New Plans Orthogonal Through the Block Factor

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Abstract

In the present paper we construct plans orthogonal through the block factor (POTBs). We describe procedures for adding blocks as well as factors to an initial plan and thus generate a bigger plan. Using these procedures we construct POTBs for symmetrical experiments with factors having three or more levels. We also construct plans inter-class orthogonal through the block factor for two-level as well as higher level factors.

Key words: Factorial experiments; Main effects; Orthogonal; Block factor.

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1. Introduction

A situation in which a treatment factor is neither orthogonal nor confounded to a nuisance factor was first explored in Morgan and Uddin (1996) in the context of nested row-column designs. They derived a sufficient condition for a treatment factor, possibly non-orthogonal to the nuisance factors, to be orthogonal to another treatment factor. They also derived a sufficient condition for optimality and constructed several series of orthogonal main effect plans (OMEPs) satisfying optimality properties. Mukherjee, Dey and Chatterjee (2002) discussed and constructed main effect plans (MEPs) on small-sized blocks, not necessarily orthogonal to all treatment factors. Their plans also satisfy optimality properties. Optimal blocked MEPs of similar type are also constructed in Das and Dey (2004).

Bose and Bagchi (2007) provided plans satisfying properties similar to those of the plans of Mukherjee, Dey and Chatterjee (2002), but requiring fewer blocks. In Bagchi (2010) the concept of orthogonality through the block factor [see Definition 2.2] is introduced. In that paper it has been shown that a plan orthogonal through the block factor (POTB) may exist in a set up, where an OMEP can not exist. Making use of the Hadamard matrices in various way, Jacroux and his co-authors (2011, 2017) have come up with a number of such plans, mostly for two-level factors, many of them satisfying optimality properties. POTBs are also constructed in Saharay and Dutta (2016).

Wang (2004) constructed plans for two-level factors on blocks of size two, estimating interaction effects also. A general method of estimating interaction effects from plans on blocks of small size is provided in Bagchi (2020). The method is applied to plans for two and three-level factors estimating two-factor interactions.

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Preece (1966) constructed 'BIBDs for two sets of treatments'. Subsequently several authors constructed similar combinatorial objects. Among these, the ones relevant to the present paper are 'balanced Graeco-Latin block designs' of Seberry (1979), 'Graeco-Latin designs of type 1' of Street (1981) and 'Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs)' of Rees and Preece (1999). We note that all these combinatorial designs are, in fact, two-factor POTBs satifying certain additional properties. We discuss these interesting combinatorial designs briefly in Section 3.

In the present paper our main objective is to provide plans in those set ups where no OMEP is available, accommodating as many factors as possible and deviating "as little as possible" from orthogonality. We construct a few series of POTBs for symmetrical experiment with factors having three or more levels. We also define plans inter-class orthogonal through the block factor (PIOTBs) [see Definition 6.1] and construct a series of such plans.

In Section 2 we present the definition of a POTB along with its attractive features. The later sections are devoted to construction. In Section 3 we obtain a few infinite series of POTBs for symmetric experiments with four or less factors, each with five or more levels [see Theorems 3.1, 3.2 and 3.3]. In Section 4 we describe methods of recursive construction. In Section 5 we use these methods and construct two series of POTBs for three-level factors on blocks of size four [see Theorems 5.1 and 5.3]. Finally, in Section 6 we construct two infinite series of PIOTBs with orthogonal classes of small size, one for factors with level ≥ 5 [see Theorem 6.1] and another for two-level factors [see Theorem 6.3]. Many of the plans constructed here are saturated. Moreover, many satisfy optimality properties.

2.   Preliminaries

We shall consider main effect plans for a symmetrical experiment with m factors, laid out on blocks of constant size.

Notation 2.1: (0) m and n will denote integers. $M = \{1, 2, \cdots m\}$. Further, $\bar{i} = M \backslash \{i\}$, $i \in M$.

(1) $\mathcal{P}$ will denote a main effect plan for a s^m experiment consisting of b blocks each of size k. n will denote the total number of runs. Thus, $n = bk$.

(2) s will denote an integer and S the set of integers modulo s. The set of levels for each factor is S unless stated otherwise. S^m will denote the following set of $m \times 1$ vectors. $S^m = \{(x_1, \cdots x_m)' : x_i \in S\}$.

(3) A_i denotes the ith factor, $i = 1, 2, \cdots m$. The vector $x = (x_1, x_2, \cdots x_m)' \in S^m$ represents a level combination or run, in which A_i is at level $x_i, i = 1, 2, \cdots m$.

(4) $\mathcal{B} = \{B_j, j = 1, \cdots b\}$ will denote the set of all blocks of $\mathcal{P}_0$, Thus, $B_j \subset S^m, |B_j| = k$, $1 \leq j \leq b$. Sometimes we describe a plan in terms of its blocks.

(5) The replication vector of A_i is denoted by the $s \times 1$ vector r_i, the pth entry of which is the number of runs x of $\mathcal{P}$ such that $x_i = p, p \in S$. R_i denotes a diagonal matrix with diagonal entries same as those of r_i in the same order, $1 \leq i \leq m$.

(6) For $1 \leq i, j \leq m$, the A_i versus A_j incidence matrix is the $s_i \times s_j$ matrix N_{ij}.

The (p, q)th entry of this matrix is $N^{ij}(p, q)$, which is the number of runs x of $\mathcal{P}$ such that $x_i = p$, $x_j = q$, $p \in S_i$, $q \in S_j$. When $j = i$, $N_{ij} = R_i$.

(7) Fix $i \in M$.

(a) L_i will denote the A_i-versus block incidence matrix. Thus, the (p, j)th entry of the L_i is

$$L^i(p, j) = |x \in B_j : x_i = p|, \ p \in S, 1 \le j \le b.$$

(b) The $s \times 1$ vector α^i will denote the vector of unknown effects of A_i. The vector of raw totals for A_i will be denoted by T_i. B will denote the vector of block totals.

(c) We define the following vectors and matrices in terms of $\bar{i}$ [recall (0)]. The vector $\mathbf{U}_{\bar{i}}$ is of order $(m-1)s \times 1$, the matrices $\mathbf{V}_{\bar{i}}$ and $W_{\bar{i}}$ are of orders $s \times (m-1)s$ and $(m-1)s \times (m-1)s$ respectively

$$\mathbf{U}_{\bar{i}} = ((T_j))_{j \in \bar{i}}, \tag{1}$$
$$\mathbf{V}_{\bar{i}} = ((N_{ij}))_{j \in \bar{i}}, \tag{2}$$
$$W_{\bar{i}} = ((N_{pq}))_{p,q \in \bar{i}}. \tag{3}$$

Consider the normal equations for a plan $\mathcal{P}$ as described above. If we eliminate the general effects and the vector of block effects from this system of equations, we get the reduced normal equation for the vectors of all (unknown) effects of all the treatment factors. This is a system of ms equations, but it is convenient to view it as m systems of s equations each, the ith system equations is of the form

$$\sum_{j=1}^{m} C_{ij;B} \widehat{\alpha^j} = \mathbf{Q}_{i;B}. \tag{4}$$

Here $C_{ij;B}$, $1 \le j \le m$ are the coefficient matrices and $\mathbf{Q}_{i;B}$ is the vector of adjusted (for the blocks) totals for A_i. The expressions of them are as follows.

