168	0.31534	3.16649	0.22777	3.07345
			-0.00361	0.33889	-0.01364	0.10609	0.0043	0.261	0.0067	0.1116	-0.0061	0.342	0.0173	0.3017	0.00812	1.01897	0.09196	1.95446	-0.00361	0.33889	-0.01364	0.10609	0.0043	0.261	0.0067	0.1116	-0.0061	0.342	0.0173	0.3017	0.00812	1.01897	0.09196	1.95446
3	6	4	1.51102	8.38895	1.63615	8.81763	1.541	8.4263	1.625	8.7421	1.4675	8.0014	1.6893	9.2976	1.58103	9.28695	1.71101	10.162	1.51102	8.38895	1.63615	8.81763	1.541	8.4263	1.625	8.7421	1.4675	8.0014	1.6893	9.2976	1.58103	9.28695	1.71101	10.162
			0.10626	1.22692	0.10878	1.29116	0.1269	1.7628	0.1031	1.7009	0.2157	1.6695	0.2149	1.8116	0.25318	2.77328	0.21167	2.76845	0.10626	1.22692	0.10878	1.29116	0.1269	1.7628	0.1031	1.7009	0.2157	1.6695	0.2149	1.8116	0.25318	2.77328	0.21167	2.76845
			0.01102	0.38895	-0.01385	0.01763	0.041	0.4263	-0.025	-0.0579	-0.0325	0.0014	0.0393	0.4976	0.08103	1.28695	0.06101	1.362	0.01102	0.38895	-0.01385	0.01763	0.041	0.4263	-0.025	-0.0579	-0.0325	0.0014	0.0393	0.4976	0.08103	1.28695	0.06101	1.362

Table 1: Continued

m_1	m_2	k	JRSS						JMnRSS						JMxRSS						JRS																
			\hat{a}_1		\hat{a}_2		\hat{b}_2		\hat{a}_1		\hat{a}_2		\hat{b}_2		\hat{a}_1		\hat{a}_2		\hat{b}_2		\hat{a}_1		\hat{a}_2		\hat{b}_2												
			RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias											
4	4	2	1.49286	8.25801	1.67274	9.53145	1.4909	8.2363	1.6479	9.1722	1.5208	8.7169	1.716	9.8436	1.66583	10.7926	1.756	10.28901	1.49286	8.25801	1.67274	9.53145	1.4909	8.2363	1.6479	9.1722	1.5208	8.7169	1.716	9.8436	1.66583	10.7926	1.756	10.28901			
			0.18434	2.27303	0.15789	1.73823	0.1784	2.4963	0.1703	2.2305	0.3614	2.9985	0.3175	2.8136	0.34001	4.36076	0.34001	4.36076	0.34107	3.5746	0.18434	2.27303	0.15789	1.73823	0.1784	2.4963	0.1703	2.2305	0.3614	2.9985	0.3175	2.8136	0.34001	4.36076	0.34001	4.36076	0.34107
		-0.00714	0.25801	0.02274	0.73145	-0.0091	0.2363	-0.0021	0.3722	0.0208	0.7169	0.066	1.0436	0.16583	2.7926	0.16583	2.7926	0.106	1.48901	-0.00714	0.25801	0.02274	0.73145	-0.0091	0.2363	-0.0021	0.3722	0.0208	0.7169	0.066	1.0436	0.16583	2.7926	0.16583	2.7926	0.106	1.48901
		1.46499	8.02776	1.64601	8.91012	1.5	8.3285	1.6341	9.1258	1.5666	9.0351	1.6795	9.4076	1.54263	9.69631	1.54263	9.69631	1.70524	10.3902	1.46499	8.02776	1.64601	8.91012	1.5	8.3285	1.6341	9.1258	1.5666	9.0351	1.6795	9.4076	1.54263	9.69631	1.54263	9.69631	1.70524	10.3902
		0.1326	1.67665	0.13566	1.44047	0.1285	1.826	0.145	1.8077	0.2881	2.6854	0.2653	2.3074	0.21483	3.1611	0.21483	3.1611	0.27166	3.50769	0.1326	1.67665	0.13566	1.44047	0.1285	1.826	0.145	1.8077	0.2881	2.6854	0.2653	2.3074	0.21483	3.1611	0.27166	3.50769		
		-0.03501	0.02776	-0.00399	0.11012	0.1285	0.3285	-0.0159	0.3258	0.0666	1.0351	0.0295	0.6076	0.04263	1.69631	0.04263	1.69631	0.05524	1.5902	-0.03501	0.02776	-0.00399	0.11012	0.1285	0.3285	-0.0159	0.3258	0.0666	1.0351	0.0295	0.6076	0.04263	1.69631	0.05524	1.5902		
4	4	4	1.53286	8.71851	1.65888	9.13196	1.4724	7.9228	1.6728	9.3161	1.5382	8.8496	1.689	9.3796	1.57266	9.55059	1.68745	9.82742	1.53286	8.71851	1.65888	9.13196	1.4724	7.9228	1.6728	9.3161	1.5382	8.8496	1.689	9.3796	1.57266	9.55059	1.68745	9.82742			
