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Inference on Stress-Strength Reliability of a System with Two Independent Stresses under Generalized Uniform Distribution

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Abstract

In this paper, we consider a system with strength X and two independent stresses Y and Z working on it. We derive the UMVUEs of the stress-strength reliability ξ = P[$X >$ max(Y,Z)] and its variance, when X, Y and Z have independent generalized uniform distributions with known shape parameters. We also discuss testing of hypothesis regarding ξ. A comparison of the UMVUE with the MLE has been carried out in terms of the mean squared error. A simulation study has also been indicated.

Key Words: Generalized uniform distribution; Stress-strength reliability; UMVUE; Hypothesis testing.

AMS Subject Classification (2000): Primary 62F30, Secondary 62N05

1. Introduction

The stress-strength model finds applicability in many areas of research, like reliability engineering, psychology, biometry, economics, medicine, environmental risk assessments, etc. The main problem is to infer about Pr[$X > Y$], where X denotes the strength and Y the stress. Estimation of Pr[$X > Y$] has been addressed by many authors for various distributions of the variables. Some studies along this line are due to Reiser and Guttman (1988), Ivshin (1996), Ali *et al.* (2005), Pal *et al.* (2005), Ng (2006), Krishnamoorthy *et al.* (2007), Kundu and Raqab (2009), Ventura and Racugno (2011), Baklizi (2014), Gunasekera (2015).

In real life situations, a system may have to withstand two or more stresses on it. For example, tension, compression, shear, bending, and torsion are the stresses on the wings, fuselage, and landing gear of an aircraft. In such situations, the stress-strength reliability will be defined by the probability that the strength of the system is more than the maximum of the stresses acting on it. There are very few studies relating to estimation of this reliability. Rinco (1983) initiated a study on the estimation of Pr[$Y_p >$ max(Y_1, Y_2, · · · , Y_{p-1})] when the random variables Y_1, ..., Y_p are independent following exponential distributions with unequal location parameters and equal scale parameters, and suggested an estimator. Gupta and Gupta (1988) derived the MLE, MVUE and Bayes estimator of the same for the case of p = 2. They carried out simulation studies to compare these estimators. Karaday *et al.* (2011) investigated the MLE of stress-strength reliability, Pr[max(Y_1,Y_2) < X], when a component with strength X following a Gamma distribution is exposed to two independent stresses Y_1,Y_2 having exponential distributions with different parameters. Kundu (2017) estimated the reliability

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function $R = \Pr[Y_3 > \max(Y_1, Y_2)]$, when Y_1, Y_2 and Y_3 are independent exponential variables with unknown location parameters. She derived several estimators of R, and compared their performance based on their risks under different loss functions. Park (2010) discussed estimation of reliability in a load sharing system.

In this paper, we find the UMVU estimator of the stress-strength reliability $\Pr[X > \max(Y, Z)]$, where the strength X and the stresses Y and Z are independently distributed, each having a generalized uniform distribution. Force of water flow, stress on venting valve, *etc.* may have generalized uniform distributions.

A generalized uniform (GU) distribution is defined by the density $f(x)$ and cumulative distribution function $F(x)$ as follows:

$$f(x) = \frac{\alpha+1}{\theta^{\alpha+1}} x^{\alpha}, 0 < x < \theta, \quad F(x) = \left(\frac{x}{\theta}\right)^{\alpha+1}, 0 < x < \theta,$$

where $\theta > 0, \alpha > -1$ (see Tiwari *et al.*, 1996). The parameters α and θ are, respectively, the shape and scale parameters of the distribution. We may write the distribution as GU(α, θ).

We also find the UMVU estimator of the variance of the UMVU estimator of $\Pr[X > \max(Y, Z)]$. We further propose a test for the stress-strength reliability, which is uniformly most powerful within the class of tests based on complete sufficient statistics.

2. Stress-Strength Reliability

Consider a system with strength X, which follows the GU(α_1, θ_1) distribution, given by

$$f_X(x) = \frac{\alpha_1+1}{\theta_1^{\alpha_1+1}} x^{\alpha_1}, \quad 0 < x < \theta_1 (< \infty). \tag{1}$$

Suppose there are two independent stresses Y and Z working on the system, which are distributed as GU(α_2, θ_2) and GU(α_3, θ_3), respectively. The system functions as long as it can withstand the two stresses.

Suppose $\theta_2 = \theta_3 = \theta$, say. Let, $\rho = \dfrac{\theta}{\theta_1}$.