$$C_{ij;B} = N_{ij} - (1/k)L_i L_j' \text{ and } \mathbf{Q}_{i;B} = T_i - (1/k)L_i B. \tag{5}$$

For a fixed i, we can eliminate $\widehat{\alpha^j}, j \in \bar{i}$ from (4) and get

the reduced normal equation for $\widehat{\alpha^i}$ as $C_{i;\bar{i}} \widehat{\alpha^i} = \mathbf{Q}_{i;\bar{i}}, \tag{6}$

where $C_{i;\bar{i}} = R_i - \mathbf{V}_{\bar{i}} W_{\bar{i}}^- \mathbf{V}_{\bar{i}}'$ and $\mathbf{Q}_{i;\bar{i}} = T_i - \mathbf{V}_{\bar{i}} W_{\bar{i}}^- \mathbf{U}_{\bar{i}}'$, U_i, V_i and W_i are as in (1) and the equations next.

With this background we present a few definitions.

Definition 2.1: An m-factor MEP is said to be 'connected' if $Rank(C_{i;\bar{i}}) = s - 1$, for every $i = 1, 2, \cdots m$.

Definition 2.2: [Bagchi (2010)] Fix $i \ne j$ $1 \le i, j \le m$. The factors A_i and A_j are said to be **orthogonal through the block factor (OTB)** if

$$kN_{ij} = L_i(L_j)'. \tag{7}$$

We denote this by $A_i \perp_{bl} A_j$.

A plan $\mathcal{P}$ is said to be a **plan orthogonal through the block factor (POTB)** if $A_i \perp_{bl} A_j$ for every pair $(i,j), i \neq j, i, j = 1, \cdots m$.

Remark 2.1: Condition (7) is equivalent to equation (7) of Morgan and Uddin (1996) in the context of nested row-column designs.

Let us try to see the implications of orthogonality through the block factor. Let $SS_{i;all}$ (respectively $SS_{i;B}$) denote sum of squares for A_i, adjusted for all other factors (respectively the block factor). The following results are known.

Theorem 2.1: Consider a plan $\mathcal{P}$. Fix $i \in \{1, \cdots m\}$.

(a) [Bagchi(2010)] If for $j \neq i$ $A_i \perp_{bl} A_j$, then

(i) $C_{ij;B} = 0$ and (ii) $Cov(l'\widehat{\alpha^i}, m'\widehat{\alpha^j}) = 0$, for $l'1_s = 0 = m'1_s$.

(b)[Bagchi (2020)] Further, $A_i \perp_{bl} A_j$, $\forall j \neq i$ is necessary and sufficient for the following.

(i) $C_{i;\bar{i}} = C_{ii;B}$ and (ii) $SS_{i;all} = SS_{i;B}$ with probability 1.

Discussion: Theorem 2.1 says the following about the inference on the factors of a connected main effect plan. The inference on a factor A_i depends only on the relationship between A_i and the block factor if and only if A_i is orthogonal to every other treatment factor through the block factor. Moreover, the data analysis of a POTB is very similar to the data analysis of a block design with s treatments.

It is well-known that the orthogonal MEP obtained from an orthogonal array is the best possible MEP in the sense that the estimates have the maximum precision among all MEPs in the same set up. The same cannot be said about an POTB since its performance also depends on the relationships of the treatment factors with the block factor. In the next theorem a guideline for the search for a 'good' POTB is provided. We omit the proof which can be obtained by going along the same lines as in the proofs of Lemma 1 and Theorem 1 of Mukherjee, Dey and Chatterjee (2002). [See Shah and Sinha (1989) for definitions, results and other details about standard optimality criteria]

Theorem 2.2: Suppose a connected POTB $\mathcal{P}^*$ satisfies the following condition. For a factor A_i and a non-increasing optimality criterion ϕ, L_i is the incidence matrix of a block design d which is ϕ-optimal in a certain class of connected block designs with s treatments and b blocks of size k each. Then, $\mathcal{P}^*$ is ϕ-optimal in a similar class of connected m-factor MEPs in the same set-up as $\mathcal{P}^*$ for the inference on A_i.

In particular, using the well-known optimality result of Kiefer (1975) we get the following result.

Corollary 2.1: Suppose $\mathcal{P}^*$ is a connected POTB. Fix $i \in \{1, \cdots m\}$.

If L_i is the incidence matrix of a BIBD, then, for the inference on A_i, $\mathcal{P}^*$ is universally optimal in the class of all m-factor connected MEP containing $\mathcal{P}^*$.

In view of the above result, we introduce the following term.

Definition 2.3: A connected POTB is said to be **balanced** if **each of its factors form a BIBD with the block factor**, that is L_i is the incidence matrix of a BIBD for each i, $1 \leq i \leq m$

We now present a small example of a balanced POTB on six blocks of size two each. It has two factors, each with four levels 0,1,2,3.

Example 1 (Bagchi and Bose, 2007) :

Blocks $\rightarrow$		B_1		B_2		B_3		B_4		B_5		B_6	
Factors ↓	A_1	0	2	1	3	0	3	1	2	0	1	3	2
	A_2	1	3	0	2	2	1	3	0	3	2	0	1

3.   Construction of Plans with a Small Number of Factors

We shall now proceed to construct POTBs for a symmetric experiment. Most of the constructions are of recursive type, in the sensethat from a given initial plan we generate a plan by adding blocks and/or factors. Before going to the actual constructions, we present a small POTB. This is not connected and therefore has no use in practice. However, its property will be used in proving the results below.

Lemma 3.1: Let $T = \{a, b, c, d\} \subset S$. Consider the following plan for a 4^2 experiment consisting of two blocks, in which the set of levels of A_1 is T, while that of A_2 is $-T$. Then, $A_1 \perp_{bl} A_2$.

Blocks $\rightarrow$		B_1		B_2	
Factors ↓	A_1	a	c	b	d
	A_2	-d	-b	-a	-c

.

Proof is by direct verification of (7).

Definition 3.1: Consider an initial plan $\mathcal{P}_0$ for an s^m experiment as described in Notation 2.1. For $B \in \mathcal{B}$ and $v \in S^m$, $B+v$ will denote the following set of k runs. $B+v = \{x+v, \ x \in B\}$. Here $x + v = [x_i + v_i : 1 \leq i \leq m]'$, where the addition in each co-ordinate is modulo s.

By the plan **generated from $\mathcal{P}_0$ by adding** S we shall mean the plan (for the same experiment) having the set of blocks $\{B + u1_m : u \in S, B \in \mathcal{B}\}$. The new plan $\mathcal{P}$ will be denoted by $\mathcal{P}_0 \oplus S$.

We present an useful result. The simple proof is omitted.

