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Randomized Block Designs, Balanced Incomplete Block Designs and Latin Square Designs with Neighbor Effects in the Presence of Covariates

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Abstract

The research work undertaken in this paper is motivated by a real life scenario in the context of agricultural experiments. It is believed that the neighboring 'plots' in a Block Design or in a Latin Square Design [LSD] tend to influence each other in terms of the mean yield through the 'neighboring effects of the treatments' applied in these plots. Further to this, there are quantifiable and controllable covariates acting linearly in the mean model. We contemplate a linear ANCOVA model and study its analysis - with special emphasis on the question of estimability of the regression coefficient(s) involving the covariates. We focus on RBDs with $b = v = 4$, on an $SBIBD(7, 7, 4, 4, 2)$ and also on an LSD of order 4.

Key words: Randomized block designs; Balanced incomplete block designs; Latin square designs; Direct treatment effects; Neighbor treatment effects; Left neighbors; Right neighbors; Top neighbors; Bottom neighbors; Linear ANCOVA model; Covariates; Optimal covariate matrices.

1. Introduction

The key reference to this article is Springer Publication by Das *et al.* (2015) titled "Optimal Covariate Designs". Generally speaking, in the context of an experimental design with covariates, each experimental unit is supposed to have attached to it a number of quantifiable and measurable covariates. Assuming that there is a large pool of units, we have a choice for selection of the units with assigned covariate-values. Optimal covariate designs are the designs which provide optimal or most efficient estimation of the covariates' effects in terms of the parameters in an assumed linear model. The experimental set-up is quite general - starting with CRDs, RBDs, BIBDs, LSDs etc. The number of covariates need not be just one or two. Optimality problems center around characterization and constructions of designs *i.e.*, choice of experimental units with 'optimally assigned' covariate values in a given experimental set-up. The reader is referred to Das *et al.*(2015) for details. This area of research grew over the last 40 years or so.

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Covariate Models or ANCOVA Models are seen as a 'blend' of 'design model' and 'regression model'. In a block design set-up, writing y_{ij} for the observation in the experimental unit corresponding to i-th block and j-th treatment, we may write the model as

$$y_{ij} = \mu + \beta_i + \tau_j + \beta_1 x_{1;(i,j)} + \beta_2 x_{2;(i,j)} + \ldots + e_{ij},$$

where it is assumed that $x_{1;(i,j)}, x_{2;(i,j)}, \ldots$ are the covariate values attached to the unit labelled (i, j) with associated linear effects parameters $\beta_1, \beta_2, \ldots$. Our purpose is to identify and select those experimental units which collectively provide optimal estimation of the covariate parameters *i.e.*, of the β's. Note that the design set- up could be very much general in nature. However, unless there is a nice combinatorial structure of the underlying design [without the covariate effects], the problem, in its most general form, is untraceable. That is why only CRDs, RBDs, BIBDs, LSDs etc have been studied in the literature. The complexity of the problem cannot be undermined if there are a number of covariates. In general terms, for any number of covariates and any experimental design set- up, it transpires that $Var(\hat{\beta}) \geq \sigma^2 / \sum x^2_{(i,j)}$. It can be argued that, without any loss of generality, we may assume $-1 \leq x_{(i,j)} \leq 1$. This takes the variance bound to σ^2/n where n is the total number of observations. We need to examine the case of 'equality' and that too, for each of the covariates and there again, we need to attain 'equality' simultaneously for all the covariates parameters' estimates. Note that we are examining the status of a design only wrt the β-parameters, ignoring other fixed-effects parameters in the model. Anyway, there are too many issues involved and, without any further digression, we refer to Das *et al.* (2015).

Specifically, if we are dealing with an RBD involving b blocks and v treatments and if there are k covariates $(X_{(1)}, X_{(2)}, \ldots, X_{(k)})$, we will attain 'equality' in the variance bound simultaneously for all the covariates if and only if the following conditions are met :

$$(i) \sum_j x_{(u;(i,j))} = 0, \quad 1 \leq i \leq b; \quad (ii) \sum_i x_{(u;(i,j))} = 0, \quad 1 \leq j \leq v;$$

$$(iii) \sum_{1 \leq i \leq b} \sum_{1 \leq j \leq v} x_{u;(i,j)} x_{u^*;(i,j)} = nI(u, u^*); \quad 1 \leq u, u^* \leq k.$$

where, in the above, $I(..)$ is the usual indicator function and $n = bv$.