The stress-strength reliability of the system is then given by

$\xi = \Pr[X > \max(Y, Z)] = g(\rho)$, say, where

$$g(\rho) = \left(\frac{\alpha_1+1}{\alpha_1+\alpha_2+\alpha_3+3}\right)\rho^{-(\alpha_2+\alpha_3+2)}, \text{ if } \rho \ge 1$$

$$= 1 - \frac{\alpha_2+\alpha_3+2}{\alpha_1+\alpha_2+\alpha_3+3}\rho^{(\alpha_1+1)}, \text{ if } \rho \le 1. \tag{2}$$

Clearly, ξ is a monotone function of ρ.

Let us assume that α_1, α_2 and α_3 are known, but θ_1 and θ are unknown.

3. MVUE of ξ

Consider independent random sample $(X_1, X_2, ..., X_n)$, $(Y_1, Y_2, ..., Y_m)$, and $(Z_1, Z_2, ..., Z_r)$ of sizes *n, m* and *r* respectively, from the distributions of *X*, *Y* and *Z*. The statistic

$$U = 1, \ \text{if} \ X_1 > \max(Y_1, Z_1) \\ = 0, \ \text{otherwise.} \tag{3}$$

is an unbiased estimator of ξ.

Let $(X_{(1)} < X_{(2)} < ... < X_{(n)})$, $(Y_{(1)} < Y_{(2)} < ... < Y_{(m)})$ and $(Z_{(1)} < Z_{(2)} < ... < Z_{(r)})$ be the ordered observations in the samples mentioned above. Then, $X_{(n)}$ is a complete sufficient statistic for θ_1. The following lemma indicates the complete sufficient statistic of θ.

Lemma 1: (i) $W = \max(Y_{(m)}, Z_{(r)})$ is a complete sufficient statistic for θ.

(ii) $\frac{Y_1}{W}$ (and also $\frac{Z_1}{W}$) is distributed independently of θ.

Proof: (i) We have

$$\Pr(W < w) = \Pr(Y_{(m)} < w, Z_{(r)} < w) = \left(\frac{w}{\theta}\right)^{m(\alpha_2+1)+r(\alpha_3+1)}, \ 0 < w < \theta,$$

since $Y_{(m)}$ and $Z_{(r)}$ are independently distributed.

Hence, $W \sim \text{GU}(m(\alpha_2+1)+r(\alpha_3+1), \theta)$.

Now, for any function $h(w)$ of w,

$$E(h(W)) = \int_0^\theta h(w)\{m(\alpha_2+1)+r(\alpha_3+1)\}\frac{w^{m(\alpha_2+1)+r(\alpha_3+1)-1}}{\theta^{m(\alpha_2+1)+r(\alpha_3+1)}}dw = 0$$

$$\Rightarrow \int_0^\theta h(w)w^{m(\alpha_2+1)+r(\alpha_3+1)-1}dw = 0. \tag{4}$$

Differentiating (4) with respect to θ gives $h(\theta) = 0$ for all θ, which implies $h(w) = 0$, for $0 < w < \theta$. Hence, W is a complete statistic.

The sufficiency part follows easily from Neyman-Fisher Factorization Theorem, by considering the joint distribution of $(Y_1, Y_2, ..., Y_m)$ and $(Z_1, Z_2, ..., Z_r)$.

(ii) Since $Y_1 \sim \mathrm{GU}(\alpha_2,\theta)$ and $Z_1 \sim \mathrm{GU}(\alpha_3,\theta)$, we have $\frac{Y_1}{\theta} \sim \mathrm{GU}(\alpha_2,1)$ and $\frac{Z_1}{\theta} \sim \mathrm{GU}(\alpha_3,1)$, which are independent of θ. Similarly, $\frac{W_1}{\theta} \sim \mathrm{GU}(m(\alpha_2+1)+r(\alpha_3+1),1)$, which is independent of θ. Hence, $\frac{Y_1}{W} = \frac{Y_1/\theta}{W/\theta}$ and $\frac{Z_1}{W} = \frac{Z_1/\theta}{W/\theta}$ are distributed independently of θ.

(Proved)

Using Lehmann-Scheffé Theorem, the UMVUE of ξ is, therefore, given by

$$E(U \mid X_{(n)},W) = \Pr(X_1 > \max(Y_1,Z_1) \mid X_{(n)},W) = \Pr\left(\frac{Y_1}{W} < \frac{X_1}{X_{(n)}}.\frac{1}{D}, \frac{Z_1}{W} < \frac{X_1}{X_{(n)}}.\frac{1}{D} \mid D\right) \quad (5)$$

where $D = \frac{W}{X_{(n)}}$.