			0.15238	1.89434	0.13485	1.73994	0.1328	1.7914	0.1093	1.3466	0.2522	2.2657	0.2335	2.0799	0.22858	2.79425	0.22858	2.79425	0.26297	3.1497	0.15238	1.89434	0.13485	1.73994	0.1328	1.7914	0.1093	1.3466	0.2522	2.2657	0.2335	2.0799	0.22858	2.79425	0.22858	2.79425	0.26297
		0.03286	0.71851	0.00888	0.33196	-0.0276	-0.0772	0.0228	0.5161	0.0382	0.8496	0.039	0.5796	0.07266	1.55059	0.07266	1.55059	0.03745	1.02742	0.03286	0.71851	0.00888	0.33196	-0.0276	-0.0772	0.0228	0.5161	0.0382	0.8496	0.039	0.5796	0.07266	1.55059	0.07266	1.55059	0.03745	1.02742
		1.53649	8.8251	1.63839	8.90336	1.5105	8.6123	1.6744	9.099	1.5353	8.9401	1.74	9.8085	1.60176	9.96134	1.60176	9.96134	1.71982	10.96764	1.53649	8.8251	1.63839	8.90336	1.5105	8.6123	1.6744	9.099	1.5353	8.9401	1.74	9.8085	1.60176	9.96134	1.60176	9.96134	1.71982	10.96764
		0.15339	1.8379	0.14193	1.61145	0.1393	2.1509	0.1612	1.7171	0.2898	2.569	0.2689	2.5554	0.27897	3.13417	0.27897	3.13417	0.24793	3.66735	0.15339	1.8379	0.14193	1.61145	0.1393	2.1509	0.1612	1.7171	0.2898	2.569	0.2689	2.5554	0.27897	3.13417	0.27897	3.13417	0.24793	3.66735
		0.03649	0.8251	-0.01161	0.10336	0.0105	0.6123	0.0244	0.299	0.0353	0.9401	0.09	1.0085	0.10176	1.96134	0.10176	1.96134	0.06982	2.16764	0.03649	0.8251	-0.01161	0.10336	0.0105	0.6123	0.0244	0.299	0.0353	0.9401	0.09	1.0085	0.10176	1.96134	0.10176	1.96134	0.06982	2.16764
4	5	3	1.51083	8.38981	1.66195	9.18154	1.5067	8.4458	1.6802	9.115	1.5758	9.0887	1.6723	9.5127	1.58959	9.83401	1.68042	9.95384	1.51083	8.38981	1.66195	9.18154	1.5067	8.4458	1.6802	9.115	1.5758	9.0887	1.6723	9.5127	1.58959	9.83401	1.68042	9.95384			
			0.13429	1.39775	0.12176	1.42818	0.1316	1.728	0.1159	1.2251	0.2998	2.6519	0.2308	1.9946	0.26432	3.20095	0.26432	3.20095	0.23664	2.88007	0.13429	1.39775	0.12176	1.42818	0.1316	1.728	0.1159	1.2251	0.2998	2.6519	0.2308	1.9946	0.26432	3.20095	0.26432	3.20095	0.23664
		0.01083	0.38981	0.01195	0.38154	0.0067	0.4458	0.0302	0.315	0.0758	1.0887	0.0223	0.7127	0.08959	1.83401	0.08959	1.83401	0.03042	1.15384	0.01083	0.38981	0.01195	0.38154	0.0067	0.4458	0.0302	0.315	0.0758	1.0887	0.0223	0.7127	0.08959	1.83401	0.08959	1.83401	0.03042	1.15384
		1.50682	8.13379	1.61718	8.66516	1.5071	8.5482	1.6511	8.7686	1.5045	8.1788	1.6547	9.0448	1.53221	9.20624	1.53221	9.20624	1.68242	9.36452	1.50682	8.13379	1.61718	8.66516	1.5071	8.5482	1.6511	8.7686	1.5045	8.1788	1.6547	9.0448	1.53221	9.20624	1.53221	9.20624	1.68242	9.36452
		0.12553	1.41001	0.12633	1.37759	0.1073	1.7723	0.1081	1.4952	0.1966	1.4313	0.1728	1.4345	0.20655	2.7503	0.20655	2.7503	0.23708	2.65352	0.12553	1.41001	0.12633	1.37759	0.1073	1.7723	0.1081	1.4952	0.1966	1.4313	0.1728	1.4345	0.20655	2.7503	0.20655	2.7503	0.23708	2.65352
		0.00682	0.13379	-0.03282	-0.13484	0.0071	0.5482	0.0011	-0.0314	0.0045	0.1788	0.0047	0.2448	0.03221	1.20624	0.03221	1.20624	0.03242	0.56452	0.00682	0.13379	-0.03282	-0.13484	0.0071	0.5482	0.0011	-0.0314	0.0045	0.1788	0.0047	0.2448	0.03221	1.20624	0.03221	1.20624	0.03242	0.56452
6	6	2	1.52455	8.46861	1.64938	8.91842	1.4945	8.5106	1.6556	9.4685	1.5893	9.0071	1.7052	9.6838	1.59438	10.08091	1.72498	10.18676	1.52455	8.46861	1.64938	8.91842	1.4945	8.5106	1.6556	9.4685	1.5893	9.0071	1.7052	9.6838	1.59438	10.08091	1.72498	10.18676			
