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Power Generalized DUS Transformation of Inverse Kumaraswamy Distribution and Stress-Strength Analysis

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Abstract

Reliability analysis, including stress-strength analysis, for given data is more widely used in the reliability literature. A large number of new distributions are available, but many of them are not showing a good fit for the data under consideration. This inspires a researcher to introduce new lifetime distributions that demonstrate superior fitness in comparison to the existing distributions. So that more accurate reliability estimates can be obtained for the given data. The DUS transformation technique is widely used in reliability literature to create better models. Power generalized DUS(PGDUS) transformation to lifetime distributions, which is found to be useful to introduce more appropriate flexible distributions for the given data. Vinyl chloride data obtained from clean upgrading and monitoring wells in mg/L have been analyzed using DUS inverse Kumaraswamy (DUS IK), inverse Kumaraswamy (IK), and Weibull distributions. As a substitute for these distributions, this paper presents a new lifetime distribution employing PGDUS transformation, utilizing the inverse Kumaraswamy distribution as the baseline. The statistical properties of the proposed distribution are derived. The parameters of the proposed distribution are estimated using the maximum likelihood (ML) method, maximum product spacing (MPS), method of moment, and method of least squares. Additionally, Bayesian parameter estimates are acquired utilizing Lindley's approximation and the Metropolis-Hastings algorithm. The consistency of the model is verified using mean squared error (MSE) and biases, which are obtained based on simulated values. Then, the proposed distribution is compared with the DUS-IK, IK, and Weibull distributions. In this paper, single-component and multi-component stress-strength reliability analyses are also conducted.

Key words: PGDUS transformation; inverse-Kumaraswamy distribution; Stress-strength reliability.

1. Introduction

An appropriate lifetime distribution is essential to conducting reliability analysis with maximum accuracy. While using existing distributions, the fitness of the distributions for the

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given data is sometimes low. To overcome these problems, several researchers introduced new distributions with more fitness characteristics. Appropriate distributions are necessary for the stress-strength analysis in statistics and reliability engineering. Reliability distributions have different failure rate properties, like increasing failure rate, decreasing failure rate, bathtub and upside-down bathtub distributions, *etc.*

There are numerous ways to suggest new distributions in the statistical literature by using some baseline distributions without incorporating scale, shape, or location parameters, so that more appropriate statistical distributions can be made available in the statistical literature. DUS transformation is one of several methods (see Kumar *et al.* (2015)). Generalizing this DUS transformation will lead to the introduction of new distributions, which could be used while dealing with reliability analysis of parallel systems with components having DUS-transformed distributions.

Kumaraswamy (1980) introduced the Kumaraswamy distribution, which is also known as a beta-like distribution due to its similarity with the beta distribution in the sense that both have the same basic shape parameter. But the probability density function (pdf), cumulative distribution function (CDF), and quantile function are in closed form, which makes Kumaraswamy distribution a more practical choice for many applications, including modeling of biomedical data, reliability engineering, finance, hydrology, *etc.* (see Kumaraswamy (1976)) over Beta distribution.

Nowadays, many researchers focus on the inverse transformation of probability distributions and their applications, which proves the increase in model flexibility. Abd Al-Fattah *et al.* (2017) introduced the inverted Kumaraswamy (IK) distribution by introducing a transformation

$$U = \frac{1-X}{X},$$

where $X \sim$ Kumaraswamy(α, β).
Iqbal (2017) generalized the IK distribution using a power transformation as

$$T = U^{\gamma},$$

where $U \sim$ IK distribution, called generalized IK distribution. All monotonic and non-monotonic failure rate patterns exhibits for this model. Jamal *et al.* (2019) proposed a new generator function based on the IK distribution and introduced a generalized IK-G family of distributions.

The DUS transformation approach was proposed by Kumar *et al.* (2015), utilizing a few baseline distributions that are sparse in computation and interpretation since they only ever contain the parameter(s) included in the baseline distribution. Let $h(u)$ and $H(u)$ be the pdf and CDF of the baseline distribution, then the pdf $g(u)$ and CDF $G(u)$ of the distribution obtained by the DUS transformation of the baseline distribution are given by

$$g(u) = \frac{1}{e-1} h(u) e^{H(u)}$$
$$G(u) = \frac{1}{e-1}\left(e^{H(u)} - 1\right)$$

Maurya *et al.* (2016) proposed the DUS transformation of the Lindley distribution, and Tripati *et al.* (2019) introduced the DUS transformation of the exponential distribution. Deepthi and Chacko (2020) introduced the DUS transformation of the Lomax distribution, which is an upside-down bathtub-shaped failure rate model. Gauthami and Chacko (2021) proposed the DUS inverse-Weibull distribution, which is also an upside-down bathtub-shaped failure rate model. Anakha and Chacko (2022) introduced a non-monotonic hazard rate distribution using the DUS transformation with the IK distribution as the baseline distribution.

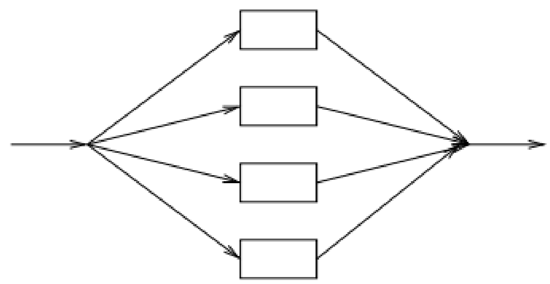

Figure 1: Parallel system

While considering a parallel system, for example, as shown in Figure 1, where each component is distributed to any DUS-transformed baseline distribution. Then the resulting distribution of parallel systems has to be investigated in detail. In order to address this problem, Thomas and Chacko (2021) introduced a method called exponentiation of DUS transformation, called PGDUS transformation, and introduced the PGDUS-Exponential distribution with exponential as the baseline distribution. Weibull and Lomax distributions are used by Thomas and Chacko (2023) to introduce new distributions using PGDUS transformation.