Lemma 3.2: Consider a t^2 experiment, say E, such that the set of levels of A_i is T_i, which is a subset of S and $|T_i| = t, i = 1, 2$. If there is a plan $\mathcal{P}_0$ for E in which $A_1 \perp_{bl} A_2$, then $A_1 \perp_{bl} A_2$ in $\mathcal{P} = \mathcal{P}_0 \oplus S$ too ; although in $\mathcal{P}$ the set of levels of each factor is S. $\mathcal{P}$ is connected, even if $\mathcal{P}_0$ is not so.

We shall now proceed to construction. We begin with plans with a small set of factors. Let S^+ denote $S \cup \{\infty\}$. The following rule will define addition in S^+.

$$u + \infty = \infty = \infty + u, \ u \in S. \tag{8}$$

Theorem 3.1: Suppose s is an integer ≥ 5. Then connected POTBs with block size two exists for the following experiments.

(a) For an s^2 experiment a POTB $\mathcal{P}$ on $2s$ blocks exists. In the case $s = 5$, $\mathcal{P}$ is balanced.

(b) For an s^4 experiment a POTB on $4s$ blocks exists, whenever $s \geq 9$. If $s = 9$, $\mathcal{P}_2$ is balanced.

(c) A POTB $\mathcal{P}$ for a $(s+1)^4$ experiment with $6s$ blocks exists, whenever $n \geq 7$.

Proof: In each case, we present the blocks, of an initial plan $\mathcal{P}_0$ The required plan is $\mathcal{P}_0 \oplus S$ [see Definition 3.1]. Here a, b, c, d are distinct members of $S \setminus \{0\}$.

(a) The blocks of $\mathcal{P}_0$ are given below.

Blocks $\rightarrow$		B_1		B_2	
Factors $\downarrow$ A_1	a	-a	b	-b
A_2	b	-b	-a	a

We apply Lemma 3.1 with $c = -a, d = -b$ and see that $A_1 \perp_{bl} A_2$ in $\mathcal{P}_0$. Then we apply Lemma 3.2 and get the result.

If $s = 5$, taking $a = 1, b = 2$ we see that $L_1 = L_2$, which is the incidence matrix of the BIBD with parameters $v = 5, b = 10, r = 4, k = 2, \lambda = 1$. Hence the result.

(b) The blocks B_l, $l = 1, \cdots 4$ of $\mathcal{P}_0$ are as follows.

Blocks $\rightarrow$		B_1		B_2		B_3		B_4	
Factors $\downarrow$ A_1	a	-a	b	-b	c	-c	-d	d
A_2	b	-b	-a	a	-d	d	-c	c
A_3	c	-c	d	-d	-a	a	b	-b
A_4	d	-d	-c	c	b	-b	a	-a

To show that the generated plan is a POTB, we proceed as follows. We consider the following subplans.

Let $\mathcal{P}_{ij} = \{B_i, B_j\}$ $i \neq j, i, j = 1, 2, 3, 4$. Further, let $\bar{\mathcal{P}}_{ij}$ denote the plan $\mathcal{P}_{kl}$, where $\{k, l\} = \{1, 2, 3, 4\} \setminus \{i, j\}$. [Thus, $\bar{\mathcal{P}}_{34} = \mathcal{P}_{12}$ and so on].

Applying Lemma 3.1 on these subplans we see that $A_i \perp_{bl} A_j$ in $\mathcal{P}_{ij}$ as well as in $\bar{\mathcal{P}}_{ij}$. Now the result follows from Lemma 3.2.

By taking $a = 1, b = 2, c = 3$ and $d = 4$ in the case $s = 9$, we get L_i is the incidence matrix of the BIBD with parameters $v = 9, b = 36, r = 8, k = 2, \lambda = 1$. Hence the result.

(c) The set of levels for each factor is S^+. The blocks B_l, $l = 1, \cdots 6$ of the initial plan are as follows.

Blocks $\rightarrow$		B_1		B_2		B_3		B_4		B_5		B_6	
Factors $\downarrow$	A_1	0	∞	a	-a	b	-b	c	-c	a	-a	a	-a
	A_2	a	-a	0	∞	c	-c	-b	b	a	-a	-a	a
	A_3	b	-b	c	-c	0	∞	a	-a	-c	c	-c	c
	A_4	c	-c	-b	b	a	-a	0	∞	-c	c	c	-c

. $\square$

By using arguments similar to that in Case (b), together with the relation (8) we get the result. $\square$

We now list a few combinatorial structures in the literature which are actually balanced POTBs (for symmetrical or asymmetrical experiments).

(a) **Balanced Graco-Latin block design** defined and constructed in Seberry (1979) heve two factors.

(b) **Graco-Latin block design of type 1** of Street (1981) are also two-factor balanced POTBs satisfying $\mathbf{N}_{12} = J$.

(c) **Perfect Graeco-Latin balanced incomplete block designs (PERGOLAs)** defined and discussed extensively in Rees and Preece (1999) are two-factor balanced POTBs satisfying

$$\mathbf{N}_{12}\mathbf{N}'_{12} = \mathbf{N}'_{12}\mathbf{N}_{12} = fI_s + gJ_s, \quad \text{where f, g are integers.} \quad (9)$$

Here I_n is the identity matrix and J_n is the all-one matrix of order n.

(d) **Mutually orthogonal BIBDs** defined and constructed by Morgan and Uddin (1996) are multi-factor balanced POTBs.

Remark 3.1: The definition of neither balanced Graco-Latin block designs nor of mutually orthogonal BIBDs include condition (9). However, it is interesting to note that all these designs constructed so far do satisfy this condition. One would, therefore, suspect that this condition is implicit in the definition. We have, however, found a balanced POTB which does not satisfy this condition, as is shown below [see Theorem 3.2 (b)].

For the next construction we need a few notations. Recall that a multiset is a set in which the number of occurrences of an element is counted.

Notation 3.1: (i) $\bigsqcup$ denotes an union counting multiplicity.

(ii) For a set A and an integer n, nA denotes the multiset in which every member of A occurs n times.

(iii) For subsets A and B of a group $(G, +)$, A_B will denote the following multiset.

$$A - B = \{a - b : a \in A, b \in B\}.$$

Notation 3.2: Consider a plan $\mathcal{P}_0$ for an S^2 experiment as in Notation 2.1.

(a) Consider the jth block B_j of $\mathcal{P}_0$. $(x_{jt}, y_{jt})'$ will denote the tth run of B_j, $j = 1, \cdots b$. $X_j = \{x_{jt}, t = 1, \cdots k\}$ (respectively $Y_j = \{y_{jt}, t = 1, \cdots k\}$) is the set of levels of A_1 (respectively A_2) appearing in B_j.

(b) We define the following multisets.

$$D_j = \{y_{jt} - x_{jt}, t = 1, \cdots k\} \text{ and } D = \bigsqcup_{1 \le j \le b} D_j. \text{ Further, } \delta = \bigsqcup_{1 \le j \le b} (Y_j - X_j).$$

The following result can be proved by direct verification.

Lemma 3.3: (a) If an element u of S appears p times in D, then the following holds for $\mathcal{P}_0 \oplus S$.

$$N_{12}(i, j) = p \text{ if } j - i = u.$$

(b) A sufficient condition for $A_1 \perp_{bl} A_2$ in $\mathcal{P}_0 \oplus S$ is that $\delta = kD$ in $\mathcal{P}_0$.