			0.15115	1.58216	0.11672	1.68982	0.149	2.1655	0.1366	2.0424	0.3316	2.65	0.2774	2.4755	0.3062	3.69095	0.3062	3.69095	0.26459	3.29431	0.15115	1.58216	0.11672	1.68982	0.149	2.1655	0.1366	2.0424	0.3316	2.65	0.2774	2.4755	0.3062	3.69095	0.3062	3.69095	0.26459
		0.02455	0.46861	-0.00062	0.11842	-0.0055	0.5106	0.0056	0.6685	0.0893	1.0071	0.0552	0.8838	0.09438	2.08091	0.09438	2.08091	0.07498	1.38676	0.02455	0.46861	-0.00062	0.11842	-0.0055	0.5106	0.0056	0.6685	0.0893	1.0071	0.0552	0.8838	0.09438	2.08091	0.09438	2.08091	0.07498	1.38676
		1.48871	8.28029	1.64183	8.95378	1.5043	8.4693	1.6437	8.6977	1.5302	8.5058	1.7057	9.4572	1.5782	9.97349	1.5782	9.97349	1.74419	10.459	1.48871	8.28029	1.64183	8.95378	1.5043	8.4693	1.6437	8.6977	1.5302	8.5058	1.7057	9.4572	1.5782	9.97349	1.5782	9.97349	1.74419	10.459
		0.11894	1.36256	0.11325	1.52915	0.0993	1.5849	0.1107	1.3881	0.2198	1.5732	0.2154	1.9084	0.24481	3.46441	0.24481	3.46441	0.24894	3.13182	0.11894	1.36256	0.11325	1.52915	0.0993	1.5849	0.1107	1.3881	0.2198	1.5732	0.2154	1.9084	0.24481	3.46441	0.24481	3.46441	0.24894	3.13182
		-0.01129	0.28029	-0.00817	0.15378	0.0043	0.4693	-0.0063	-0.1023	0.0302	0.5058	0.0557	0.6572	0.0782	1.97349	0.0782	1.97349	0.09419	1.659	-0.01129	0.28029	-0.00817	0.15378	0.0043	0.4693	-0.0063	-0.1023	0.0302	0.5058	0.0557	0.6572	0.0782	1.97349	0.0782	1.97349	0.09419	1.659
4	4	4	1.49008	8.12498	1.63823	8.83481	1.4869	7.9199	1.6483	9.0197	1.4993	8.4833	1.6761	9.3754	1.51529	8.76422	1.70084	9.82427	1.49008	8.12498	1.63823	8.83481	1.4869	7.9199	1.6483	9.0197	1.4993	8.4833	1.6761	9.3754	1.51529	8.76422	1.70084	9.82427			
			0.11396	1.40001	0.10384	1.3087	0.1162	1.4423	0.1031	1.5729	0.1941	1.7086	0.1837	1.617	2.69504	0.1941	2.69504	0.19635	2.62445	0.11396	1.40001	0.10384	1.3087	0.1162	1.4423	0.1031	1.5729	0.1941	1.7086	0.1837	1.617	2.69504	0.1941	2.69504	0.19635	2.62445	
		-0.00992	0.12498	-0.01177	0.03481	-0.0131	-0.0801	-0.0017	0.2197	-0.0007	0.4833	0.0261	0.5754	0.01529	0.76422	0.01529	0.76422	0.05084	1.02427	-0.00992	0.12498	-0.01177	0.03481	-0.0131	-0.0801	-0.0017	0.2197	-0.0007	0.4833	0.0261	0.5754	0.01529	0.76422	0.01529	0.76422	0.05084	1.02427

Table 1: Continued

m_1	m_2	k	JRSS			JMnRSS			JMxRSS			JRSR						
			\hat{a}_1 RMSE Bias	\hat{b}_1 RMSE Bias	\hat{a}_2 RMSE Bias	\hat{a}_1 RMSE Bias	\hat{b}_1 RMSE Bias	\hat{a}_2 RMSE Bias	\hat{a}_1 RMSE Bias	\hat{b}_1 RMSE Bias	\hat{a}_2 RMSE Bias	\hat{a}_1 RMSE Bias	\hat{b}_1 RMSE Bias	\hat{a}_2 RMSE Bias	\hat{b}_2 RMSE Bias			
4	3	2	1.48816	8.27453	1.66497	8.9578	1.5227	8.7664	1.6813	8.7665	1.6	9.3217	1.6564	9.4102	1.59855	9.93289	1.73751	10.6201
		3	0.12018	1.56676	0.12897	1.09736	0.1301	2.0904	0.1614	1.6084	0.2888	2.6692	0.3089	2.7612	0.26986	3.49422	0.29757	3.35348
		4	-0.01184	0.27453	0.01497	0.1578	0.0227	0.7664	0.0313	-0.0335	0.1	1.3217	0.0064	0.6102	0.09855	1.93289	0.08751	1.8201
		5	1.49904	8.26458	1.64212	8.85833	1.4819	7.7353	1.648	8.7964	1.5608	9.1097	1.7019	9.7415	1.60716	10.0496	1.77386	11.01777