This paper introduces a new lifetime distribution for a system with components connected in parallel in which each of the components follows the DUS transformation of the IK distribution to study the distributions having monotone and non-monotone failure rate functions.

Consider a random variable U with pdf $h(u)$ and CDF $H(u)$. Then the pdf $q(u)$ and CDF $Q(u)$ of the PGDUS-IK(α, β, λ) distribution can be obtained as

$$q(u) = \frac{\lambda}{(e-1)^{\lambda}}(e^{H(u)} - 1)^{\lambda-1} e^{H(u)} h(u), \qquad \lambda > 0, \quad u > 0 \tag{1}$$

and

$$Q(u) = \left(\frac{e^{H(u)} - 1}{e - 1}\right)^{\lambda}, \qquad \lambda > 0, \quad u > 0. \tag{2}$$

respectively.

Similarly, the failure rate function of the PGDUS-IK distribution can be written as

$$r(u) = \frac{\lambda h(u) e^{H(u)} (e^{H(u)} - 1)^{\lambda - 1}}{(e-1)^{\lambda} - (e^{H(u)} - 1)^{\lambda}}, \qquad \lambda > 0, \quad u > 0. \tag{3}$$

This paper is divided into 10 sections: the PGDUS-IK distribution is proposed in Section 2. In Section 3, a detailed investigation into the properties of the PGDUS-IK distribution is undertaken. Section 4 discusses the mean residual life function of the PGDUS-IK distribution. In Section 5, the estimation of parameters for the proposed distribution has been done using the methods of maximum likelihood (ML), maximum product spacing (MPS), moments, and least squares. Also, bayesian estimators of α, β, and λ based on the squared error loss function, by taking gamma priors, are derived. The asymptotic confidence interval and bootstrap confidence interval for the unknown parameters of PGDUS-IK are derived in Section 6. The efficacy of the proposed estimators is investigated in terms of their bias and mean squared error (MSE) values in Section 7. Section 8 illustrates the applications of proposed estimators using the vinyl chloride data given in Bhaumik *et al.* (2009). In Section 9, stress-strength reliability for single components and for multi-components for the proposed distribution is investigated. A Simulation study to investigate and compare the performance of the reliability estimators is conducted, and data analysis for estimating single component and multi-component reliability is given, in the same section. Conclusions are provided in Section 10.

2. Power generalized DUS transformation of inverse-Kumaraswamy distribution

Kumaraswamy (1980) introduced the Kumaraswamy (K) distribution, which is empirically useful for a wide range of reliability applications. The pdf of the K distribution is given as

$$f(y; \alpha, \beta) = \alpha\beta y^{\alpha-1}(1 - y^{\alpha})^{\beta-1}, \qquad 0 < y < 1, \ \ \alpha > 0, \ \ \beta > 0. \tag{4}$$

IK distribution has the following pdf, CDF, and failure rate function

$$h(u) = \alpha\beta(1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta-1}, \quad u > 0\ , \ \ \alpha > 0, \ \ \beta > 0, \tag{5}$$

$$H(u; \alpha, \beta) = (1-(1+u)^{-\alpha})^{\beta}, \quad u > 0\ , \ \ \alpha > 0, \ \ \beta > 0, \tag{6}$$

and

$$r(u) = \frac{\alpha\beta(1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta}}{1-(1-(1+u)^{-\alpha})^{\beta}}, \qquad u > 0, \ \ \alpha > 0, \ \ \beta > 0 \tag{7}$$

respectively.

The DUS-IK distribution with pdf and CDF can be defined as

$$g(u) = \frac{\alpha\beta}{e-1}(1+u)^{-(\alpha+1)}\Big(1-(1+u)^{-\alpha}\Big)^{\beta-1} e^{(1-(1+u)^{-\alpha})^{\beta}}, \quad u > 0, \ \ \alpha > 0, \ \ \beta > 0, \tag{8}$$

and

$$G(u) = \frac{e^{(1-(1+u)^{-\alpha})^{\beta}} - 1}{e-1}, \quad u > 0, \ \ \alpha > 0, \quad \beta > 0 \tag{9}$$

respectively. The survival function will be

$$\bar{G}(u) = \frac{e - e^{(1-(1+u)^{-\alpha})^{\beta}}}{e-1}, \quad \text{for} \quad u > 0,\ \alpha > 0, \quad \beta > 0. \tag{10}$$

PGDUS-IK(α, β, λ)

By using the PGDUS transformation to IK distribution, the pdf, CDF, and failure rate functions can be written as

$$q(u) = \frac{\alpha\beta\lambda}{(e-1)^{\lambda}}(1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta-1}e^{(1-(1+u)^{-\alpha})^{\beta}}(e^{(1-(1+u)^{-\alpha})^{\beta}}-1)^{\lambda-1}, \tag{11}$$

$$Q(u) = \left(\frac{e^{(1-(1+u)^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda}, \tag{12}$$

and

$$r(u) = \frac{\alpha\beta\lambda(1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta-1}e^{(1-(1+u)^{-\alpha})^{\beta}}(e^{(1-(1+u)^{-\alpha})^{\beta}}-1)^{\lambda-1}}{(e-1)^{\lambda}-(e^{(1-(1+u)^{-\alpha})^{\beta}}-1)^{\lambda}} \tag{13}$$

respectively, where, $u > 0, \quad \lambda > 0, \quad \alpha > 0, \quad \beta > 0.$

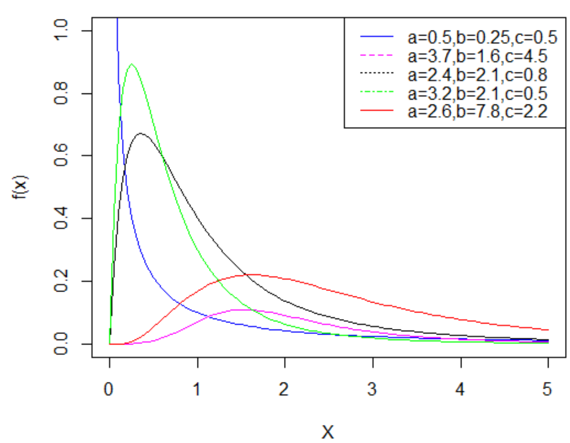

Figure 2: pdf plot for PGDUS-IK distribution

The PGDUS-IK(α, β, λ) distribution has both monotonic and non-monotonic hazard rates.