Clearly, (5) is a function of only D, α_1, α_2 and α_3, since the distributions of $\frac{X_1}{X_{(n)}}, \frac{Y_1}{W}$ and $\frac{Z_1}{W}$ are independent of θ_1 and θ. Thus, any unbiased estimator of ξ, which is a function of D will be UMVUE of ξ, for α_1, α_2 and α_3 given.

To find the expression of the UMVUE of ξ, we obtain the density function of the distribution of D, which comes out as

$$\begin{aligned} f_D(d) &= \frac{C}{\rho}\left(\frac{d}{\rho}\right)^{m(\alpha_2+1)+r(\alpha_3+1)-1}, \text{ if } 0 < d < \rho \\ &= \frac{C}{\rho}\left(\frac{d}{\rho}\right)^{-n(\alpha_1+1)-1}, \text{ if } d \geq \rho, \end{aligned}$$

where

$$C = \frac{n(\alpha_1+1)\{m(\alpha_2+1)+r(\alpha_3+1)\}}{n(\alpha_1+1)+m(\alpha_2+1)+r(\alpha_3+1)}. \quad (6)$$

Inspecting possible estimators of ξ based on D, we arrive at the following theorem:

Theorem 1: The UMVU estimator of ξ is given by

$$\begin{aligned} V &= C_1 D^{-(\alpha_2+\alpha_3+2)}, \text{ if } D \geq 1 \\ &= 1 - C_2 D^{\alpha_1+1}, \text{ if } D < 1, \end{aligned}$$

where

$$C_1 = \frac{\{n(\alpha_1+1)+\alpha_2+\alpha_3+2\}\{(m-1)(\alpha_2+1)+(r-1)(\alpha_3+1)\}}{n\{m(\alpha_2+1)+r(\alpha_3+1)\}(\alpha_1+\alpha_2+\alpha_3+3)}$$

$$C_2 = \frac{(n-1)(\alpha_2+\alpha_3+2)\{(\alpha_1+1)+m(\alpha_2+1)+r(\alpha_3+1)\}}{n(\alpha_1+\alpha_2+\alpha_3+3)\{m(\alpha_2+1)+r(\alpha_3+1)\}}.$$

To prove the theorem, it is sufficient to show that V is an unbiased estimator of ξ. For $\rho \geq 1$,

$$E(V) = C[\int_0^1 (1 - C_2 d^{\alpha_1+1}) f_D(d)dd + \int_1^{\rho} C_1 d^{-(\alpha_2+\alpha_3+2)} f_D(d)dd$$
$$+ \int_{\rho}^{\infty} C_1 d^{-(\alpha_2+\alpha_3+2)} f_D(d)dd],$$

which reduces to $\left(\dfrac{\alpha_1+1}{\alpha_1+\alpha_2+\alpha_3+3}\right)\rho^{-(\alpha_2+\alpha_3+2)}$ on simplification.

Similarly, for $\rho < 1$, it can be shown that V is an unbiased estimator of ξ.

4. UMVUE of ξ^k

Consider $k \geq 2$ to be an integer. We have

$$\xi^k = \{\Pr[X > \max(Y,Z)]\}^k$$
$$= \left(\frac{\alpha_1+1}{\alpha_1+\alpha_2+\alpha_3+3}\right)^k \rho^{-k(\alpha_2+\alpha_3+2)}, \text{ if } \rho \geq 1$$
$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} \left(\frac{\alpha_2+\alpha_3+2}{\alpha_1+\alpha_2+\alpha_3+3}\right)^i \rho^{(\alpha_1+1)i}, \text{ if } \rho < 1.$$

We find the UMVUE of ξ^k for $k < \min(n, m, r)$ when α_1, α_2 and α_3 are known. An unbiased estimator of ξ^k is given by

$$U_k = 1, \text{ if } X_1 > \max(Y_1, Z_1), X_2 > \max(Y_2, Z_2), \ldots, X_k > \max(Y_k, Z_k),$$
$$= 0, \text{ otherwise}.$$