In this paper we will deal with an $RBD(b = 4, v = 4)$, a $BIBD(7, 7, 4, 4, 2)$ and an LSD of order 4. Moreover, we will adopt a model where, besides the block effects/row- column effects and treatment effects, we also have neighbor effects - designated as Left-Neighbor (LN)-Effects, Right-Neighbor (RN)-Effects etc. Naturally, we will require more conditions to be satisfied by the collection of the $x_{(u;i,j)}$'s. Note that (iii) requires that $x_{(u;(i,j))} = +1/-1$ for all choices of $(_{u;(i,j)})$'s. With this background, we will proceed to derive/present the results on optimal covariates designs in a model with N-Effects. In doing so, our target will be to cover maximum number of such covariates with most efficient estimation for each one. Once for all, we refer to systematic study of four- sided RN- and CN- effects as proposed and discussed in Varghese *et al.* (2014) for an explanation of neighbor effects. There are two follow- up papers in this direction as well. [Sapam *et al.* (2019a, 2019b)]. We may mention another related paper by Jaggi *et al.* (2018).

2. RBD with b=v=4

We start with the following RBD in Table 1 wherein we also display the Left-sided and Right-sided Neighbor Effects, assuming a circular model. [Vide Kunert (1984)].

Table 1: *RBD* with b=v=4: First Choice

LN 4	1	2	3	4	RN 1
LN 1	2	3	4	1	RN 2
LN 2	3	4	1	2	RN 3
LN 3	4	1	2	3	RN 4

We assume the existence of a controllable and quantifiable covariate (X) attached to every plot in the block design. We denote by x_{ij} the value of the covariate attached to the plot labelled (i, j) which corresponds to plot number i in block number j; $i, j = 1, 2, 3, 4$.

Without any loss of generality, we further assume that $-1 \leq x_{ij} \leq 1$ for each of the covariate values.

Under the assumed linear model, it follows that $I(\beta) \leq \sum\sum x_{ij}^2 \leq bv = 16$, dropping the error variance σ^2 in the model. The case of 'equality' has been studied earlier in our papers in easier settings. We refer to Das *et al.* (2015) for details. However, the present setting is a bit complicated since there are block effects, (direct) treatment effects and both LN- and RN- Effects of the treatments. Consider the following $X_{(1)}$-matrix in Table 2 for one choice of the covariate values.

Table 2: Covariate matrix for *RBD* with v=b=4 in Table 1

$$X_{(1)} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

It can be verified that this choice of the X-matrix provides equality in the above wrt information on β. As a matter of fact, the column vector of order 16×1 consisting of the covariate values is seen to be orthogonal to each of the $4+4+4+4 = 16$ vectors corresponding to 4 block effects parameters, 4 treatment effects parameters, 4 LN-Effects parameters and 4 RN-Effects parameters. It would be an interesting exercise to figure out how many such X-matrices can be made available which are (i) orthogonal to those listed in the above and (ii) themselves mutually orthogonal. Here are two others *i.e.*, $X_{(2)}$ and $X_{(3)}$ displayed in Table 3.

We now refer to Das *et al.* (2015) Monograph on 'Optimal Covariate Designs'. Specifically, subsection 3.2 lists 9 matrices, denoted as $W^{(1)}, W^{(2)}, \ldots, W^{(9)}$, in the context of an

Table 3: Covariate matrices for *RBD* with *v*=*b*=4 in Table 1

$X_{(2)}$ =

1	−1	−1	1
1	1	−1	−1
1	−1	−1	1
1	1	−1	−1

$X_{(3)}$=

1	1	−1	−1
1	−1	−1	1
1	1	−1	−1
1	−1	−1	1

RBD with parameters $b = v = 4$. It turns out that all these 9 matrices serve our purpose in the present context. As a matter of fact, we have already listed 3 of them $[W^1,\ W^2,\ W^3]$ in the above - suitably rewritten to fit in our framework as $X_{(1)}$, $X_{(2)}$, $X_{(3)}$ in Table 2 and Table 3. The rest are shown in the Appendix - A.

Remark 1: It must be noted that not all block design structures are amenable to this kind of allocation of covariate values with desirable orthogonality properties. Take, for example, the following RBD in Table 4 with associated LN- and RN-Effects shown along the margins. We may try to convert W into X-matrix, hoping that it would serve the purpose ! We show it below in Table 5.