3. Statistical properties

Statistical properties are discussed in this section.

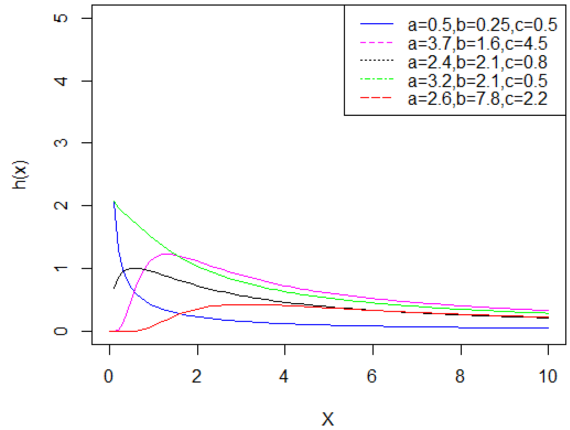

Figure 3: Failure rate plot of PGDUS-IK distribution

3.1. Moments

The rth raw moment of the PGDUS-IK(α, β, λ) distribution can be derived as follows

$$\mu_r^1 = E(U^r)$$
$$= \int_0^\infty u^r \frac{\alpha\beta\lambda}{(e-1)^\lambda}(1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta-1}e^{(1-(1+u)^{-\alpha})^\beta}(e^{(1-(1+u)^{-\alpha})^\beta}-1)^{\lambda-1}du$$

Substitute $a = (1-(1+u)^{-\alpha})^\beta$ in the above integral,

$$\mu_r^1 = \int_0^1 \frac{\lambda}{(e-1)^\lambda}e^a(e^a-1)^{\lambda-1}\left(\left(1-a^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}-1\right)^r da$$
$$= \frac{\lambda}{(e-1)^\lambda}\int_0^1 e^a \sum_{s=0}^{\lambda-1}(-1)^s \binom{\lambda-1}{s}(e^a)^{\lambda-s-1}\sum_{z=0}^{r}(-1)^{r-z}\binom{r}{z}\left(\left(1-a^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}\right)^s da$$
$$= \frac{\lambda}{(e-1)^\lambda}\sum_{s=0}^{\lambda-1}\sum_{z=0}^{r}(-1)^{s+r-z}\binom{\lambda-1}{s}\binom{r}{z}\int_0^1\left(1-a^{\frac{1}{\beta}}\right)^{-\frac{s}{\alpha}}e^{a(\lambda-s)}da$$
$$= \frac{\beta\lambda}{(e-1)^\lambda}\sum_{s=0}^{\lambda-1}\sum_{z=0}^{r}\sum_{l=0}^{\infty}\frac{(-1)^{s+r-z+l}}{l!}\binom{\lambda-1}{s}\binom{r}{z}(\lambda-s)^l\int_0^1 a^l\left(1-a^{\frac{1}{\beta}}\right)^{-\frac{s}{\alpha}}da$$
$$= \frac{\beta\lambda}{(e-1)^\lambda}\sum_{s=0}^{\lambda-1}\sum_{z=0}^{r}\sum_{l=0}^{\infty}\frac{(-1)^{s+r-z+l}}{l!}\binom{\lambda-1}{s}\binom{r}{z}(\lambda-s)^l\beta\left(1-\frac{s}{\alpha}, \beta l+\beta\right). \tag{14}$$

By putting $r = 1, 2, ...$, we get the corresponding raw moments as

$$\mu_1^1 = \frac{\beta\lambda}{(e-1)^\lambda}\sum_{s=0}^{\lambda-1}\sum_{l=0}^{\infty}\{(-1)^{s+l+1}+(-1)^{s+l}\}\binom{\lambda-1}{s}\frac{(\lambda-s)^l}{l!}\beta\left(1-\frac{s}{\alpha}, \beta l+\beta\right). \tag{15}$$

$$\mu_2^1 = \frac{\beta\lambda}{(e-1)^\lambda}\sum_{s=0}^{\lambda-1}\sum_{l=0}^{\infty}\{(-1)^{s+l+2}+2.(-1)^{s+l+1}+(-1)^{s+l}\}\binom{\lambda-1}{s}\frac{(\lambda-s)^l}{l!}$$
$$\beta\left(1-\frac{s}{\alpha}, \beta l+\beta\right). \tag{16}$$

3.2. Moment generating function

Let $U \sim$ PGDUS-IK(α, β, λ), it's moments generating function is derived as

$$\begin{aligned}
M_U(t) &= E(e^{tu}) \\
&= \int_0^\infty e^{tu} \frac{\alpha\beta\lambda}{(e-1)^\lambda}(1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta-1} e^{(1-(1+u)^{-\alpha})^\beta}(e^{(1-(1+u)^{-\alpha})^\beta}-1)^{\lambda-1} du \\
&= \frac{\alpha\beta\lambda}{(e-1)^\lambda}\sum_{j=0}^\infty \frac{t^j}{j!} \\
&\quad \int_0^\infty u^j (1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta-1} e^{(1-(1+u)^{-\alpha})^\beta}(e^{(1-(1+u)^{-\alpha})^\beta}-1)^{\lambda-1} du. \qquad (17)
\end{aligned}$$