5	4	2	0.12487	1.55068	0.12663	1.36741	0.1233	1.4287	0.1274	1.1432	0.2362	2.1425	0.2959	2.5743	0.26576	3.4895	0.27533	3.58305
		3	-0.00096	0.26458	-0.00788	0.05833	-0.0181	-0.2647	-0.002	-0.0036	0.0608	1.1097	0.0519	0.9415	0.10716	2.0496	0.12386	2.21777
		4	1.50033	8.30283	1.66359	8.99673	1.5122	8.725	1.68	9.5934	1.5431	8.5502	1.6501	8.9983	1.53403	9.26047	1.75299	10.80342
		5	0.10876	1.40566	0.12898	1.53228	0.1184	2.2533	0.1047	1.6158	0.2314	1.8853	0.1753	1.5094	0.21066	2.92393	0.23196	3.12815
6	3	2	0.00033	0.30283	0.01359	0.19673	0.0122	0.725	0.03	0.7934	0.0431	0.5502	0.0001	0.1983	0.03403	1.26047	0.10299	2.00342
		3	1.47711	7.85813	1.63547	8.78846	1.5055	8.2648	1.6518	9.2239	1.5891	9.3453	1.7194	9.7755	1.63624	10.45833	1.70631	10.60979
		4	0.1315	1.36304	0.13285	1.33712	0.1203	1.7857	0.1141	1.8308	0.305	2.7984	0.2442	2.2355	0.30936	3.70549	0.27717	3.60894
		5	-0.02289	-0.14187	-0.01453	-0.01154	0.0055	0.2648	0.0018	0.4239	0.0891	1.3453	0.0694	0.9755	0.13624	2.45833	0.05631	1.80979
6	2	2	1.53389	8.76255	1.65234	9.0756	1.5126	8.4426	1.6603	9.2183	1.56	8.8246	1.6635	9.3273	1.5746	9.55388	1.7006	10.68063
		3	0.13852	1.68649	0.12719	1.60717	0.1027	1.8195	0.1151	1.7292	0.2973	2.4951	0.2351	2.0674	0.2255	3.01462	0.23814	3.00685
		4	0.03389	0.76255	0.00234	0.2756	0.0126	0.4426	0.0103	0.4183	0.06	0.8246	0.0135	0.5273	0.0746	1.55388	0.0506	1.88063
		5	1.49763	8.11658	1.64213	8.85041	1.5327	8.9147	1.6572	9.015	1.4898	8.0667	1.6636	9.0496	1.57873	9.72109	1.69001	9.75746
6	4	2	0.09675	1.18347	0.08422	1.04815	0.0962	1.7947	0.1222	1.6516	0.1732	1.3127	0.1714	1.3468	0.23349	3.07959	0.20892	2.54929
		3	-0.00237	0.11658	-0.00787	0.05041	0.0327	0.9147	0.0072	0.215	-0.0102	0.0667	0.0136	0.2496	0.07873	1.72109	0.04001	0.95746
		4	1.45535	7.88926	1.65669	9.00768	1.522	8.8238	1.6791	9.2446	1.5775	9.0796	1.6785	9.4493	1.60715	10.43475	1.72228	10.41793
		5	0.15067	1.77778	0.13747	1.72694	0.1273	2.0251	0.1215	1.66	0.3161	2.4909	0.3163	2.6428	0.28067	3.68499	0.25893	3.48022
6	3	2	-0.04465	-0.11074	0.00669	0.20768	0.022	0.8238	0.0291	0.4446	0.0775	1.0796	0.0285	0.6493	0.10715	2.43475	0.07228	1.61793
		3	1.47545	7.86434	1.62126	8.72511	1.4818	8.0136	1.6558	8.8154	1.5533	8.4833	1.6483	9.1125	1.59503	9.75264	1.69195	10.41235
		4	0.1182	1.28614	0.10698	1.20663	0.1206	1.7415	0.1074	1.0814	0.2293	1.6834	0.1923	1.4398	0.22472	2.81341	0.24977	3.30357
		5	-0.02455	-0.13566	-0.02874	-0.07489	-0.0182	0.0136	0.0058	0.0154	0.0533	0.4833	-0.0017	0.3125	0.09503	1.75264	0.04195	1.61235
6	4	2	1.52193	8.35878	1.63265	8.75807	1.4997	8.4966	1.6487	8.7416	1.497	8.1003	1.673	9.1451	1.59803	9.80179	1.66894	9.73596
		3	0.10536	1.26196	0.09571	1.17483	0.0832	1.3748	0.0985	1.2126	0.1861	1.408	0.1719	1.43	0.23598	2.92769	0.18036	2.46181
		4	0.02193	0.35878	-0.01735	-0.04193	-0.0003	0.4966	-0.0013	-0.0584	-0.003	0.1003	0.023	0.3451	0.09803	1.80179	0.01894	0.93596
		5	1.52193	8.35878	1.63265	8.75807	1.4997	8.4966	1.6487	8.7416	1.497	8.1003	1.673	9.1451	1.59803	9.80179	1.66894	9.73596

Table 2: Simulation Result of MLE of Kumaraswamy parameters for change in difference d between parameters $a_1 = 1.5, b_1 = 8$ and $a_2 = d \times a_1, b_2 = d \times b_1$