Table 4: *RBD* with *b*=*v*=4: Second Choice

LN 4	1	2	3	4	RN 1
LN 3	1	2	4	3	RN 1
LN 4	2	1	3	4	RN 2
LN 3	2	1	4	3	RN 2

Table 5: Non-conformative Covariate Matrix for *RBD* with *b*=*v*=4: Second Choice

1	−1	1	−1
−1	1	1	−1
−1	1	1	−1
1	−1	1	−1

It turns out that (i) block total of x-values is zero for each block;
(ii) treatment total of x-values is zero for each treatment.
However, orthogonality fails wrt LN- and RN-Effects. The message is clear. We have to study the structure of allocation of the treatments in the RBD and proceed accordingly. For the RBD in Table 4, we are able to establish that there are at the most 4 X-matrices - satisfying the desirable properties. We provide a proof of this statement as also display all the available X-matrices in the Appendix - B.

Remark 2: Every layout of an $RBD(b = v = 4)$ is special and has to be dealt with due attention to its structure. Here we have one more in the 'affirmative' sense displayed in Table 6. It has at least one underlying X-matrix and we display one X-matrix in Table 7.

Table 6: Covariate Matrix for *RBD* with *b*=*v*=4: Third Choice

LN 4	1	2	3	4	RN 1
LN 3	2	1	4	3	RN 2
LN 2	3	4	1	2	RN 3
LN 1	4	3	2	1	RN 4

Table 7: *RBD* with *b*=*v*=4: Third Choice

−1	1	−1	1
−1	1	−1	1
−1	1	−1	1
−1	1	−1	1

Remark 3: It is tempting to conjecture that for any given layout of an RBD, there is at least one X-matrix available satisfying all the properties stipulated.

3. BIBD with b=v=7, r=k=4, λ=2

We borrow necessary results from Das *et al.* (2015), Chapter 4, Subsection 4.2. We take up the $SBIBD(7,7,4,4,2)$ and display the incidence matrix in a slightly modified form below in Table 8. We also show the LN- and RN-Effects in the same table, assuming a circular model. We now display the X-matrix of $((+1,-1))$'s in Table 9.

Table 8: *SBIBD* (7, 7, 4, 4, 2)

LN 7	1	4	6	7	RN 1
LN 7	1	2	5	7	RN 1
LN 2	1	6	3	2	RN 1
LN 7	2	3	4	7	RN 2
LN 3	1	5	4	3	RN 1
LN 6	2	4	5	6	RN 2
LN 7	3	5	6	7	RN 3

It is readily verified that this X-matrix is one desired solution to provide most efficient estimation of the β-coefficient even in the presence of LN- and RN-effects of the treatments. As a ready reckoner, we display below in Table 10, the LN-and RN-effects of the treatments, assuming a circular model. Note that the positions of the treatments within the blocks are important for assessing the properties of the X-matrix. It would be interesting to investigate if there are other such X-matrices and mutually orthogonal to the one just found.

Table 9: X-matrix for *SBIBD* (7,7,4,4,2)

1	0	0	−1	0	1	−1
−1	1	0	0	−1	0	1
1	−1	1	0	0	−1	0
0	1	−1	1	0	0	−1
−1	0	1	−1	1	0	0
0	−1	0	1	−1	1	0
0	0	−1	0	1	−1	1

Table 10: LN- and RN-Effects under a circular model

LN Tr 1	Coeff	RN Tr 1	Coeff.
4	−1	7	−1
2	1	7	1
6	−1	2	−1
5	1	3	1
*	*	*	*
LN Tr 2	Coeff	RN Tr 2	Coeff.
5	−1	1	−1
1	1	3	1
3	−1	7	−1
4	1	6	1
*	*	*	*
LN Tr 3	Coeff	RN Tr 3	Coeff.
2	−1	6	−1
4	1	2	1
1	−1	4	−1
5	1	7	1
*	*	*	*

LN Tr 4	Coeff	RN Tr 4	Coeff.
6	1	1	1
7	−1	3	−1
3	1	5	1
5	−1	2	−1
*	*	*	*
LN Tr 5	Coeff	RN Tr 5	Coeff.
7	1	2	1
4	−1	1	−1
6	1	4	1
6	−1	3	−1
*	*	*	*
LN Tr 6	Coeff	RN Tr 6	Coeff.
7	−1	4	−1
3	1	3	1
2	−1	5	−1
7	1	5	1
*	*	*	*
LN Tr 7	Coeff	RN Tr 7	Coeff.
6	1	6	1
5	−1	5	−1
2	1	4	1
3	−1	6	−1
*	*	*	*

4. Latin Square Design of Order 4

So far we have developed study of RBDs and BIBDs with covariates and in the presence of neighbor- effects. Now we focus on an LSD of order 4. We refer to Das *et al.* (2015), pages 155 - 159. In Example 8.2.3 (page 155), an LSD of order 4 has been laid out. We reproduce it here in Table 11 along with all the four-sided neighbor-effects : Left-sided Neighbor Effects (LN), Right-sided Neighbor Effects (RN), Top-sided Neighbor Effects (TN) and Down-sided Neighbor Effects (DN). We assume a circular model - covering all sides.