Substituting $a = (1-(1+u)^{-\alpha})^\beta$ and by solving, we get

$$M_U(t) = \frac{\beta\lambda}{(e-1)^\lambda}\sum_{j=0}^\infty \frac{t^j}{j!}\sum_{s=0}^{\lambda-1}\sum_{z=0}^{j}\sum_{l=0}^\infty \frac{(-1)^{s+j-z+l}}{l!}\binom{\lambda-1}{s}\binom{j}{z}(\lambda-s)^l \beta\Big(1-\frac{s}{\alpha}, \beta l+\beta\Big). \qquad (18)$$

3.3. Characteristic function

The characteristic function of the distribution is derived as

$$\phi_U(t) = \frac{\beta\lambda}{(e-1)^\lambda}\sum_{j=0}^\infty \sum_{s=0}^{\lambda-1}\sum_{z=0}^{j}\sum_{l=0}^\infty \frac{(-1)^{s+j-z+l}}{j!l!}\binom{\lambda-1}{s}\binom{j}{z}(it)^j(\lambda-s)^l \beta\Big(1-\frac{s}{\alpha}, \beta l+\beta\Big), \qquad (19)$$

where $i = \sqrt{-1}$.

3.4. Cumulant generating function

The cumulant generating function of the distribution is derived as

$$\begin{aligned}
K_U(t) &= \log \phi_U(t) \\
&= \log\left(\frac{\beta\lambda}{(e-1)^\lambda}\sum_{j=0}^\infty \sum_{s=0}^{\lambda-1}\sum_{z=0}^{j}\sum_{l=0}^\infty \frac{(-1)^{s+j-z+l}}{j!l!}\binom{\lambda-1}{s}\binom{j}{z}(it)^j(\lambda-s)^l \right. \\
&\qquad \left. \beta\Big(1-\frac{s}{\alpha}, \beta l+\beta\Big)\right) \\
&= \log \frac{\beta\lambda}{(e-1)^\lambda} \\
&\quad + \log\left(\sum_{j=0}^\infty \sum_{s=0}^{\lambda-1}\sum_{z=0}^{j}\sum_{l=0}^\infty \frac{(-1)^{s+j-z+l}}{j!l!}\binom{\lambda-1}{s}\binom{j}{z}(it)^j(\lambda-s)^l \beta\Big(1-\frac{s}{\alpha}, \beta l+\beta\Big)\right), \qquad (20)
\end{aligned}$$

where $i = \sqrt{-1}$.

3.5. Quantile function

The i^{th} quantile function, denoted by $D(i)$, of PGDUS-IK(α,β,λ) distribution is obtained by solving

$$Q(D(i)) = i, \qquad 0 < i < 1.$$

That is,

$$\left(\frac{e^{(1-(1+D)^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda} = i$$

By solving this, the quantile function of the distribution is obtained as

$$D(i) = \left(1-\left(\log(1+i^{\frac{1}{\lambda}}(e-1))\right)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - 1, \qquad i\epsilon(0,1). \tag{21}$$

Median of PGDUS-IK(α,β,λ) can be derived by substituting $i=\frac{1}{2}$ in $D(i)$. That is,

$$Median = \left(1-\left(\log(1+0.5^{\frac{1}{\lambda}}(e-1))\right)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - 1. \tag{22}$$

Similarly, the inter-quartile range (IQR) of the distribution is,

$$IQR = \left(1-\left[\log(1+0.75^{\frac{1}{\lambda}}(e-1))\right]^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - \left(1-\left[\log(1+0.25^{\frac{1}{\lambda}}(e-1))\right]^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}}. \tag{23}$$

3.6. Order statistics

Let $U_{(1)}, U_{(2)}, ..., U_{(l)}$ be the order statistics for the random sample $U = (U_1, U_2, ..., U_l)$ taken from PGDUS-IK(α,β,λ). The pdf and CDF are given as

$$\begin{aligned} q_{(r)}(u) &= \frac{l!}{(r-1)!(l-r)!} q(u)(Q(u))^{r-1}(1-Q(u))^{l-r} \\ &= \frac{l!\alpha\beta\lambda}{(r-1)!(l-r)!}\left((e-1)^{\lambda} - (e^{(1-(1+u)^{-\alpha})^{\beta}}-1)^{\lambda}\right)^{l-r} \\ &\quad \frac{(1+u)^{-(\alpha+1)}(1-(1+u)^{-\alpha})^{\beta-1}e^{(1-(1+u)^{-\alpha})^{\beta}}(e^{(1-(1+u)^{-\alpha})^{\beta}}-1)^{\lambda r-1}}{(e-1)^{l\lambda}} \end{aligned} \tag{24}$$

and

$$\begin{aligned} Q_{(r)}(u) &= \sum_{s=r}^{l}\binom{l}{s}(Q(u))^{s}(1-Q(u))^{l-s} \\ &= \sum_{l=r}^{l}\binom{l}{s}\left(\frac{e^{(1-(1+u)^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda s}\left(1-\left(\frac{e^{(1-(1+u)^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda}\right)^{l-s}, \end{aligned} \tag{25}$$

respectively. Substituting $r=1$ and $r=l$ into equations (24) and (25) allows us to derive the pdf and the CDF of the 1^{st} and l^{th} order statistics, respectively.