Now we go to another construction.

Theorem 3.2: Let s be a positive integer ≥ 5. Then

(a) there exists a symmetric POTB $\mathcal{P}$ with three factors each having $s + 1$ levels on $b = 6s$ blocks of size two.

(b) In the case $s = 5$, we get a Balanced POTB. The restriction to any two of the factors reduces it to a PERGOLA, except that condition (9) is not satisfied.

Proof : Let S^+ be the set of levels for each factor. Consider an initial plan $\mathcal{P}_0$ with the set of factors $\{A_0, A_1, A_2\}$ and $\mathcal{B} = \{B_{ij}, i = 1, 2, j = 0, 1, 2\}$, where B_{ij}' are as shown in the table below. The required plan $\mathcal{P} = \mathcal{P}_0 \oplus S$.

Blocks $\rightarrow$		B_{10}		B_{11}		B_{11}		B_{20}		B_{21}		B_{22}	
Factors $\downarrow$ A_0	∞	0	-1	1	0	1	∞	0	1	2	0	2
A_1	0	1	∞	0	-1	1	0	2	∞	0	1	2
A_2	-1	1	0	1	∞	0	1	2	0	2	∞	0

(a) Consider a pair of factors of $\mathcal{P}_0$, say A_i, A_j. One can verify that the following are satisfied.

$$D = \{\infty^4, 0^2, 1^2, -1^2, 2, -2\} \text{ and } \delta = 2D \tag{10}$$

Here the superscript denote the multiplicity. In the computation we have used (8).

Now an application of Lemma 3.3 implies that $A_i \perp_{bl} A_j$ in $\mathcal{P}$. Since i, j are arbitrary, $\mathcal{P}$ is a POTB.

(b) From (10), in view of Lemma 3.3 (a) it follows that in $\mathcal{P}$

$$N_{pq} = \begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 2 \\ 2 & 2 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 2 & 2 \end{bmatrix}, \; p \neq q, p, q = 0, 1, 2. \tag{11}$$

Moreover, by the construction,

$$L_i(L_i)' = 8I_6 + 2J_6, \; i = 0, 1, 2.$$

From the relation above, we see that each L_i is the incidence matrix of a BIBD with parameters $(v = 6, b = 30, r = 10, k = 2, \lambda = 2)$. Thus, by Definition 2.3 $\mathcal{P}$ is a balanced POTB. However, N_{pq} does not satisfy (9), $p \neq q$, $p, q = 0, 1, 2$. $\square$

Next we construct a series of balanced POTBs using finite fields. We need the following notation.

Notation 3.3: (i) s is an odd prime power. $t = (s - 1)/2$. F denotes the Galois field of order s. Further, $F^* = F \setminus \{0\}$ and $F^+ = F \cup \{\infty\}$.

(ii) α denotes a primitive element of F.

(iii) C_0 denotes the subgroup of order t of the multiplicative group of F and C_1 the coset of C_0. Thus, C_0 is the set of all non-zero squares of F, while C_1 is the set of all non-zero non-squares of F.

(iv) (i, j) denotes the number of ordered pairs of integers (k,l) such that the following equation is satisfied in F. [This notation is borrowed from the theory of cyclotomy]

$$1 + \alpha^k = \alpha^l, \; k \equiv i, l \equiv j \pmod 2.$$

We present the following well-known result for ready reference. [See equations (11.6.30), (11.6.40) and (11.6.43) of Hall (1986)].

Lemma 3.4: The difference between the cosets of F^* can be expressed in terms of the cyclotomy numbers as follows.

$$C_1 - C_0 = \bigcup_{k=0}^{1} (k, 1) C_k.$$

The following cyclotomy numbers are known.

Case 1: t odd. $(0, 0) = (1, 1) = (1, 0) = (t - 1)/2, (0, 1) = (t + 1)/2$.

Case 2: t even. $(0, 0) = t/2 - 1, (0, 1) = (1, 0) = (1, 1) = t/2$.

A series of two-factor balanced POTBs :

Theorem 3.3: Suppose s is an odd prime or a prime power. Then there exists a balanced POTB $\mathcal{P}^*$ for a $(s+1)^2$ experiment on $b = 2s$ blocks of size $(s+1)/2$.

Proof: The set of levels of each factor is F^+. We shall present the initial plan $\mathcal{P}_0$ consisting of a pair of blocks. The required POTB is $\mathcal{P}^* = \mathcal{P}_0 \oplus F$.

Let $\delta \in C_1$. Consider three $2 \times (t+1)$ arrays R^0, R^1 and R^2, the rows of which are indexed by $\{0,1\}$ and the columns by $C_0 \cup \{0\}$. The entries of the arrays are as given below.

$$R^0(1,0) = R^1(0,0) = R^2(0,0) = 0 \text{ and } R^0(0,0) = R^1(1,0) = R^2(1,0) = \infty. \qquad (12)$$
$$\text{For } x = 0, 1, \ y \in C_0, \ R^0(x,y) = \delta^x y, R^1(x,y) = \delta^{-x}y \text{ and } R^2(x,y) = \delta^{x-1}y. \qquad (13)$$

For $i = 0, 1, 2$, let B_i be the block, the runs of which are the columns of R^i. When t is even, B_0 and B_1 constitute $\mathcal{P}_0$, while B_0 and B_2 constitute $\mathcal{P}_0$ when t is odd.

Clearly block size is $t+1 = (s+1)/2$. To show that $\mathcal{P}^*$ satisfies the required property, we have to show that

(a) $\mathcal{P}^*$ is a POTB and (b) each factor forms a BIBD with the block factor.

Condition (b) follows from the construction in view of Lemma 3.4. So, we prove (a). Let us write N for N_{12}. The rows and columns of N are indexed by F^+. From (12) and (3.9), we see that

$$N(ii) = 0, \ i \in F^+ \text{ and } N(\infty, i) = N(i, \infty) = 1, \ i \in F. \qquad (14)$$

So, we assume $i \neq j \in F$. Let $u = j - i$. Then, $N(ij)$ is the number of times u appears in the multiset

$$\begin{cases} (\delta - 1)C_0 \sqcup (\delta^{-1} - 1)C_0 & \text{if t is even} \\ (\delta - 1)C_0 \sqcup (1 - \delta^{-1})C_0 & \text{if t is odd} \end{cases}$$

Since $-1 \in C_0$ if and only if t is even, $\delta^{-1} - 1$ is in the same coset as $\delta - 1$ if and only if t is odd. Therefore, the relations above together with (14) above imply that

$$N = J_{s+1} - I_{s+1}. \qquad (15)$$

Now we take up $L_1 L_2' = H$ (say). From (12) and (3.9), we see that

$$H(ii) = 0, \ i \in F^+. \qquad (16)$$

Further, for every $i \in F$, $H(\infty, i)$ is the replication number of i in the block design generated by the initial block $\{0\} \cup C_1$. Similarly, $H(i, \infty)$ is the replication number of i in the block design generated by the initial block $\{0\} \cup C_0$ if t is odd and $\{0\} \cup C_1$ otherwise. Thus,

$$H(\infty, i) = H(i, \infty) = t + 1, \ i \in F. \qquad (17)$$

We, therefore, assume $i \neq j, \ i, j \in F$. Let $u = j - i$. Then, $H(ij)$ is the number of times u appears in the multiset

$$\begin{cases} ((\{0\} \cup C_1) - C_0) \sqcup (C_1 - (\{0\} \cup C_0)) & \text{if } t \text{ is even} \\ ((\{0\} \cup C_1) - C_0) \sqcup (C_0 - (\{0\} \cup C_1)) & \text{if } t \text{ is odd} \end{cases}$$

These relations, together with (16), (17) and Lemma 3.4 imply that $H = (t+1)(J_{s+1} - I_{s+1})$. Therefore, in view of (15), (7) follows and we are done. □

4. More on Recursive Construction

In this section we describe procedures for adding factors as well as blocks to an initial plan.