Since in an LSD of order 4, there are six (6) orthogonal linear error functions (*i.e.*, 6 error df), in the Example 8.2.3, six (6) orthogonal X-matrices have been shown. Vide the

Table 11: *LSD* of order 4 with 4-sided NEs

TN	4	3	2	1	Effects
LN-Effects					RN-Effects
4	1	2	3	4	1
3	2	1	4	3	2
2	3	4	1	2	3
1	4	3	2	1	4
DN	1	2	3	4	Effects

bottom part of the matrix shown in the expression for E^{LSD}. These represent optimal choices of six orthogonal covariate matrices for estimation of the same number of beta-coefficients. This, however, holds without the presence of any sort of neighbor effects. While we introduce the N-Effects on all sides (*i.e.*, in all directions), it follows that only four (4) of them are valid X-matrices. These are the 2nd, 4th, 5th and 6th X-matrices in the bottom part of the table for E^{LSD}. These are reproduced below for the sake of completeness in Table 12. Moreover, as in the case of the RBD in Table 4, we prove that for the LSD under consideration, there exist only 4 distinct and mutually orthogonal X-matrices as are found out and displayed in Table 12. This is taken up in Appendix - C.

Table 12: Optimal X-matrices

X(1)	1	−1	−1	1;	−1	1	1	−1;	1	−1	−1	1;	−1	1	1	−1
X(2)	1	−1	1	−1;	−1	1	−1	1;	−1	1	−1	1;	1	−1	1	−1
X(3)	1	−1	−1	1;	1	−1	−1	1;	−1	1	1	−1;	−1	1	1	−1
X(4)	1	1	−1	−1;	−1	−1	1	1;	1	1	−1	−1;	−1	−1	1	1

Remark 4: We must note that the choice of the specific form of the LSD is very crucial for existence of such X-matrices. For example, if we adopt the LSD shown in Table 13 [reproduced as L_2 on Page 29 of Das *et al.* (2015)], then we can find one X-matrix comfortably and it is shown in Table 14. However, our attempt to find one more did not succeed.

Remark 5: Even though we are discussing about LSDs of order 4, very general treatments of row-column designs are available in the literature. Vide, for example, Shah and Sinha (1996). The reader might like to study such general patterns in the light of Neighbor- Effects and covariates.

Table 13: *LSD* of order 4 from Das *et al.* (2015) Page 29 L_2

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

Table 14: X-matrix for *LSD* in Table 13

1	−1	−1	1
1	−1	−1	1
−1	1	1	−1
−1	1	1	−1

5. Conclusion

In this paper we have examined the existence of 'optimal covariates designs' in the presence of neighbor-effects. The designs considered are (i) $RBD(b = v = 4)$, (ii) $BIBD(b = v = 7, r = k = 4, \lambda = 2)$ and (iii) $LSD\ of\ Order\ 4$. The model adopted is linear in the general mean, block - effects / row-column effects, treatment effects and circularly located neighbor- efffects. The presence of covariates makes the analysis complicated unless their effects are optimally and orthogonally estimated. This study shows that at times we are in a position to achieve this by suitably allocating the covariates values in the experimental units. Even though the experimental set-ups are simple, the results are non-trivial and worth noting.

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References

Das, P., Dutta, G., Mandal, N. K. and Sinha, Bikas K. (2015). *Optimal Covariate Designs.* Springer-Verlag Text Book Series.

Jaggi, S., Pateria, D. K., Varghese, C., Varghese, E. and Bhowmik A. (2018). A note on circular neighbor balanced designs. *Communication in Statistics- Simulation and Computation*, **47(10)**, 2896-2905.

Kunert, J. (1984). Optimality of balanced uniform repeated measurements designs. *Annals of Statistics*, **12(3)**, 1006 - 1017.