$m_1 = 3, m_2 = 4$		$d = 1.1$				$d = 1.3$				
		\hat{a}_1	\hat{b}_1	\hat{a}_2	\hat{b}_2	\hat{a}_1	\hat{b}_1	\hat{a}_2	\hat{b}_2	
		RMSE Bias	RMSE Bias	RMSE Bias	RMSE Bias	RMSE Bias	RMSE Bias	RMSE Bias	RMSE Bias	
JRSS	2	1.52042	8.74694	1.67808	9.4451	1.5291	9.17915	1.9949	11.49791	
		0.20423	2.53662	0.17952	2.16132	0.23957	2.97852	0.23274	3.13233	
		0.02042	0.74694	0.02808	0.6451	0.0291	1.17915	0.0449	1.09791	
	3	1.48903	8.08997	1.65839	9.17293	1.49837	8.5496	1.97007	11.09915	
		0.15559	1.61661	0.15126	1.83125	0.17681	2.19129	0.17885	2.30199	
		-0.01097	0.08997	0.00839	0.37293	-0.00163	0.5496	0.02007	0.69915	
	4	1.52597	8.46756	1.62412	8.74326	1.51706	8.35237	1.92898	10.36807	
		0.15298	1.6016	0.12414	1.38213	0.15302	1.65903	0.13385	1.59539	
		0.02597	0.46756	-0.02588	-0.05674	0.01706	0.35237	-0.02102	-0.03193	
	JMnRSS	2	1.5071	8.3475	1.6543	8.8928	1.499	8.3764	1.9692	10.6688
			0.1801	2.3415	0.1776	2.3886	0.2181	2.8794	0.2134	2.0536
			0.0071	0.3475	0.0043	0.0928	-0.001	0.3764	0.0192	0.2688
3		1.4781	8.1915	1.6839	9.3758	1.4968	8.364	1.9436	10.2982	
		0.1503	2.4149	0.1275	1.8764	0.1433	2.0406	0.1723	1.88	
		-0.0219	0.1915	0.0339	0.5758	-0.0032	0.364	-0.0064	-0.1018	
4		1.4869	7.9214	1.6448	9.0335	1.5142	8.5056	1.9409	10.9441	
		0.1444	1.4674	0.1245	1.3401	0.1194	1.6181	0.1243	1.242	
		-0.0131	-0.0786	-0.0052	0.2335	0.0142	0.5056	-0.0091	0.5441	
JMxRSS		2	1.6105	9.4153	1.725	10.2453	1.5455	8.7778	1.9005	10.712
			0.3301	2.823	0.3112	2.9758	0.331	2.7325	0.3005	2.77
			0.1105	1.4153	0.075	1.4453	0.0455	0.7778	-0.0495	0.312
	3	1.535	8.7852	1.6761	9.4917	1.5749	8.9757	2.0615	11.7672	
		0.2918	2.535	0.2893	2.4968	0.3347	2.743	0.3207	3.1144	
		0.035	0.7852	0.0261	0.6917	0.0749	0.9757	0.1115	1.3672	
	4	1.569	9.0162	1.6414	9.0979	1.5316	8.595	1.9448	10.7263	
		0.2402	2.1269	0.2198	1.9701	0.2831	2.3068	0.2157	2.2432	
		0.069	1.0162	-0.0086	0.2979	0.0316	0.595	-0.0052	0.3263	
	JSRS	2	1.70427	11.48768	1.82056	11.82987	1.66509	11.07387	2.07283	13.03948
			0.37004	4.69534	0.34346	4.32441	0.41662	5.32529	0.38875	4.95431
			0.20427	3.48768	0.17056	3.02987	0.16509	3.07387	0.12283	2.63948
3		1.59852	10.16494	1.72934	10.64433	1.54414	9.31753	2.02295	11.61412	
		0.32773	3.52876	0.29465	3.55104	0.23431	3.11811	0.24579	2.92413	
		0.09852	2.16494	0.07934	1.84433	0.04414	1.31753	0.07295	1.21412	
4		1.55196	9.22715	1.71098	10.45493	1.56442	9.45817	2.02109	11.83446	
		0.22626	2.82601	0.25781	3.27956	0.23593	3.14709	0.24714	3.31259	
		0.05196	1.22715	0.06098	1.65493	0.06442	1.45817	0.07109	1.43446	

Table 3: Simulation Result of power of LRT test for equality of parameters of Kumaraswamy populations $a_1 = a_2 = a$, when b_1 and b_2 are known

			JRSS		JMnRSS		JMxRSS		JSRS		
m_1	m_2	k	$\lambda(W)$	Power	$\lambda(W)$	Power	$\lambda(W)$	Power	$\lambda(W)$	Power	
3	4	2	0.605946	0.109	0.634739	0.1	0.640683	0.097	0.683245	0.071	
		3	0.560637	0.17	0.602584	0.127	0.597573	0.118	0.66023	0.087	
		4	0.511984	0.2	0.55775	0.172	0.596695	0.125	0.654486	0.089	
	5	2	0.561658	0.148	0.616627	0.123	0.624577	0.122	0.683656	0.066	
		3	0.527265	0.193	0.581667	0.157	0.612644	0.113	0.665368	0.077	
		4	0.480041	0.251	0.518879	0.194	0.563974	0.17	0.651837	0.096	