Notation 4.1: Consider a subset V of S^m. For every $i, 1 \leq i \leq m$, V_i will denote the following multiset of $|V|$ members of S. $V_i = \{v_i : v = (v_1, \cdots v_m)' \in V\}$.

Definition 4.1: Consider an initial plan $\mathcal{P}_0$ for an s^m experiment as described in Notation 2.1. Let V be as in Notation 4.1. By the plan $\mathcal{P}_0 + V$ **generated from** $\mathcal{P}_0$ **along** V we shall mean the plan (for the same experiment) having the set of blocks $\mathcal{B} + V = \{B + v : v \in V, B \in \mathcal{B}\}$, where $B + v$ is as in Definition 3.1. Usually, V will contain the 0-vector, so that the blocks of $\mathcal{P}_0$ will also be blocks of $\mathcal{P}_0 + V$.

The next lemma provides a few sufficient conditions on $\mathcal{P}_0$ and V so that a given pair of factors are orthogonal through the block factor in $\mathcal{P}_0 + V$. The proof is by direct verification.

Remark 4.1: In an initial plan, say $\mathcal{P}_0$, one or more levels of one or more factors may be absent. $\mathcal{P}_0$ may still be a POTB if (7) holds (with one or more row/column of N'_{ij}s being null vectors) for every unordered pair of (i, j). In such cases one has to choose V such that all levels of all factors do appear in $\mathcal{P}_0 + V$.

Lemma 4.1: Consider an initial plan $\mathcal{P}_0$ for an s^2 experiment and a subset V of S^2. The following conditions on $\mathcal{P}_0$ and V are sufficient for $\mathcal{P}_0 + V$ to be a POTB.

(a) In $\mathcal{P}_0$ all the levels of the first factor appear and $V = \{(0, i), i \in S\}$.

(b) $\mathcal{P}_0$ is arbitrary and $V = \{(i, j), i, j \in S\}$.

(c) $\mathcal{P}_0$ has a pair of blocks B_0, B_1 each of size 2, as described below. Let $i \neq j$, $k \neq l \in S$. Let $x_0 = (i, i)'$, $y_0 = (j, j)$, $x_1 = (k, l)'$ and $y_1 = (l, k)'$. B_i consists of runs x_i and y_i, $i = 0, 1$. $V = (u, u)$, $u \in S$.

(d) $\mathcal{P}_0$ is a POTB in which with one or more levels of one or both factors may be absent. V is such that every member of S appears at least once in each $V_i, i = 1, 2$.

Our next procedure enlarges the set of factors of a given plan, while keeping the number of blocks fixed.

Definition 4.2: (a) Consider a plan $\mathcal{P}$ as in Notation 2.1. Suppose there is another plan $\mathcal{P}'$ having b blocks of size k each. We shall combine these two plans to get another one with a larger set of factors.

Let x_{ij} (respectively $\tilde{x}_{ij}$) denote the jth run in the ith block of $\mathcal{P}$ (respectively $\mathcal{P}'$), $1 \leq j \leq k$, $1 \leq i \leq b$. Let $y_{ij} = \begin{bmatrix} x_{ij} & \tilde{x}_{ij} \end{bmatrix}'$, $1 \leq j \leq k$, $1 \leq i \leq b$. Then, the plan on b blocks of size k with y_{ij} as the jth run in the ith block, $1 \leq j \leq k$, $1 \leq i \leq b$ is said

to obtained by joining the factors of $\mathcal{P}$ and $\mathcal{P}'$ together. The new plan will be denoted by $\begin{bmatrix} \mathcal{P} & \mathcal{P}' \end{bmatrix}$.

(b) In case $\mathcal{P}'$ is a copy of $\mathcal{P}$ then $\begin{bmatrix} \mathcal{P} & \mathcal{P}' \end{bmatrix}$ is denoted by $\mathcal{P}^2$. For $t \geq 3$, the plan $\mathcal{P}^t$ is defined in the same way.

We find it convenient to name the factors of plan $\mathcal{P}$ in a manner different from Notation 2.1. The names the factors of $\mathcal{P}$ and its power $\mathcal{P}^t$ will be as in the notation below.

Notation 4.2: Consider a plan $\mathcal{P}$ having the set of factors $\mathcal{F}_0 = \{A, B, \cdots H\}$. The set of factors of $\mathcal{P}^t$ will be named as

$$\mathcal{F} = \bigcup_{i=1}^{t} \mathcal{F}_i, \text{where } \mathcal{F}_i = \{A_i, B_i, \cdots H_i\}.$$

Combining Definitions 4.1 and 4.2 we get a recursive construction described below.

Definition 4.3: Consider an initial plan $\mathcal{P}_0$ for an s^m experiment laid on b blocks of size k each. Consider a $p \times q$ array $H = ((h_{ij}))_{1 \leq i \leq p, 1 \leq j \leq q}$. We now obtain a plan for an s^{mq} experiment on bp blocks of size k using the array H as follows. We first obtain $\mathcal{P}_0^q$ following Definition 4.2.

Let $v_i = \begin{bmatrix} h_{i1}.1_t' & h_{i2}.1_t' & \cdots & h_{iq}.1_t' \end{bmatrix}'$, $1 \leq i \leq p$ and $V_H = \{v_i, 1 \leq i \leq p\}$.

Our required plan $\mathcal{P}$ is $\mathcal{P}_0^q + V_H$ and it will be denoted by $H \diamond \mathcal{P}$. Symbolically,

$$\mathcal{P} = H \diamond \mathcal{P}_0 = \mathcal{P}_0^q + V_H. \tag{18}$$

Our task is to find a suitable array H so that the plan $H \diamond \mathcal{P}_0$ satisfies certain desirable properties. A natural choice for H is an orthogonal array of strength 2. We shall use a modification of such an orthogonal array so as to accommodate a few more factors.

Definition 4.4: (Rao, 1946) Let $m, N, t \geq 2$ be integers and s is an integer ≥ 2. Then an orthogonal array of strength t is an $m \times N$ array, with the entries from a set S of s symbols satisfying the following. All the s^t t-tuples with symbols from S appear equally often as columns in every $t \times N$ subarray. Such an array is denoted by $OA(N, m, s, t)$.