Sapam, Sobita, Mandal, Nripes K. and Sinha, Bikas K. (2019a). Latin Square designs with neighbour effects. *The Journal of Indian Society of Agricultural Statistics*, **73(2)**, 91-98.

Sapam, Sobita, Mandal, Nripes K. and Sinha, Bikas K. (2019b). Latin Square designs with neighbor effects - part II, *Communications in Statistics - Theory and Methods.* Published online 2019. https://doi.org/10.1080/03610926.2019.1702694.

Shah, K. R. and Sinha, Bikas K. (1996). Row-column designs. In *Design and Analysis of Experiments*. 903-937, *Handbook of Statistics*, **Vol. 13**., North-Holland, Amsterdam.

Varghese, E., Jaggi, S. and Varghese, C. (2014). Neighbor- Balanced Row Column Designs. *Communications in Statistics - Theory and Methods*, **43(6)**, 1261-1276.

APPENDIX

A.I : Choice of Six Additional and Mutually Orthogonal Optimal Covariate Matrices for the RBD(b=v=4) in Table 1

Table 15: Covariate matrices for *RBD* with v=b=4 in Table 1

$X_{(4)}=$

1	−1	1	−1
−1	1	−1	1
−1	1	−1	1
1	−1	1	−1

$X_{(5)}=$

1	−1	−1	1
1	1	−1	−1
−1	1	1	−1
−1	−1	1	1

$X_{(6)}=$

1	1	−1	−1
−1	1	1	−1
1	1	−1	−1
−1	1	1	−1

$X_{(7)}=$

1	−1	1	−1
1	−1	1	−1
−1	1	−1	1
−1	1	−1	1

$X_{(8)}=$

1	−1	−1	1
−1	−1	1	1
1	−1	−1	1
−1	−1	1	1

$X_{(9)}=$

1	1	−1	−1
−1	1	1	−1
−1	−1	1	1
1	−1	−1	1

A.II : Verification of Orthogonality wrt LN- and RN-effects of each of the treatments

We take up the verification wrt $X_{(1)}$ below in Table 16.

The nature of incidence of the treatments as LN- and RN-Effects is very special. That is clearly visible in Table 16. The conditions relating to orthogonality wrt these N-Effects are the same as orthogonality wrt (direct) treatment effects which is true. Therefore, all the X-matrices satisfy the stipulated conditions of orthogonality.

Table 16: Coefficients of LN- and RN- Effects for *RBD* in Table 1 corresponding to the covariate matrix $X_{(1)}$

Blocks	Tr. 1 as LNE	as LNE coeff	Tr. 1 as RNE	as RNE coeff
1	2	−1	4	−1
2	2	1	4	1
3	2	−1	4	−1
4	2	1	4	1
Total		0		0
Blocks	Tr. 2 as LNE	as LNE coeff	Tr. 2 RNE	as RNE coeff
1	3	1	1	1
2	3	−1	1	−1
3	3	1	1	1
4	3	−1	1	−1
Total		0		0
Blocks	Tr. 3 as LNE	as LNE coeff	Tr. 3 RNE	as RNE coeff
1	4	−1	2	−1
2	4	1	2	1
3	4	−1	2	−1
4	4	1	2	1
Total		0		0
Blocks	Tr. 4 as LNE	as LNE coeff	Tr. 4 RNE	as RNE coeff
1	1	1	3	1
2	1	−1	3	−1
3	1	1	3	1
4	1	−1	3	−1
Total		0		0

B. X- matrices for RBD : Second Choice

We have displayed four mutually orthogonal covariate matrices for the $RBD(b = v = 4)$: *Second Choice* in the Table 17. We now establish that no further X-matrices exist in this context. Let us start with a general form of an X-matrix given in Table 18.

Table 17: Four covariate matrices for *RBD* v=b=4: Second Choice

$X_{(1)}$=

1	−1	1	−1
1	−1	1	−1
1	−1	1	−1
1	−1	1	−1

$X_{(2)}$=

1	−1	1	−1
−1	1	−1	1
−1	1	−1	1
1	−1	1	−1

$X_{(3)}$=

1	−1	−1	1
−1	1	1	−1
−1	1	1	−1
1	−1	−1	1

$X_{(4)}$=

1	1	−1	−1
−1	−1	1	1
−1	−1	1	1
1	1	−1	−1

We realize that there are too many restrictions on the elements of X. It may be noted that WOLG, we may assume $a = 1$. The restrictions are listed below in Table 19. By examining

Table 18: General form of a covariate matrix X for *RBD* (*v*=*b*=4): Second Choice

a	b	c	d
e	f	g	h
i	j	k	l
m	n	o	p

the triplet (b, e, f) and all the $2^3 = 8$ combinations along with $a = 1$, we can argue that the following are the only feasible combinations in this context.
$(a, b, e, f) = (1, 1, -1, -1), X_{(4)} Matrix;$
$(a, b, e, f) = (1, -1, 1, -1), X_{(1)} Matrix;$
$(a, b, e, f) = (1, -1, -1, 1), X_{(2)}$ *and* $X_{(3)}$ *Matrices.*
Hence the stated claim is established.