	6	2	0.579479	0.151	0.604477	0.103	0.61886	0.118	0.677142	0.08	
		3	0.507203	0.205	0.553525	0.17	0.577514	0.133	0.661561	0.086	
		4	0.45349	0.276	0.510304	0.21	0.542722	0.172	0.647128	0.106	
	4	4	2	0.59251	0.146	0.610892	0.12	0.619238	0.122	0.690272	0.074
			3	0.510793	0.213	0.57404	0.14	0.593448	0.117	0.653123	0.09
			4	0.462144	0.254	0.514641	0.204	0.551085	0.165	0.643129	0.08
5		2	0.548538	0.195	0.600239	0.115	0.602536	0.141	0.655993	0.086	
		3	0.490216	0.243	0.553969	0.179	0.571588	0.155	0.657353	0.079	
		4	0.408267	0.307	0.517983	0.2	0.53759	0.181	0.641473	0.1	
6		2	0.51064	0.217	0.576494	0.144	0.570677	0.142	0.661429	0.079	
		3	0.450653	0.281	0.527459	0.188	0.543078	0.175	0.645314	0.089	
		4	0.373734	0.379	0.468754	0.256	0.518053	0.208	0.621441	0.111	
5	4	2	0.548698	0.159	0.602322	0.126	0.617442	0.121	0.676036	0.072	
		3	0.46877	0.25	0.531177	0.177	0.572671	0.15	0.647266	0.091	
		4	0.428425	0.294	0.49494	0.209	0.538261	0.18	0.640677	0.097	
	5	2	0.519791	0.214	0.572252	0.144	0.590827	0.131	0.677568	0.077	
		3	0.449507	0.277	0.513034	0.201	0.547614	0.177	0.638682	0.095	
		4	0.361311	0.366	0.464509	0.241	0.497757	0.239	0.620769	0.113	
	6	2	0.475895	0.231	0.564788	0.162	0.598058	0.133	0.669942	0.082	
		3	0.393947	0.341	0.50123	0.226	0.530053	0.192	0.644247	0.099	
		4	0.31849	0.439	0.450126	0.285	0.484016	0.217	0.624834	0.117	

Table 5: Result of MLE Kumaraswamy parameters (a_1, b_1) and (a_2, b_2)

m_1	m_2	k	JRSS						JMnRSS						JMxRSS						JSRS																																													
			\hat{a}_1		\hat{b}_2		\hat{a}_2		\hat{b}_1		\hat{a}_1		\hat{b}_2		\hat{a}_2		\hat{b}_1		\hat{a}_1		\hat{b}_2		\hat{a}_2		\hat{b}_1		\hat{a}_1		\hat{b}_2																																					
			RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias																																				
5	4	2	2.19255	0.16734	0.76217	0.09424	1.5109	0.1998	0.806	0.1542	7.449	0.1502	55.0472	0.1175	5.01078	0.18472	1.97813	0.10901	2.19255	0.16734	0.76217	0.09424	1.5109	0.1998	0.806	0.1542	7.449	0.1502	55.0472	0.1175	5.01078	0.18472	1.97813	0.10901	2.19255	0.16734	0.76217	0.09424	1.5109	0.1998	0.806	0.1542	7.449	0.1502	55.0472	0.1175	5.01078	0.18472	1.97813	0.10901	2.19255	0.16734	0.76217	0.09424	1.5109	0.1998	0.806	0.1542	7.449	0.1502	55.0472	0.1175	5.01078	0.18472	1.97813	0.10901
			1.23843	0.02735	0.45422	0.01486	0.5805	0.0882	0.232	0.0711	6.3491	0.0255	54.3904	0.0202	4.08135	0.0537	1.50313	0.02366	1.23843	0.02735	0.45422	0.01486	0.5805	0.0882	0.232	0.0711	6.3491	0.0255	54.3904	0.0202	4.08135	0.0537	1.50313	0.02366	1.23843	0.02735	0.45422	0.01486	0.5805	0.0882	0.232	0.0711	6.3491	0.0255	54.3904	0.0202	4.08135	0.0537	1.50313	0.02366	1.23843	0.02735	0.45422	0.01486	0.5805	0.0882	0.232	0.0711	6.3491	0.0255	54.3904	0.0202	4.08135	0.0537	1.50313	0.02366
		0.98311	0.01672	0.10532	-0.0039	0.3015	0.0492	0.1491	0.0561	6.2396	0.0004	54.3903	0.0194	3.80134	0.0341	1.32128	0.01092	0.98311	0.01672	0.10532	-0.0039	0.3015	0.0492	0.1491	0.0561	6.2396	0.0004	54.3903	0.0194	3.80134	0.0341	1.32128	0.01092	0.98311	0.01672	0.10532	-0.0039	0.3015	0.0492	0.1491	0.0561	6.2396	0.0004	54.3903	0.0194	3.80134	0.0341	1.32128	0.01092	0.98311	0.01672	0.10532	-0.0039	0.3015	0.0492	0.1491	0.0561	6.2396	0.0004	54.3903	0.0194	3.80134	0.0341	1.32128	0.01092	