From Lehmann-Scheffé Theorem, the UMVUE of ξ^k is

$$E(U_k \mid X_{(n)}, W) = \Pr[\frac{Y_1}{W} < \frac{X_1}{X_{(n)}}.\frac{1}{D}, \frac{Z_1}{W} < \frac{X_1}{X_{(n)}}.\frac{1}{D}; \ldots; \frac{Y_m}{W} < \frac{X_n}{X_{(n)}}.\frac{1}{D}, \frac{Z_r}{W} < \frac{X_1}{X_{(n)}}.\frac{1}{D} \mid D],$$

which is again only a function of $D = \dfrac{W}{X_{(n)}}$. α_1, α_2 and α_3. Hence, an unbiased estimator of ξ^k based on D will be the UMVUE of ξ^k when α_1, α_2 and α_3 are known.

Theorem 2: For positive integer $k < \min(n, m, r)$, let

$$V_k = G_{1k} D^{-k(\alpha_2+\alpha_3+2)}, \text{ if } D \geq 1$$
$$= \sum_{i=0}^{k} (-1)^i \binom{k}{i} G_{2i} D^{i(\alpha_1+1)}, \text{if } D < 1,$$

where

$$G_{1k} = \left(\frac{\alpha_1+1}{\alpha_1+\alpha_2+\alpha_3+3}\right)^k \frac{[(m-k)(\alpha_2+1)+(r-k)(n_3+1)][n(\alpha_1+1)+k(n_2+1)+k(n_3+1)]}{n(\alpha_1+1)[m(\alpha_2+1)+r(\alpha_3+1)]}$$

$$G_{2i} = \left(\frac{\alpha_2+\alpha_3+2}{\alpha_1+\alpha_2+\alpha_3+3}\right)^i \frac{(n-i)[i(\alpha_1+1)+m(\alpha_2+1)+r(\alpha_3+1)]}{n[m(\alpha_2+1)+r(\alpha_3+1)]}, i=0(1)k.$$

Then, V_k is the UMVUE of ξ^k.

Proof: For $\rho \geq 1$,

$$E(V_k) = C[\int_0^1 \sum_{i=0}^{k} (-1)^i \binom{k}{i} G_{2i} d^{i(\alpha_1+1)} f_D(d)dd + \int_1^\rho G_{1k} d^{-k(\alpha_2+\alpha_3+2)} f_D(d)dd$$

$$+ \int_\rho^\infty G_{1k} d^{-k(\alpha_2+\alpha_3+2)} f_D(d)dd]$$

$$= \xi^k + CA\rho^{-m(\alpha_2+1)-r(\alpha_3+1)}, \tag{7}$$

where $A = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{G_{2i}}{i(\alpha_1+1)+k(\alpha_2+\alpha_3+2)} - \frac{G_{1k}}{(m-k)(\alpha_2+1)+(r-k)(\alpha_3+1)}$.

On simplification, the second term of (7) is zero.

Similarly, for $\rho < 1$we get $E(V_k) = \xi^k$.Thus, V_k is an unbiased, and hence UMVU estimator, of ξ^k.

Remark: For $k = 2$, the UMVUE of $Var(V)$ is $\hat{V}ar(V) = V^2 - V_2$, provided min $(m,n,r) > 2$, where V_2 is given by

$$V_2 = G_{12} D^{-2(\alpha_2+\alpha_3+2)}, \text{if } D \geq 1$$

$$= G_{20} - 2G_{21} D^{\alpha_1+1} + G_{22} D^{2(\alpha_1+1)}, \text{ if } D < 1,$$

with

$$G_{12} = \left(\frac{\alpha_1+1}{\alpha_1+\alpha_2+\alpha_3+3}\right)^2 \frac{[(m-2)(n_2+1)+(r-2)(n_3+1)][n(\alpha_1+1)+2(n_2+1)+2(n_3+1)]}{n(\alpha_1+1)[m(\alpha_2+1)+r(\alpha_3+1)]}$$

$$G_{20} = 1, \quad G_{21} = \left(\frac{\alpha_2+\alpha_3+2}{\alpha_1+\alpha_2+\alpha_3+3}\right) \frac{(n-1)[(\alpha_1+1)+m(\alpha_2+1)+r(\alpha_3+1)]}{n[m(\alpha_2+1)+r(\alpha_3+1)]}$$

$$G_{22} = \left(\frac{\alpha_2+\alpha_3+2}{\alpha_1+\alpha_2+\alpha_3+3}\right)^2 \frac{(n-2)[2(\alpha_1+1)+m(\alpha_2+1)+r(\alpha_3+1)]}{n[m(\alpha_2+1)+r(\alpha_3+1)]}.$$

It is easy to check that $V^2 - V_2 > 0$.