Table 19: Restrictions on the elements of X

Sl. No.	Restriction
Tr1	a+e+j+n = 0
Tr2	b+f+i+m = 0
Tr3	c+h+k+p = 0
Tr4	d+g+l+o = 0
Bl1	a+b+c+d = 0
Bl2	e+f+g+h = 0
Bl3	i+j+k+l = 0
Bl4	m+n+o+p = 0
LN1	b+f+k+o = 0
LN2	c+g+j+h = 0
LN3	d+e+l+m = 0
LN4	a+h+i+p = 0
RN1	d+h+i+m = 0
RN2	a+e+l+p = 0
RN3	b+g+j+o=0
RN4	c+f+k+n=0

C : Existence of four mutually orthogonal X-matrices for the LSD in Table 11

We refer to Table 11 for the particular LSD of order 4 and also to Table 18 for a general structure of an X-matrix. We now incorporate the conditions for optimality.

(a) Consideration of Treatment Effects :

$$a + f + k + p = 0 \quad (1); \quad b + e + l + o = 0 \quad (2);$$
$$c + h + i + n = 0 \quad (3); \quad d + g + j + m = 0 \quad (4).$$

(b) Consideration of Row Effects :

$$a + b + c + d = 0 \quad (5); \quad e + f + g + h = 0 \quad (6);$$

$$i + j + k + l = 0 \quad (7); \quad m + n + o + p = 0 \quad (8).$$

(c) Consideration of Column Effects :

$$a + e + i + m = 0 \quad (9); \quad b + f + j + n = 0 \quad (10);$$

$$c + g + k + o = 0 \quad (11); \quad d + h + l + p = 0 \quad (12).$$

(d) Consideration of Left-Neighbor Effects :

$$b + g + l + m = 0 \quad (13); \quad c + f + i + p = 0 \quad (14);$$

$$d + e + j + o = 0 \quad (15); \quad a + h + k + n = 0 \quad (16).$$

(e) Consideration of Right-Neighbor Effects :

$$b + g + l + m = 0 \quad (17); \quad c + f + i + p = 0 \quad (18);$$

$$e + j + o + d = 0 \quad (19); \quad a + h + k + n = 0 \quad (20).$$

(f) Consideration of Top-Neighbor Effects

$$b + g + l + m = 0 \quad (21); \quad a + h + k + n = 0 \quad (22);$$

$$e + j + o + d = 0 \quad (23); \quad c + f + i + p = 0 \quad (24).$$

(g) Consideration on Down-Neighbor Effects

$$e + j + o + d = 0 \quad (25); \quad c + f + i + p = 0 \quad (26);$$

$$b + g + l + m = 0 \quad (27); \quad a + h + k + n = 0 \quad (28).$$

From the above, we find that the 4 equation sets, *viz.*, those arising out of LN-sum, RN-sum, TN-sum and DN-sum, each of 4 equations, are the same. So, we consider only the 4 equation sets, *viz.*, those arising from Treatment- sum, Row- sum, Column- sum and LN-sum. If there exists a solution of these equations with solution space $[1, -1]$, an X-matrix exists. As in the case of RBD set-up, WOLG, we set $a = 1$ and examine all the 8 combinations corresponding to choices of (b, e, f). The results are stated below.

Case 1. $b = e = f = 1$: *no solution*;
Case 2. $b = -1, \quad e = f = 1$: *no solution*;
Case 3. $b = f = -1, e = 1$: *one solution viz.,* $X(1)$;
Case 4. $e = -1, b = f = 1$: *no solution*;
Case 5. $f = -1, b = e = 1$: *no solution*;
Case 6. $b = e = -1, f = 1$: $X(3)$ *and* $X(2)$ *are the two solutions*;
Case 7. $e = f = -1, b = 1$: *one solution viz.,* $X(4)$;
Case 8. $b = e = f = -1$: *no solution.*

Hence the claim is justified.