3.7. Entropy

Renyi entropy is derived as

$$\tau_R(\zeta) = \frac{1}{1-\zeta}\log\Big(\int q^{\zeta}(u)du\Big), \quad \zeta > 0, \quad \zeta \neq 1.$$

$$\begin{aligned}
\int_0^{\infty} q^{\zeta}(u)du = {} & \left(\frac{\alpha\beta\lambda}{(e-1)^{\lambda}}\right)^{\zeta} \\
& \textstyle\int_0^{\infty}(1+u)^{-\zeta(\alpha+1)}(1-(1+u)^{-\alpha})^{\zeta(\beta-1)}e^{\zeta(1-(1+u)^{-\alpha})^{\beta}}(e^{(1-(1+u)^{-\alpha})^{\beta}}-1)^{\zeta(\lambda-1)}du \\
& = \left(\frac{\alpha\beta\lambda}{(e-1)^{\lambda}}\right)^{\zeta}\sum_{s=0}^{\zeta(\lambda-1)}(-1)^{\zeta(\lambda-1)-s}\binom{\zeta(\lambda-1)}{s} \\
& \quad \int_0^{\infty} e^{s(1-(1+u)^{-\alpha})^{\beta}}(1+u)^{-\zeta(\alpha+1)}(1-(1+u)^{-\alpha})^{\zeta(\beta-1)}e^{\zeta(1-(1+u)^{-\alpha})^{\beta}}du \\
& = \left(\frac{\alpha\beta\lambda}{(e-1)^{\lambda}}\right)^{\zeta}\sum_{z=0}^{\infty}\sum_{s=0}^{\zeta(\lambda-1)}\frac{(-1)^{\zeta(\lambda-1)-s}}{z!}\binom{\zeta(\lambda-1)}{s}(\zeta+s)^{z} \\
& \quad \int_0^{\infty}(1+u)^{-\zeta(\alpha+1)}(1-(1+u)^{-\alpha})^{\zeta(\beta-1)+\beta z}du.
\end{aligned}$$

Using the transformation $a = 1-(1+u)^{-\alpha}$,

$$\begin{aligned}
\int_0^{\infty} q^{\zeta}(u)du = {} & \left(\frac{\alpha\beta\lambda}{(e-1)^{\lambda}}\right)^{\zeta}\frac{1}{\alpha}\sum_{z=0}^{\infty}\sum_{s=0}^{\zeta(\lambda-1)}\frac{(-1)^{\zeta(\lambda-1)-s}}{z!}\binom{\zeta(\lambda-1)}{s}(\zeta+s)^{z} \\
& \beta\Big(\zeta(\beta-1)+\beta z+1, \zeta\Big(1+\frac{1}{\alpha}\Big)-\frac{1}{\alpha}\Big).
\end{aligned}$$

Then Renyi entropy form will be

$$\begin{aligned}
\tau_R(\zeta) = {} & \frac{1}{1-\zeta}\log\left(\frac{\alpha\beta\lambda}{(e-1)^{\lambda}}\right)^{\zeta}\frac{1}{\alpha}\sum_{z=0}^{\infty}\sum_{s=0}^{\zeta(\lambda-1)}\frac{(-1)^{\zeta(\lambda-1)-s}}{z!}\binom{\zeta(\lambda-1)}{s}(\zeta+s)^{z} \\
& \beta\Big(\zeta(\beta-1)+\beta z+1, \zeta(1+\frac{1}{\alpha})-\frac{1}{\alpha}\Big) \\
= {} & \frac{1}{1-\zeta}\log\left(\frac{\alpha\beta\lambda}{(e-1)^{\lambda}}\right)^{\zeta}\frac{1}{\alpha}+\frac{1}{1-\zeta}\log\left(\sum_{z=0}^{\infty}\sum_{s=0}^{\zeta(\lambda-1)}\frac{(-1)^{\zeta(\lambda-1)-s}}{z!}\binom{\zeta(\lambda-1)}{s}(\zeta+s)^{z}\right. \\
& \left.\beta\Big(\zeta(\beta-1)+\beta z+1, \zeta(1+\frac{1}{\alpha})-\frac{1}{\alpha}\Big)\right)
\end{aligned} \tag{26}$$

where $\alpha > 0, \beta > 0, \lambda > 0, \zeta > 0, \zeta \neq 1$.

4. Mean residual life function

The mean residual life function at age ν is defined as the expected remaining life given survival at age ν and it is expressed as

$$\begin{aligned} MRL(\nu) &= \frac{1}{\bar{Q}(\nu)} \int_{\nu}^{\infty} u dQ(u) - \nu \\ &= \frac{\beta\lambda}{(e-1)^{\lambda} - (e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1)^{\lambda}} \sum_{s=0}^{\lambda-1} \sum_{l=0}^{\infty} \{(-1)^{s+l+1} + (-1)^{s+l}\} \binom{\lambda-1}{s} \frac{(\lambda-s)^l}{l!} \\ &\qquad \beta\Big(1 - \frac{s}{\alpha}, \beta l + \beta\Big) - \nu. \end{aligned} \tag{27}$$

5. Estimation

To estimate the unknown parameters, methods of maximum likelihood, maximum product spacing, moments, and least squares are described below. Let $U = (U_1, U_2, ..., U_l)$ be a random sample of size l taken from PGDUS-IK(α, β, λ).

5.1. Maximum likelihood estimation

To obtain the maximum likelihood estimate (MLE) of unknown parameters α, β, and λ, consider

$$\begin{aligned} LF(u) &= \prod_{i=1}^{l} q(u) \\ &= \prod_{i=1}^{l} \frac{\alpha\beta\lambda}{(e-1)^{\lambda}} (1+u_i)^{-(\alpha+1)} (1-(1+u_i)^{-\alpha})^{\beta-1} e^{(1-(1+u_i)^{-\alpha})^{\beta}} (e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1)^{\lambda-1} \\ &= \left(\frac{\alpha\beta\lambda}{(e-1)^{\lambda}}\right)^{l} \prod_{i=1}^{l} (1+u_i)^{-(\alpha+1)} (1-(1+u_i)^{-\alpha})^{\beta-1} e^{(1-(1+u_i)^{-\alpha})^{\beta}} (e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1)^{\lambda-1}, \end{aligned} \tag{28}$$

the likelihood function and its logarithm will be

$$\begin{aligned} \log LF(u) = l\Bigg(\log\alpha + \log\beta + \log\lambda - \lambda\log(e-1)\Bigg) - (\alpha+1)\sum_{i=1}^{l} \log(1+u_i) \\ + (\beta-1)\sum_{i=1}^{l} \log(1-(1+u_i)^{-\alpha}) + \sum_{i=1}^{l} (1-(1+u_i)^{-\alpha})^{\beta} \\ + (\lambda-1)\sum_{i=1}^{l} \log\Big(e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1\Big). \end{aligned}$$