5. Comparison with MLE of ξ

The maximum likelihood estimator (MLE) of ρ is given by $\tilde{\rho} = \frac{\max(Y_{(m)}, Z_{(r)})}{X_{(n)}}$, where $\tilde{\theta} = \max(Y_{(m)}, Z_{(r)})$ and $\tilde{\theta}_1 = X_{(n)}$ are the MLEs of θ and θ_1, respectively. And, the MLE of ξ is given by $\tilde{\xi} = g(\tilde{\rho})$. Clearly, $\tilde{\xi}$ is a biased estimator of ξ. To compare it with the UMVUE of ξ, a Monte Carlo simulation study has been carried out with 5000 replications. Without loss of generality, we set $(\alpha_1, \alpha_2, \alpha_3) = (0.5, 0.25, 0.75)$, and compute the mean squared errors (MSEs) of the estimators for different settings of the parameters, which are shown in Table 1.

Table 1: Comparison of MSEs of UMVUE and MLE of ξ

		(*n*,*m*,*r*)			
(θ, θ_1)	Estimator	**(5, 5, 5)**	**(5, 10, 10)**	**(10, 10, 10)**	**(10, 5, 5)**
(1, 1.5)	UMVUE	0.01227	0.04535	0.00245	0.00248
	MLE	0.01278	0.08158	0.00476	0.00347
(2, 2)	UMVUE	0.00981	0.01005	0.00156	0. 00241
	MLE	0.01388	0.00825	0.00451	0. 00323
(3, 2)	UMVUE	0.03371	0.03119	0.02912	0. 03433
	MLE	0.03270	0.02756	0.02691	0. 05062

From Table 1 it is clear that the UMVUE does not perform uniformly better than the MLE, though in most situations considered, the MSE of UMVUE is lower than that of the MLE.

4. Test of ξ Based on *D*

As *D* is the key statistic in finding the UMVUE of ξ, we find the best test for H_0: $\xi = \xi_0$ among the class of tests based on *D*, when α_1, α_2 and α_3 are known.

Suppose we want to test the null hypothesis H_0: $\xi = \xi_0$ against $H_A: \xi < \xi_0$, As ξ is monotone decreasing in ρ, this is equivalent to testing H_0^*: $\rho = \rho_0$ against H_A^*: $\rho > \rho_0$, where, from (2), we have

$$\rho_0 = \left(\frac{\alpha_1 + 1}{\xi_0 \{\alpha_1 + \alpha_2 + \alpha_3 + 3\}} \right)^{\frac{1}{(\alpha_2 + \alpha_3 + 2)}}, \text{if } \xi_0 \le \frac{\alpha_1 + 1}{\alpha_1 + \alpha_2 + \alpha_3 + 3}$$

$$= \left((1 - \xi_0) \frac{\alpha_1 + \alpha_2 + \alpha_3 + 3}{\alpha_2 + \alpha_3 + 2} \right)^{\frac{1}{\alpha_1 + 1}}, \text{if } \xi_0 > \frac{\alpha_1 + 1}{\alpha_1 + \alpha_2 + \alpha_3 + 3}.$$

Within the class of tests based on *D,* the MP test for testing H_0^*: $\rho = \rho_0$ against H_A^*: $\rho = \rho_1 (> \rho_0)$, has the critical region $W = \{d : \lambda(d) > k\}$, where *k* is determined from the size condition, and $\lambda(d)$ is given by

$$\lambda(d) = \frac{f_D(d;\rho_1)}{f_D(d;\rho_0)} = \left(\frac{\rho_1}{\rho_0}\right)^{n(\alpha_1+1)}, \text{ if } d \ge \rho_1$$
$$= \rho_0^{-n(\alpha_1+1)} \rho_1^{-m(\alpha_2+1)-r(\alpha_3+1)} d^{n(\alpha_1+1)+m(\alpha_2+1)+r(\alpha_3+1)}, \text{ if } \rho_0 < d < \rho_1$$
$$= \left(\frac{\rho_1}{\rho_0}\right)^{-m(\alpha_2+1)-r(\alpha_3+1)}, \text{ if } d < \rho_0.$$