		1.58935	0.15835	0.66964	0.09419	1.3191	0.1647	0.7887	0.1456	1.039	0.1469	8.4275	0.1153	2.46861	0.16429	0.99721	0.10256	1.58935	0.15835	0.66964	0.09419	1.3191	0.1647	0.7887	0.1456	1.039	0.1469	8.4275	0.1153	2.46861	0.16429	0.99721	0.10256	1.58935	0.15835	0.66964	0.09419	1.3191	0.1647	0.7887	0.1456	1.039	0.1469	8.4275	0.1153	2.46861	0.16429	0.99721	0.10256	1.58935	0.15835	0.66964	0.09419	1.3191	0.1647	0.7887	0.1456	1.039	0.1469	8.4275	0.1153	2.46861	0.16429	0.99721	0.10256	
		0.66195	0.02429	0.29454	0.01267	0.3176	0.0416	0.1803	0.053	0.7655	0.0172	7.7707	0.0172	1.56602	0.03721	0.50017	0.01493	0.66195	0.02429	0.29454	0.01267	0.3176	0.0416	0.1803	0.053	0.7655	0.0172	7.7707	0.0172	1.56602	0.03721	0.50017	0.01493	0.66195	0.02429	0.29454	0.01267	0.3176	0.0416	0.1803	0.053	0.7655	0.0172	7.7707	0.0172	1.56602	0.03721	0.50017	0.01493	0.66195	0.02429	0.29454	0.01267	0.3176	0.0416	0.1803	0.053	0.7655	0.0172	7.7707	0.0172	1.56602	0.03721	0.50017	0.01493	
		0.37991	0.00773	0.01279	-0.0039	0.1097	0.0141	0.1318	0.0475	-0.1704	-0.0037	7.7706	0.0172	1.25917	0.01367	0.34036	0.00447	0.37991	0.00773	0.01279	-0.0039	0.1097	0.0141	0.1318	0.0475	-0.1704	-0.0037	7.7706	0.0172	1.25917	0.01367	0.34036	0.00447	0.37991	0.00773	0.01279	-0.0039	0.1097	0.0141	0.1318	0.0475	-0.1704	-0.0037	7.7706	0.0172	1.25917	0.01367	0.34036	0.00447	0.37991	0.00773	0.01279	-0.0039	0.1097	0.0141	0.1318	0.0475	-0.1704	-0.0037	7.7706	0.0172	1.25917	0.01367	0.34036	0.00447	
1.56245	0.1558	0.57733	0.09586	1.1889	0.1484	0.7675	0.1386	1.1218	0.1419	2.2007	0.1117	2.01851	0.16664	0.82286	0.10514	1.56245	0.1558	0.57733	0.09586	1.1889	0.1484	0.7675	0.1386	1.1218	0.1419	2.2007	0.1117	2.01851	0.16664	0.82286	0.10514	1.56245	0.1558	0.57733	0.09586	1.1889	0.1484	0.7675	0.1386	1.1218	0.1419	2.2007	0.1117	2.01851	0.16664	0.82286	0.10514	1.56245	0.1558	0.57733	0.09586	1.1889	0.1484	0.7675	0.1386	1.1218	0.1419	2.2007	0.1117	2.01851	0.16664	0.82286	0.10514			
0.56322	0.01677	0.25792	0.00969	0.2295	0.032	0.1503	0.0437	0.4233	0.0109	1.5438	0.0144	1.06235	0.03341	0.30901	0.01675	0.56322	0.01677	0.25792	0.00969	0.2295	0.032	0.1503	0.0437	0.4233	0.0109	1.5438	0.0144	1.06235	0.03341	0.30901	0.01675	0.56322	0.01677	0.25792	0.00969	0.2295	0.032	0.1503	0.0437	0.4233	0.0109	1.5438	0.0144	1.06235	0.03341	0.30901	0.01675	0.56322	0.01677	0.25792	0.00969	0.2295	0.032	0.1503	0.0437	0.4233	0.0109	1.5438	0.0144	1.06235	0.03341	0.30901	0.01675			
0.35301	0.00518	-0.0795	-0.0022	-0.0205	-0.0022	0.1106	0.0405	-0.0876	-0.0087	1.5438	0.0136	0.80907	0.01602	0.16601	0.00705	0.35301	0.00518	-0.0795	-0.0022	-0.0205	-0.0022	0.1106	0.0405	-0.0876	-0.0087	1.5438	0.0136	0.80907	0.01602	0.16601	0.00705	0.35301	0.00518	-0.0795	-0.0022	-0.0205	-0.0022	0.1106	0.0405	-0.0876	-0.0087	1.5438	0.0136	0.80907	0.01602	0.16601	0.00705	0.35301	0.00518	-0.0795	-0.0022	-0.0205	-0.0022	0.1106	0.0405	-0.0876	-0.0087	1.5438	0.0136	0.80907	0.01602	0.16601	0.00705			