Notation 4.3: (a) The set of symbols of an $OA(N, m, s, 2)$ is assumed to be the set of integers modulo s.

(b) The array obtained by adding a column of all zeros (in the mth position, say) to an $OA(N, m-1, s, 2)$ will be denoted by $Q(N, m, s)$.

Exploring the properties of an orthogonal array of strength 2, we get the following result from the recursive construction described in Definition 4.3.

Theorem 4.1: Consider a plan $\mathcal{P}_0$ for an s^t experiment on b blocks of size k each. If an $OA(N, m-1, s, 2)$ exists, then $\exists$ a plan $\mathcal{P}$ with a set of s^{mt} factors on bN blocks of size k each with the following properties. Here the factors of $\mathcal{P}_0^q$ as well as $\mathcal{P}$ are named according to Notation 4.2. Let $P \neq Q, P, Q \in \mathcal{F}_0$.

(a) $P_i \perp_{bl} Q_j$, $i \neq j, 1 \leq i, j \leq m$.

(b) $P_i \perp_{bl} Q_i$ for every i, $1 \leq i \leq m$, if and only if $P \perp_{bl} Q$ in $\mathcal{P}_0$.

Proof: By assumption $Q = Q(N, m, s)$ exists. The required plan $\mathcal{P}$ is $Q \diamond \mathcal{P}_0$. Property (a) follows from the construction while (b) follows from (b) of Lemma 4.1. □

Remark 4.2: Table 1 of Rees and Preece (1999) presents a number of examples of two-factor POTBs (which they call PERGOLAs). An Application of Theorem 4.1 on each of them would yield a balanced POTB for a larger set of factors.

Finally, we describe a procedure of modifying the sets of levels of factors. Specifically, given a pair of plans with the same number of factors and the same block size, we obtain a plan by merging the sets of levels of the corresponding factors of the given plans.

Definition 4.5: Consider a pair of plans $\mathcal{P}_1$ and $\mathcal{P}_2$ each having t factors and blocks of size k. Let S_i denote the set of levels of each factor of $\mathcal{P}_i$, $s_i = |S_i|$, $i = 1, 2$. We assume that $S_1 \neq S_2$. Let $U = S_1 \cup S_2$ and $u = |U|$. The plan consisting of all the blocks of $\mathcal{P}_1$ and $\mathcal{P}_2$ taken together will be viewed as a plan, say $\mathcal{P}_1 \cup \mathcal{P}_2$, for an u^t experiment in the following sense.

(a) Each factor of $\mathcal{P}_1 \cup \mathcal{P}_2$ will have U as the set of levels.

(b) Fix $p \in U$. Let $\mathcal{R}_p^{ij}$ denote the set of runs of $\mathcal{P}_j$, in which the level p of the ith factor appears, $j = 1, 2, 1 \leq i \leq t$. [Needless to mention that $\mathcal{R}_p^{ij} = \phi$ if p is not in S_j.] Then, the level p of the ith factor of $\mathcal{P}_1 \cup \mathcal{P}_2$ appears in exactly the runs in $\mathcal{R}_p^{i1} \sqcup \mathcal{R}_p^{i2}, 1 \leq i \leq t$.

Remark 4.3: From Definition 4.5 we see that for $p \in U$, the replication number of level p of the ith factor of $\mathcal{P}_1 \cup \mathcal{P}_2$ is $r^{i1}(p) + r^{i2}(p)$, where $r^{ij}(p)$ is the replication number of level p of the ith factor of $\mathcal{P}_j$.

For instance, in Theorem 5.1 below, Definition 4.5 is used to construct the plan $\mathcal{P}_h$ by merging the corresponding factors of $\mathcal{P}_{1h}$ and $\mathcal{P}_{2h}$. There, $S_1 = \{0, 1\}$, while $S_2 = \{0, 2\}$. Thus, while both $\mathcal{P}_{1h}$ and $\mathcal{P}_{2h}$ are equireplicate, the replication number of level 0 of each factor of $\mathcal{P}_h$ is double of that of the levels 1 and 2 of the same factor.

The following result is an immediate consequence of Definition 4.5 .

Lemma 4.2: Consider a pair of connected plans $\mathcal{P}_1$ and $\mathcal{P}_2$, as in Definition 4.5 (recall Definition 2.1). Then, we can say the following about the plan $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$.

(a) If both $\mathcal{P}_1$ and $\mathcal{P}_2$ are POTB, then so is $\mathcal{P}$.

(b) $\mathcal{P}$ is connected, if and only if $S_1 \cap S_2 \neq \phi$.

5.   Construction of POTBs for Three-level Factors

In this section we make use of the tools described in Section 4 to generate plans for three-level factors. The factors of the initial and final plans are named in accordance with Notation 4.2.

Theorem 5.1: If h is the order of a Hadamard matrix, then there exists a connected and saturated POTB $\mathcal{P}_h$ for a 3^{3h} experiment in $2h$ blocks of size 4 each.

Proof: Let $O_4 = OA(4, 3, 2, 2)$ with $S = \{0, 1\}$. Let $\mathcal{P}_0$ be the plan consisting of a single block consisting of the four columns of O_4 as runs. Thus, $\mathcal{P}_0$ is an OMEP for a 2^3 experiment.

Using the relation between orthogonal arrays of strength two and Hadamard matrices, [see Theorem 7.5 in Hedayat, Sloane and Stuffken (1999), for instance], we see that a $Q(h, h, 2)$ exists, in view of the hypothesis.

Let $\mathcal{P}_{1h} = Q(h, h, 2) \diamond \mathcal{P}_0$ and $\mathcal{P}_{2h}$ be obtained from $\mathcal{P}_{1h}$ by replacing level 1 of every factor by the level 2. Next we construct our required plan $\mathcal{P}_h = \mathcal{P}_{1h} \cup \mathcal{P}_{2h}$ by using Definition 4.5. By construction $\mathcal{P}_h$ has $2h$ blocks of size 4 each.

We now show that $\mathcal{P}_h$ is a POTB. We note that by Theorem 4.1, each of $\mathcal{P}_{1h}$ and $\mathcal{P}_{2h}$ is a POTB for a 2^{3h} experiment on h blocks of size 4 each. The sets of levels of each factor of them are $\{0, 1\}$ and $\{0, 2\}$ respectively. It follows from Lemma 4.2 that $\mathcal{P}_h$ is a connected POTB for an experiment with $3h$ factors, the set of levels of each factor being $\{0, 1, 2\}$. Since the available degrees of freedom for the treatment factors is $6h$ which is the same as the required degrees of freedom, the plan is saturated.   □

We now take $h = 2$ and present the plan $\mathcal{P}_2$ for a 3^6 experiment on four blocks of size four each.

Table 1 : The plan $\mathcal{P}_2$

Blocks $\rightarrow$		B_{01}	B_{02}	B_{11}	B_{12}
Factors $\downarrow$	A_1	00 11	00 11	00 22	00 22
	B_1	01 01	01 01	02 02	02 02
	C_1	01 10	01 10	02 20	02 20
	A_2	00 11	11 00	00 22	22 00
	B_2	01 01	10 10	02 02	20 20
	C_2	01 10	10 01	02 20	20 02

For the next construction we need some more notations.