Clearly $\lambda(d)$ is non-decreasing in *d*. Hence, $\lambda(d) > k \Leftrightarrow d > d^*$, where d^* satisfies the size condition, *i.e.*, $\Pr[D > d^* \mid H_0^*] = \alpha$, α specified. As $\Pr[D \ge \rho_0 \mid H_0^*] = C / n(\alpha_1 + 1)$, we get

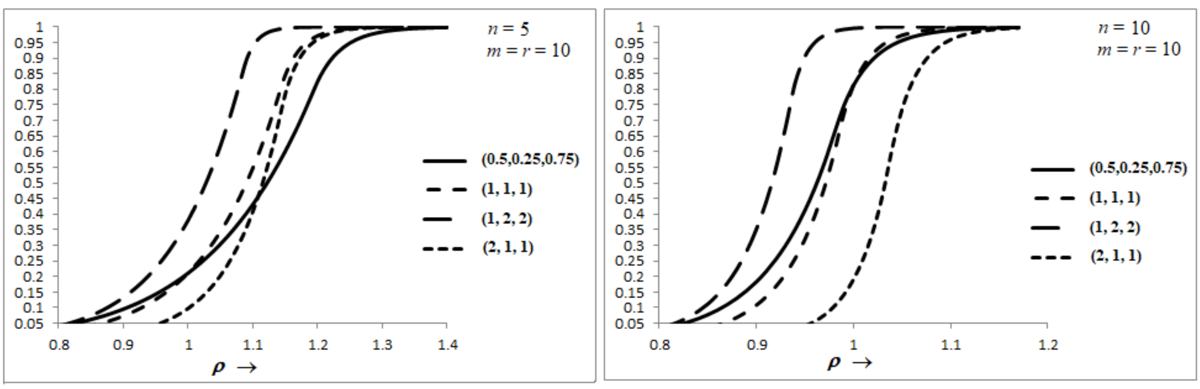
$$d^* = \rho_0 \left[\alpha \frac{n(\alpha_1+1)}{C}\right]^{-\frac{1}{n(\alpha_1+1)}} . \text{if } C/n(\alpha_1+1) > \alpha$$
$$= \rho_0 \left[(1-\alpha)\frac{m(\alpha_2+1)+r(\alpha_3+1)}{C}\right]^{\frac{1}{m(\alpha_2+1)+r(\alpha_3+1)}}, \text{if } C/n(\alpha_1+1) < \alpha, \qquad (8)$$

where $C = \dfrac{n(\alpha_1+1)\{m(\alpha_2+1)+r(\alpha_3+1)\}}{n(\alpha_1+1)+m(\alpha_2+1)+r(\alpha_3+1)}$.

Since d^* is independent of ρ_1, $W^*=\{d \mid d > d^*\}$ will be the critical region of the UMP test of size α among all tests based on *D*.

For any $\rho > \rho_0$, the power of the test is given by

$$\beta(\rho) = \frac{C}{n(\alpha_1+1)}\left(\frac{d^*}{\rho}\right)^{-n(\alpha_1+1)}, \text{ if } d^* \ge \rho$$
$$= 1 - \frac{C}{m(\alpha_2+1)+r(\alpha_2+1)}\left(\frac{d^*}{\rho}\right)^{m(\alpha_2+1)+r(\alpha_2+1)}, \text{ if } d^* < \rho.$$



(i) (ii)

Figure 1: Power curves of tests for testing H_0: $\xi = 0.5$ against $H_A : \xi < 0.5$ for some combinations of (α_1, α_2, α_3), when $\theta = 2$, $\theta_1 = 3$ and $\alpha = 0.05$

For testing H_0: $\xi = \xi_0$ against H_A: $\xi > \xi_0$ (or H_A: $\xi \neq \xi_0$), which is equivalent to testing H_0^*: $\rho = \rho_0$ against H_A^*: $\rho < \rho_0$ (or H_A: $\rho \neq \rho_0$), we proceed as above and obtain the UMP test of size α among all tests based on D as follows:

(i) H_A^*: $\rho < \rho_0$

The critical region of the size α test is $W^* = \{d \mid d < d^*\}$, where

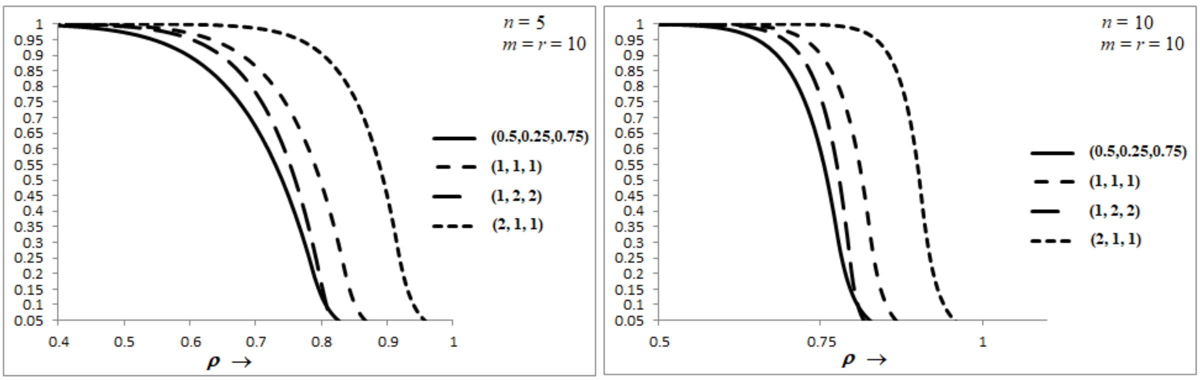
$$d^* = \rho_0\left[\alpha \frac{m(\alpha_2+1)+r(\alpha_3+1)}{C}\right]^{\frac{1}{m(\alpha_2+1)+r(\alpha_3+1)}}, \text{ if } \frac{C}{m(\alpha_2+1)+r(\alpha_3+1)} > \alpha$$

$$= \rho_0\left[(1-\alpha)\frac{n(\alpha_1+1)}{C}\right]^{-\frac{1}{n(\alpha_1+1)}}, \text{ if } \frac{C}{m(\alpha_2+1)+r(\alpha_3+1)} < \alpha. \quad (9)$$

And the power of the test is

$$\beta(\rho) = 1 - \frac{C}{n(\alpha_1+1)}\left(\frac{d^*}{\rho}\right)^{-n(\alpha_1+1)}], \text{ if } d^* \geq \rho$$

$$= \frac{C}{m(\alpha_2+1)+r(\alpha_2+1)}\left(\frac{d^*}{\rho}\right)^{m(\alpha_2+1)+r(\alpha_2+1)}, \text{ if } d^* < \rho.$$



(i) (ii)

Figure 2: Power curves of tests for testing H_0: $\xi = 0.5$ against $H_A: \xi > 0.5$ for some combinations of (α_1, α_2, α_3), when $\theta = 2$, $\theta_1 = 3$ and $\alpha = 0.05$

(ii) H_A: $\rho \neq \rho_0$

The critical region of the size α test is $W^* = \{d \mid d < d_1^* \text{ or } d > d_2^*\}$, where d_1^* and d_2^* ($d_1^* < d_2^*$) are determined from the size condition. Assuming $\Pr[D < d_1^* \mid H_0^*] = \Pr[D > d_2^* \mid H_0^*] = \alpha/2$, d_1^* is given by (9) and d_2^* by (8), after replacing α by $\alpha/2$ in each case.

The power of the test is:

$$\beta(\rho) = 1 - \frac{C}{n(\alpha_1+1)}\left(\frac{d_1^*}{\rho}\right)^{-n(\alpha_1+1)} + \frac{C}{n(\alpha_1+1)}\left(\frac{d_2^*}{\rho}\right)^{-n(\alpha_1+1)}, \text{ if } d_2^* > d_1^* \geq \rho$$

$$= 1 + \frac{C}{m(\alpha_2+1)+r(\alpha_2+1)}\left(\frac{d_1^*}{\rho}\right)^{m(\alpha_2+1)+r(\alpha_2+1)} - \frac{C}{m(\alpha_2+1)+r(\alpha_2+1)}\left(\frac{d_2^*}{\rho}\right)^{m(\alpha_2+1)+r(\alpha_2+1)},$$

$$\text{if } d_1^* < d_2^* < \rho$$

$$= \frac{C}{n(\alpha_1+1)}\left(\frac{d_2^*}{\rho}\right)^{-n(\alpha_1+1)} + \frac{C}{m(\alpha_2+1)+r(\alpha_2+1)}\left(\frac{d_1^*}{\rho}\right)^{m(\alpha_2+1)+r(\alpha_2+1)}, \text{ if } d_1^* < \rho < d_2^*.$$