To obtaine the MLEs, we find the first-order derivative of $\log L$ and equate it with zero.

$$\begin{aligned}\frac{\partial \log LF}{\partial \alpha} &= \frac{l}{\alpha} - \sum_{i=1}^{l} \log(1+u_i) + (\beta-1)\sum_{i=1}^{l} \frac{(1+u_i)^{-\alpha}\log(1+u_i)}{(1-(1+u_i)^{-\alpha})} \\ &+ \beta \sum_{i=1}^{l} (1+u_i)^{-\alpha} \log(1+u_i)(1-(1+u_i)^{-\alpha})^{\beta-1} \\ &+ \beta(\lambda-1)\sum_{i=1}^{l} \frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}\log(1+u_i)e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1} = 0\end{aligned} \tag{29}$$

$$\begin{aligned}\frac{\partial \log LF}{\partial \beta} &= \frac{l}{\beta} + \sum_{i=1}^{l} \log(1-(1+u_i)^{-\alpha}) + \sum_{i=1}^{l} (1-(1+u_i)^{-\alpha})^{\beta} \log(1-(1+u_i)^{-\alpha}) \\ &+ (\lambda-1)\sum_{i=1}^{l} \frac{(1-(1+u_i)^{-\alpha})^{\beta}\log(1-(1+u_i)^{-\alpha})e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1} = 0\end{aligned} \tag{30}$$

$$\frac{\partial \log LF}{\partial \lambda} = \frac{l}{\lambda} - l\log(e-1) + \sum_{i=1}^{l} \log\left(e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1\right) = 0. \tag{31}$$

To solve equations (29), (30), (31) simultaneously, statistical software has to be used.

5.2. Maximum product spacing estimation

The maximum product spacing (MPS) estimation method was introduced by Cheng and Amin (1983) and explored in detailed by Ranneby (1984). The MPS estimation method ensures consistent estimators whether the MLE method exists or not.

To find the MPS estimators of α, β, and λ, first define the spacings

$$D_i = Q(u_i, \alpha, \beta, \lambda) - Q(u_{i-1}, \alpha, \beta, \lambda); i = 1, 2, ..., l+1.$$

Hence, MPS estimators are nothing but parameter values that maximize the geometric mean of the spacings obtained from the observed samples. That is,

$$\begin{aligned}A &= \left(\prod_{i=1}^{l+1} D_i\right)^{1/l+1} \\ &= \left(\prod_{i=1}^{l+1}\left(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda} - \left(\frac{e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda}\right)^{1/l+1}.\end{aligned} \tag{32}$$

$$\log A = \frac{1}{l+1}\sum_{i=1}^{l+1} \log\left(\left(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda} - \left(\frac{e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1}{e-1}\right)^{\lambda}\right).$$

$$\frac{\partial \log A}{\partial \alpha} = \frac{\beta\lambda}{l+1}\sum_{i=1}^{l+1}\Bigg(\frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log(1+u_i)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}-(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda}}$$
$$(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda-1}-(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda-1}$$
$$\frac{(1+u_{i-1})^{-\alpha}(1-(1+u_{i-1})^{-\alpha})^{\beta-1}e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}\log(1+u_{i-1})}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}-(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda}}\Bigg), \tag{33}$$

$$\frac{\partial \log A}{\partial \beta} = \frac{\lambda}{l+1}\sum_{i=1}^{l+1}\Bigg(\frac{(1-(1+u_i)^{-\alpha})^{\beta}e^{(1-(1+u_i)^{-\alpha})^{\beta}}(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda-1}\log(1-(1+u_i)^{-\alpha})}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}-(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda}}$$
$$-\frac{(1-(1+u_{i-1})^{-\alpha})^{\beta}e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda-1}\log(1-(1+u_{i-1})^{-\alpha})}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}-(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda}}\Bigg), \tag{34}$$

and

$$\frac{\partial \log A}{\partial \lambda} = \frac{1}{l+1}\sum_{i=1}^{l+1}\Bigg(\frac{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}\log\left(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}{e-1}\right)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}-(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda}}$$
$$-\frac{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}\log\left(\frac{e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1}{e-1}\right)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^{\lambda}-(e^{(1-(1+u_{i-1})^{-\alpha})^{\beta}}-1)^{\lambda}}\Bigg). \tag{35}$$

Setting the equations (33), (34) and (35) to zero, and solving simultaneously we get the MPS estimates of α, β, and λ. It is easy to obtain estimates using R software by numerical methods.

5.3. Method of moment estimation

The r^{th} order moment of PGDUS-IK(α, β, λ) is

$$\mu_r^1 = \frac{\beta\lambda}{(e-1)^{\lambda}}\sum_{s=0}^{\lambda-1}\sum_{z=0}^{r}\sum_{l=0}^{\infty}\frac{(-1)^{s+r-z+l}}{l!}\binom{\lambda-1}{s}\binom{r}{z}(\lambda-s)^{l}\beta\Big(1-\frac{s}{\alpha}, \beta l+\beta\Big)$$

Taking $r = 1, 2$, and 3 we get first 3 raw moments of the PGDUS-IK distribution. Then, by equating these raw moments to corresponding sample moments, we get

$$\mu_1^1 = \frac{1}{l}\sum_{i=1}^{l} u_i \tag{36}$$

$$\mu_2^1 = \frac{1}{l}\sum_{i=1}^{l} u_i^2 \tag{37}$$

$$\mu_3^1 = \frac{1}{l}\sum_{i=1}^{l} u_i^3 \tag{38}$$

and solving these equations (36), (37), (38) simultaneously we get moment estimators. Statistical software can be used to solve these equations.