Notation 5.1: O_4 is as in the proof of Theorem 5.1. T_4 will denote the array obtained from O_4 by replacing each 1 by 2 and $\tilde{T}_4$ the array obtained from T_4 by interchanging 0 and 2.

Theorem 5.2: A POTB for a 3^3 experiment on two blocks of size four exists.

Proof: Let $B_{10} = O_4$, $B_{20} = T_4$ and $B_{02} = \tilde{T}_4$. The set of columns of each of these 3×4 array constitutes an OMEP for a 2^3 experiment, the set of levels of factors being $\{0, 1\}$ for B_{10}, while $\{0, 2\}$ for the other two.

Let ρ_1 (respectively ρ_2) denote the plan consisting of the pair of blocks B_{10}, B_{20} (respectively B_{10}, B_{02}). By Lemma 4.2, each of ρ_1 and ρ_2 is a POTB for a 3^3 experiment.   □

Using the pair of plans constructed above, we generate a bigger plan.

Theorem 5.3: (a) If there exists an $OA(N, m, 3, 2)$, then there exists a connected POTB $\mathcal{P}_m$ for a $3^{3(2m+1)}$ experiment in $2N$ blocks of size 4 each.

In particular $\mathcal{P}_m$ is saturated whenever $N = 3^n$ and $m = (3^{n-1} - 1)/2$, for an integer $n \geq 2$.

(b) There exists a connected POTB $\mathcal{P}_1$ for a 3^9 experiment in 6 blocks of size 4 each.

Proof: Part (a): Let the factors of ρ_1 and ρ_2 be named as A, B, C and $\tilde{A}, \tilde{B}, \tilde{C}$ respectively. Let $O = OA(N, m, 3, 2)$ and $Q = Q(N, m, 3)$. We now use Definition 4.5 to generate bigger plans $\mathcal{P}_{1m}$ and $\mathcal{P}_{2m}$ as follows.

$$\mathcal{P}_{1m} = Q \diamond \rho_1 \text{ and } \mathcal{P}_{2m} = O \diamond \rho_2.$$

Clearly, $\mathcal{P}_{1m}$ and $\mathcal{P}_{2m}$ are plans for $3^{3(m+1)}$ and 3^{3m} experiments respectively, each on $2N$ blocks of size 4. Following Notation 4.2, we name of the factors of these plans as follows.

$$\text{The factors of } \mathcal{P}_{1m} \text{ are} \quad A_0, B_0, C_0, A_1, B_1, C_1, \cdots A_m, B_m, C_m$$
$$\text{and the factors of } \mathcal{P}_{2m} \text{ are} \quad \tilde{A}_1, \tilde{B}_1, \tilde{C}_1, \cdots \tilde{A}_m, \tilde{B}_m, \tilde{C}_m.$$

Now we combine the factors of $\mathcal{P}_{1m}$ and $\mathcal{P}_{2m}$ following Definition 4.2 (a) and thus obtain our required plan $\mathcal{P}_m$. Symbolically,

$$\mathcal{P}_m = \begin{bmatrix} \mathcal{P}_{1m} & \mathcal{P}_{2m} \end{bmatrix}.$$

By construction, $\mathcal{P}_m$ is a plan for $2m+1$ three-level factors on $2N$ blocks of size 4 each. We shall now show that it is a POTB.

Theorems 4.1 and 5.2 imply that each one of $\mathcal{P}_{1m}$ and $\mathcal{P}_{2m}$ is a POTB. Therefore, if we show the following relation, then we are done.

$$P_i \perp_{bl} \tilde{Q}_j, \ P, Q \in \{A, B, C\}, \ i \in I \cup \{0\}, \ j \in I, \text{ where } I = \{1, \cdots m\}. \tag{19}$$

To show this relation, we fix P_i and $\tilde{Q}_j$ as above.

Case 1. $i, j \in I$: Since ρ_1 and ρ_2 are POTBs, (19) follows from Lemma 4.1 (d), whenever $Q \neq P$. Again, (c) of the same Lemma proves (19) for the case $Q = P$.

Case 2. $i = 0, \ j \in I$: We take P_0 as the first and $\tilde{Q}_j$ as the second factor. Then applying Lemma 4.1 (a) we get (19).

Hence the proof of the first part is complete.

To prove the second part, we see that $\mathcal{P}_m$ is saturated when $N = 2m + 1$. Now Rao (1946) has shown that an $OA(s^n, (s^n - 1)/(s - 1), s, 2)$ exists whenever $n \geq 2$. (see Theorem 3.20 of Hedayat, Sloane and Stufken (1999) for instance). Putting $s = 3$, we get the result.

Proof of (b) : Let $O = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. and $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. Now the construction for the plan, say $\mathcal{P}_1$, is just like that in Case (a). The verification is also exactly like the same in Case (a) with $I = \{1\}$. $\hfill\square$

We now present the plan of Theorem 5.3 (b).

Table 2 : The plan $\mathcal{P}_1$

Blocks $\rightarrow$		B_{10}	B_{20}	B_{11}	B_{21}	B_{12}	B_{22}
Factors $\downarrow$	A_0	00 11	00 22	00 11	00 22	00 11	00 22
	B_0	01 01	02 02	01 01	02 02	01 01	02 02
	C_0	01 10	02 20	01 10	02 20	01 10	02 20
	A_1	00 11	00 22	11 22	11 00	22 00	22 11
	B_1	01 01	02 02	12 12	10 10	20 20	21 21
	C_1	01 10	02 20	12 21	10 01	20 02	21 12
	$\tilde{A}_1$	00 11	22 00	11 22	00 11	22 00	11 22
	$\tilde{B}_1$	01 01	20 20	12 12	01 01	20 20	12 12
	$\tilde{C}_1$	01 10	20 02	12 21	01 10	20 02	12 21

6.   Inter-class Orthogonal Plans

Inter-class orthogonal plans are defined in Bagchi (2019) in the context of plans without any blocking factor. Here we extend the definition to the present context - the orthogonality being through the block factor.

Definition 6.1: Let us consider a plan ρ. Suppose the set of all factors of ρ can be divided into several classes in such a way that if two factors belong to different classes, then they are orthogonal through the block factor. Such a plan ρ is called a **"Plan Inter-class Orthogonal through the Blocks (PIOTB)"** and the classes will be referred to as "orthogonal classes".

We shall now proceed towards the construction of PIOTBs.

Theorem 6.1: If s is an integer ≥ 5, then a PIOTBs with $4s$ blocks of size two exists for an s^4 experiment.

Proof: The blocks B_l, $l = 1, \cdots 4$ of an initial plan $\mathcal{P}_0$ are presented below. Here a, b are distinct members of $S \setminus \{0\}$. The required plan $\mathcal{P}$ is $\mathcal{P}_0 \oplus S$ [see Definition 3.1].