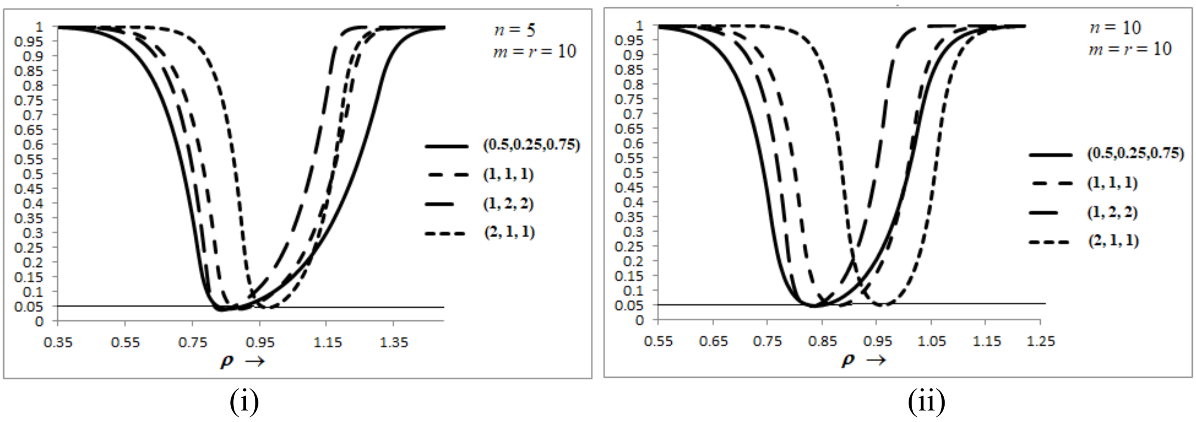


Figure 3: Power curves of tests for testing H$_0$: $\xi = 0.5$ against $\mathrm{H_A}: \xi \neq 0.5$ for some combinations of (α_1, α_2, α_3), when $\theta = 2$, $\theta_1 = 3$ and $\alpha = 0.05$

Figures 2 and 3 show that the one-sided tests are unbiased. But Figure 3 indicates that the suggested two-sided test is not unbiased. However, actual calculation shows that the power falls very slightly below the level of significance for some alternative values of ρ, so that the test may be regarded as an almost unbiased test.

6. A Simulation Study

Consider X ~ GU(1, 3), Y ~ GU(1, 2) and Z ~ GU(2, 2). To obtain the UMVUE of ξ, random samples of sizes $m = n = r = 20$ are taken on X, Y and Z respectively. The sample observations are as follows:

Variable	Sample observations									
X	2.1207	2.5148	1.6567	2.9090	2.1738	2.4589	1.2106	0.3185	1.6929	2.9582
	0.8061	2.4919	1.6372	2.8557	2.8237	2.9984	2.9964	0.9302	2.8248	0.1385
Y	1.3415	1.7143	1.0931	1.9151	1.9157	1.0229	0.8568	0.8378	1.6839	1.5374
	1.3346	1.3565	0.1223	1.7003	0.1179	1.9553	1.7329	1.9037	1.2483	0.9266
Z	1.1165	1.3229	1.2098	1.1777	1.8805	1.9025	1.8780	1.7501	1.9573	1.7706
	1.4741	1.9629	1.6615	1.4704	1.7707	1.8960	1.5299	1.6479	0.5454	0.9176

The UMVUE of the stress-strength reliability ξ (= 0.8589 $H_A: \xi < 0.5.$) is V = 0.8729, and the UMVUE of its variance is given by $\hat{Var}(V)$ = 0.0267.

Suppose we want to test the hypothesis H_0: $\xi = 0.5$ against H_A:ξ < 0.5. The size 0.05 UMP test rejects H_0 if observed D > 0.8942. For the given samples, the observed D is 0.6546. Hence, H_0 is accepted.

7. Discussion

The paper studies the UMVU estimator of the stress-strength reliability and its variance when there are two independent stresses acting on a system. The strength of the system and the stresses are assumed to be independent of one another, and follow generalized uniform distributions with known shape parameters, but unknown scale parameters. The UMVU estimator is obtained as a function of the ratio of the complete sufficient statistics of the scale parameters. Tests regarding the stress-strength reliability have been discussed, and the UMP test has been obtained among those based on this ratio. The study has been carried out assuming the scale parameters of the stress distributions to be equal. A natural extension would, therefore, be to assume the scale parameters to be completely unknown. Further, it would be interesting to extend the problem to the case of p (> 2) stresses.

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