5.4. Method of least square estimation

The least-square estimators for the parameters in PGDUS-IK(α, β, λ) can be derived as follows:

$$LS = \sum_{i=1}^{l} \Big(Q(u_i) - \mathcal{Q}_i\Big)^2.$$

where, $Q(U_i)$ - theoretical CDF of the observation u_i
and $\mathcal{Q}_i$ - empirical CDF which is usually estimated by

$$\hat{\mathcal{Q}}_i = \frac{i}{l+1}.$$

There for,

$$LS = \sum_{i=1}^{l} \left(\left(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1}{e-1}\right)^{\lambda} - \frac{i}{l+1}\right)^2.$$

$$\frac{\partial LS}{\partial \alpha} = 0 => \sum_{i=1}^{l}(1+u_i)^{-\alpha}\log(1+u_i)\Big(1-(1+u_i)^{-\alpha}\Big)^{\beta-1} e^{(1-(1+u_i)^{-\alpha})^{\beta}}\Big(e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1\Big)^{\lambda-1} \left(\Big(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1}{e-1}\Big)^{\lambda} - \frac{i}{l+1}\right) = 0. \tag{39}$$

$$\frac{\partial LS}{\partial \beta} = 0 => \sum_{i=1}^{l}\Big(1-(1+u_i)^{-\alpha}\Big)^{\beta}\log(1-(1+u_i)^{-\alpha}) e^{(1-(1+u_i)^{-\alpha})^{\beta}}\Big(e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1\Big)^{\lambda-1} \left(\Big(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1}{e-1}\Big)^{\lambda} - \frac{i}{l+1}\right) = 0. \tag{40}$$

$$\frac{\partial LS}{\partial \lambda} = 0 => \sum_{i=1}^{l}\Big(e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1\Big)^{\lambda}\log\left(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1}{e-1}\right)\left(\Big(\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1}{e-1}\Big)^{\lambda} - \frac{i}{l+1}\right) = 0. \tag{41}$$

Solving (39), (40), and (41) simultaneosuly with respect to α, β and λ gives the least squares estimators. By using statistical softwares, we can find estimated values.

5.5. Bayesian analysis

The joint posterior density function of (α, β, λ) can be written as

$$\Phi(\alpha, \beta, \lambda|U) = \frac{L(\alpha, \beta, \lambda|U)w(\alpha, \beta, \lambda)}{\int_\alpha \int_\beta \int_\lambda L(\alpha, \beta, \lambda|U)w(\alpha, \beta, \lambda)d\alpha d\beta d\lambda},$$

where $w(\alpha, \beta, \lambda)$ is the joint prior density function of the parameters.

Then, the Bayes estimator under the squared error loss function is

$$I(U) = \hat{z}_B = E_{\underline{\theta}|U}(z(\underline{\theta})) = \frac{\int_{\underline{\theta}} z(\underline{\theta})L(\underline{\theta}|U)w(\underline{\theta})d\underline{\theta}}{\int_{\underline{\theta}} L(\underline{\theta}|U)w(\underline{\theta})d\underline{\theta}}.$$

There is no easy closed form for this estimated value since it involves an integral ration.

Lindley (1980) proposed the procedure to approximate the ratio of the two integrals. For a three-parameter distribution, Lindley's approximation can be written as (see Ali and Kanani (2021))

$$\begin{aligned} I(U) = v + (v_1\theta_1 + v_2\theta_2 + v_3\theta_3 + \theta_4 + \theta_5) + \frac{1}{2}\Big(B_1(v_1\sigma_{11} + v_2\sigma_{12} + v_3\sigma_{13})\Big) \\ + \frac{1}{2}\Big(B_2(v_1\sigma_{21} + v_2\sigma_{22} + v_3\sigma_{23})\Big) + \frac{1}{2}\Big(B_3(v_1\sigma_{31} + v_2\sigma_{32} + v_3\sigma_{33})\Big). \end{aligned} \tag{42}$$

where,

$$\begin{aligned} B_1 &= \sigma_{11}M_{111} + 2\sigma_{12}M_{121} + 2\sigma_{13}M_{131} + 2\sigma_{23}M_{231} + \sigma_{22}M_{221} + \sigma_{33}M_{331} \\ B_2 &= \sigma_{11}M_{112} + 2\sigma_{12}M_{122} + 2\sigma_{13}M_{132} + 2\sigma_{23}M_{232} + \sigma_{22}M_{222} + \sigma_{33}M_{332} \\ B_3 &= \sigma_{11}M_{113} + 2\sigma_{12}M_{123} + 2\sigma_{13}M_{133} + 2\sigma_{23}M_{233} + \sigma_{22}M_{223} + \sigma_{33}M_{333} \\ v_1 &= \frac{\partial v(\alpha, \beta, \lambda)}{\partial \alpha}, \qquad v_2 = \frac{\partial v(\alpha, \beta, \lambda)}{\partial \beta}, \qquad v_3 = \frac{\partial v(\alpha, \beta, \lambda)}{\partial \lambda} \\ v_{11} &= \frac{\partial^2 v(\alpha, \beta, \lambda)}{\partial^2 \alpha}, \qquad v_{22} = \frac{\partial^2 v(\alpha, \beta, \lambda)}{\partial^2 \beta}, \qquad v_{33} = \frac{\partial^2 v(\alpha, \beta, \lambda)}{\partial^2 \lambda} \end{aligned}$$