Blocks $\rightarrow$		B_1		B_2		B_3		B_4	
Factors $\downarrow$	A_1	0	a	a	-a	0	b	-b	b
	A_2	a	-a	0	-a	-b	b	0	-b
	A_3	0	-b	b	-b	-a	0	a	-a
	A_4	b	-b	0	b	a	-a	a	0

We define at subplans $\mathcal{P}_{ij}$, $i,j = 1,2,3,4$ in the same way as in the proof of Theorem 3.1 (b). Using Lemma 3.1 we see that in each of the subplans $\mathcal{P}_{12}$ and $\mathcal{P}_{34}$ $A_1 \perp_{bl} A_2$ and $A_3 \perp_{bl} A_4$. Similarly, in each of the subplans $\mathcal{P}_{13}$ and $\mathcal{P}_{24}$ $A_1 \perp_{bl} A_3$ and $A_2 \perp_{bl} A_4$. Therefore, by Lemma 3.2 $A_i \perp_{bl} A_j$ in $\mathcal{P}$, where $i \in \{1,4\}$ and $j \in \{2,3\}$. However, A_1 is not orthogonal to A_4 and A_2 is not orthogonal to A_3. It, therefore, follows that $\mathcal{P}$ is interclass orthogonal with classes $\{A_1, A_4\}$ and $A_2, A_3\}$. □

Next we present a PIOTB for three-level factors.

Theorem 6.2: A saturated PIOTB exists for a 3^6 experiment on four blocks of size four each.

Proof: Consider the following plan $\mathcal{P}$. It is easy to see that it is a PIOTB with non-orthgonal classes $\{P_1, P_2\}$, $P = A, B, C$.

Table 1 : Plan $\mathcal{P}$

Blocks $\rightarrow$		B_1	B_2	B_3	B_4
Factors $\downarrow$	A_1	00 12	00 21	00 12	00 21
	B_1	01 02	02 01	10 20	20 10
	C_1	01 20	02 10	02 10	01 20
	A_2	01 01	02 02	01 01	02 02
	B_2	01 10	02 20	10 01	20 02
	C_2	00 11	00 22	11 00	22 00

□

Remark 6.1: A POTB for a 4^4 experiment on 4 blocks of size 4 is well-known [can be obtained by treating a row of OA(16,5,4,2) as the block factor]. By collapsing two of the levels of each factor to one level one gets a POTB for a 3^4 experiment on the same set up. Allowing non-orthogonality we have been able to accommodate two more three-level factors, making it saturated.

Theorem 6.3: Suppose Hadamard matrices of orders m and n exist. Then, there exists a saturated PIOTB $\mathcal{P}_{(m,n)}$ for a 2^{mn} experiment on n blocks of size $m+1$ each. There are n orthogonal classes of size m each.

Proof: By hypothesis $Q_m = Q(m,m,2)$ exists. Let R be the $m \times m+1$ array obtained by juxtaposing a column of all-ones to Q_m. Let $\mathcal{P}_0$ be the plan for a 2^m experiment on a single block consisting of $m+1$ runs, which are the columns of R. Let us name the factors of $\mathcal{P}_0$ as $A, B, \cdots H\}$. Note that the column added to Q_m saves H from being confounded with the block. However, this column also destroys the relation of orthogonality enjoyed by the factors in Q_m (represented by its rows).

By hypothesis, $Q_n = Q(n,n,2)$ exists. Let $\mathcal{P}_{(m,n)} = Q_n \diamond \mathcal{P}_0$. Clearly, $\mathcal{P}_{(m,n)}$ is an main effect plan for a 2^{mn} experiment with parameters as in the statement. By construction, no factor is confounded with the block factor. Using Theorem 4.1 and the property of $\mathcal{P}_0$, we

see that $\mathcal{P}_n$ is interclass orthogonal with orthogonal classes $\{A_i, B_i, \cdots H_i\}$, $1 \leq i \leq n$ (recall Notation 4.2). Hence the result.   □

We now present the plan $\mathcal{P}_{(4,2)}$.

Table 6.2 : The plan $\mathcal{P}_{4,2}$

Blocks →		B_1	B_1
Factors ↓	A_1	00 11 1	11 00 0
	B_1	01 01 1	10 10 0
	C_1	01 10 1	10 01 0
	D_1	00 00 1	11 11 0
	A_1	00 11 1	00 11 1
	B_1	01 01 1	01 01 1
	C_1	01 10 1	01 10 1
	D_1	00 00 1	00 00 1

There are two orthogonal classes, which are $\{A_i, B_i, C_i, D_i\}$, $i = 1, 2$.

A situation when a PIOTB is better than a POTB.

Since in a POTB the factors are orthogonal to each other, one expects that its performance would be better than that of a PIOTB, in which orthogonality does not hold for one or more pairs of factors. However, we find an example, where a PIOTB is better than a POTB w.r.t. the commonly used optimality criteria.

Consider the following three plans for two-level factors on the set up with two blocks of size 5 each.

$\mathcal{P}_1$ is a plan for four factors constructed in Jacroux and Kealy-Dichone (2015). A POTB having seven factors is constructed in Bagchi and Bagchi (2020). Let $\mathcal{P}_2$ denote the POTB obtained by deleting the last factor of it. Let $\mathcal{P}_3$ denote the plan obtained from $\mathcal{P}_{4,2}$ by deleting the factors D_1 and D_2.

We compare these plans in terms of the C-matrix for the contrasts. For the definition of this C-matrix we refer to Jacroux and Kealy-Dichone (2015). We find the following.

Theorem 6.4: (a) $\mathcal{P}_3$ is A- and D-better than $\mathcal{P}_2$.

(b) Let $\tilde{\mathcal{P}}_2$ denote the restriction of $\mathcal{P}_2$ to four factors. Let $\tilde{\mathcal{P}}_3$ be the plan obtained from $\mathcal{P}_{4,2}$ consisting of the factors A_1, B_1, A_2, B_2. Then, $\tilde{\mathcal{P}}_3$ is A- and D-better than $\mathcal{P}_1$ as well as $\tilde{\mathcal{P}}_2$.

(c) All the three plans are E-optimal.

Proof: We compute the C-matrix for the contrasts of each plan and find the following.

$$C_{\mathcal{P}_1} = 8I_4 + (8/5)J_4 \quad \text{and} \quad C_{\mathcal{P}_2} = 8I_6.$$

$$C_{\mathcal{P}_3} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \text{where} \quad A = 8I_3 + (8/5)J_3.$$

The spectrums of these matrices are as follows.

$$\begin{aligned}
C_{\mathcal{P}_1} &: \quad 8^3(8 + 32/5)^1, \\
C_{\mathcal{P}_2} &: \quad 8^5, \\
C_{\mathcal{P}_3} &: \quad 8^4(8 + 24/5)^2 \\
\text{and } C_{\tilde{\mathcal{P}}_3} &: \quad 8^2(8 + 16/5)^2.
\end{aligned}$$

The rest is by simple computation. $\qquad\square$

Conjecture : An universally optimal plan can not exist in the above set up and $\mathcal{P}_3$ is optimal w.r.t. a large class of optimality criteria.

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