2.09839	0.16378	0.71042	0.09722	1.3139	0.1683	0.8624	0.171	3.5029	0.1511	8.4418	0.1171	4.94195	0.19485	2.3303	0.11005	2.09839	0.16378	0.71042	0.09722	1.3139	0.1683	0.8624	0.171	3.5029	0.1511	8.4418	0.1171	4.94195	0.19485	2.3303	0.11005	2.09839	0.16378	0.71042	0.09722	1.3139	0.1683	0.8624	0.171	3.5029	0.1511	8.4418	0.1171	4.94195	0.19485	2.3303	0.11005	2.09839	0.16378	0.71042	0.09722	1.3139	0.1683	0.8624	0.171	3.5029	0.1511	8.4418	0.1171	4.94195	0.19485	2.3303	0.11005			
1.10089	0.02768	0.3219	0.01217	0.3283	0.0607	0.2439	0.0779	2.6262	0.0132	7.785	0.021	4.03999	0.06423	1.88627	0.02353	1.10089	0.02768	0.3219	0.01217	0.3283	0.0607	0.2439	0.0779	2.6262	0.0132	7.785	0.021	4.03999	0.06423	1.88627	0.02353	1.10089	0.02768	0.3219	0.01217	0.3283	0.0607	0.2439	0.0779	2.6262	0.0132	7.785	0.021	4.03999	0.06423	1.88627	0.02353	1.10089	0.02768	0.3219	0.01217	0.3283	0.0607	0.2439	0.0779	2.6262	0.0132	7.785	0.021	4.03999	0.06423	1.88627	0.02353			
0.88895	0.01316	0.05357	-0.0009	0.1045	0.0177	0.2055	0.0729	2.2935	0.0005	7.7849	0.019	3.73251	0.04423	1.67345	0.01196	0.88895	0.01316	0.05357	-0.0009	0.1045	0.0177	0.2055	0.0729	2.2935	0.0005	7.7849	0.019	3.73251	0.04423	1.67345	0.01196	0.88895	0.01316	0.05357	-0.0009	0.1045	0.0177	0.2055	0.0729	2.2935	0.0005	7.7849	0.019	3.73251	0.04423	1.67345	0.01196	0.88895	0.01316	0.05357	-0.0009	0.1045	0.0177	0.2055	0.0729	2.2935	0.0005	7.7849	0.019	3.73251	0.04423	1.67345	0.01196			
1.68475	0.16108	0.56554	0.09429	1.191	0.1503	0.8027	0.1519	6.4043	0.1458	11.4464	0.117	2.61316	0.17801	0.9468	0.10569	1.68475	0.16108	0.56554	0.09429	1.191	0.1503	0.8027	0.1519	6.4043	0.1458	11.4464	0.117	2.61316	0.17801	0.9468	0.10569	1.68475	0.16108	0.56554	0.09429	1.191	0.1503	0.8027	0.1519	6.4043	0.1458	11.4464	0.117	2.61316	0.17801	0.9468	0.10569	1.68475	0.16108	0.56554	0.09429	1.191	0.1503	0.8027	0.1519	6.4043	0.1458	11.4464	0.117	2.61316	0.17801	0.9468	0.10569			
0.64269	0.01964	0.23871	0.01013	0.2495	0.0419	0.1776	0.0573	5.5168	0.0146	10.7895	0.0189	1.66118	0.05019	0.48115	0.01968	0.64269	0.01964	0.23871	0.01013	0.2495	0.0419	0.1776	0.0573	5.5168	0.0146	10.7895	0.0189	1.66118	0.05019	0.48115	0.01968	0.64269	0.01964	0.23871	0.01013	0.2495	0.0419	0.1776	0.0573	5.5168	0.0146	10.7895	0.0189	1.66118	0.05019	0.48115	0.01968	0.64269	0.01964	0.23871	0.01013	0.2495	0.0419	0.1776	0.0573	5.5168	0.0146	10.7895	0.0189	1.66118	0.05019	0.48115	0.01968			
0.47531	0.01046	-0.0913	-0.0038	-0.0184	-0.0003	0.1458	0.0538	5.1949	-0.0048	10.7895	0.0189	1.40372	0.02739	0.28995	0.0076	0.47531	0.01046	-0.0913	-0.0038	-0.0184	-0.0003	0.1458	0.0538	5.1949	-0.0048	10.7895	0.0189	1.40372	0.02739	0.28995	0.0076	0.47531	0.01046	-0.0913	-0.0038	-0.0184	-0.0003	0.1458	0.0538	5.1949	-0.0048	10.7895	0.0189	1.40372	0.02739	0.28995	0.0076	0.47531	0.01046	-0.0913	-0.0038	-0.0184	-0.0003	0.1458	0.0538	5.1949	-0.0048	10.7895	0.0189	1.40372	0.02739	0.28995	0.0076			
1.54937	0.15834	0.58177	0.09218	1.2054	0.1568	0.7744	0.1475	2.5569	0.1451	8.523	0.1112	1.64065	0.16345	0.96234	0.10367	1.54937	0.15834	0.58177	0.09218	1.2054	0.1568	0.7744	0.1475	2.5569	0.1451	8.523	0.1112	1.64065	0.16345	0.96234	0.10367	1.54937	0.15834																																	