where M - the logarithm of the likelihood function. Then

$$\begin{aligned} M_1 &= \frac{\partial M}{\partial \alpha}, M_2 = \frac{\partial M}{\partial \beta}, \quad M_3 = \frac{\partial M}{\partial \lambda} \\ M_{ij} &= \frac{\partial^2 M}{\partial \tau_i \partial \tau_j}, \quad i, j = 1, 2, 3, \quad (\tau_i, \tau_j) = (\alpha, \beta, \lambda) \\ M_{ijk} &= \frac{\partial^3 M}{\partial \tau_i \partial \tau_j \partial \tau_k}, \quad (i, j, k) = 1, 2, 3, \quad (\tau_i, \tau_j, \tau_k) = (\alpha, \beta, \lambda) \\ \sigma_{ij} &= -\frac{1}{M_{ij}} \\ \theta_i &= \rho_1\sigma_{i1} + \rho_2\sigma_{i2} + \rho_3\sigma_{i3}, \qquad i = 1, 2, 3. \\ \theta_4 &= v_{12}\sigma_{12} + v_{13}\sigma_{13} + v_{23}\sigma_{23}, \quad \theta_5 = \frac{1}{2}(v_{11}\sigma_{11} + v_{22}\sigma_{22} + v_{33}\sigma_{33}). \\ \rho &= \log(w(\alpha, \beta, \lambda)), \qquad \rho_1 = \frac{\partial \rho}{\partial \alpha}, \qquad \rho_2 = \frac{\partial \rho}{\partial \beta}, \qquad \rho_3 = \frac{\partial \rho}{\partial \lambda} \end{aligned}$$

The detailed derivations of equation (42) are given in the appendix.

When information is unavailable for the parameters we use non-informative prior, like Uniform prior, where

$$w(\underline{\theta}) \propto 1.$$

Since the parameter ranges from 0 to ∞, we can choose the gamma distribution as the prior distribution. Therefore,

$$w(\underline{\theta}) \propto \alpha^{r-1}\beta^{p-1}\lambda^{t-1}e^{-(\alpha s+\beta q+\lambda v)}$$

Then the Bayes estimators of the parameters become

$$\hat{\alpha}_B = \hat{\alpha} + \theta_1 + \frac{1}{2}\Big(B_1\sigma_{11} + B_2\sigma_{21} + B_3\sigma_{31}\Big). \tag{43}$$

$$\hat{\beta}_B = \hat{\beta} + \theta_2 + \frac{1}{2}\Big(B_1\sigma_{12} + B_2\sigma_{22} + B_3\sigma_{32}\Big). \tag{44}$$

$$\hat{\lambda}_B = \hat{\lambda} + \theta_3 + \frac{1}{2}\Big(B_1\sigma_{13} + B_2\sigma_{23} + B_3\sigma_{33}\Big). \tag{45}$$

Metropolis-Hasting algorithm

The Metropolis-Hasting (MH) algorithm (see Tobias(2014)), a general Markov Chain Monte Carlo (MCMC) technique, is used to generate samples from models that are complicated. Metropolis *et al.* (1953) developed it initially, then Hastings (1970) developed it afterwards. The MH algorithm, for sampling from a target distribution, let it be π, and let $q(\theta_1^*|\theta_2, ..., \theta_k, \underline{x})$ denotes a proposal density that generates a candidate θ_1^*.

Algorithm:

The MH algorithm is used to simulate a probability distribution p from another probability distribution q, which is easier to simulate. Here p is called target distribution and q is the proposal. Let $\theta^{(t)}$ be the current draw from $p(\theta)$. The MH algorithm performs as follows:

1. Draw θ^* from $q(\theta|\theta^{(1)})$.
2. Accept $\theta^{(t+1)} = \theta^*$ with the probability $\min(1, p^*)$ where

$$p^* = \frac{p(\theta^*)q(\theta^{(t)}|\theta^*)}{p(\theta^{(t)})q(\theta^*|\theta^{(t)})}.$$

Otherwise, set $\theta^{(t+1)} = \theta^{(t)}$.

That is, accepting with the probability $\min(1, p^*)$ means that we will be drawing u according to a uniform distribution on (0,1), and if $u < \min(1, p^*)$, then accept θ^* is accepted; otherwise, it's not.

In the Bayesian context, the MH algorithm can be defined as follows: For that, the posterior distribution will be the form

$$p(\theta|y) \propto LF(y|\theta)w(\theta).$$

where LF is the likelihood function and w is the prior distribution. The MH algorithm can be used to simulate $p(\theta|y)$, by using $t(\theta|y) = LF(y|\theta) * w(\theta)$ and a proposal distribution $q(\theta_1|\theta_2, y)$, as follows.

1. Draw θ^* from $f(\theta|\theta^{(t)}, y)$.
2. Accept $\theta^{(t+1)} = \theta^*$ with the probability $\min(1, p^*)$, where

$$p^* = \frac{q(\theta^*|y)g(\theta^{(t)}|\theta^*, y)}{q(\theta^{(t)}|y)g(\theta^*|\theta^{(t)}, y)}.$$

Otherwise set $\theta^{(t+1)} = \theta^{(t)}$.

6. Confidence interval

In this section, we propose the asymptotic confidence interval and the bootstrap confidence interval, for the unknown parameters α, β, and λ of the PGDUS-IK distribution.

6.1. Asymptotic confidence interval

The asymptotic confidence intervals can be used when the MLEs are not in the closed form. Let us consider the Fisher information matrix I as

$$I = E\begin{bmatrix} \frac{-\partial^2 \log LF}{\partial\alpha^2} & \frac{-\partial^2 \log LF}{\partial\alpha\partial\beta} & \frac{-\partial^2 \log LF}{\partial\alpha\partial\lambda} \\ \frac{-\partial^2 \log LF}{\partial\alpha\partial\beta} & \frac{-\partial^2 \log LF}{\partial\beta^2} & \frac{-\partial^2 \log LF}{\partial\beta\partial\lambda} \\ \frac{-\partial^2 \log LF}{\partial\alpha\partial\lambda} & \frac{-\partial^2 \log LF}{\partial\beta\partial\lambda} & \frac{-\partial^2 \log LF}{\partial\lambda^2} \end{bmatrix}.$$

The second partial derivative of $\log LF$ is briefly given in the appendix.

The asymptotic distribution of MLEs $\tau = (\alpha, \beta, \lambda)$ is normal, with mean zero and variance-covariance matrix I^{-1}. That is,

$$l(\hat{\tau} - \tau) \to N(0, I^{-1}).$$

Hence, the asymptotic $100(1-\eta)\%$ confidence interval of α, β, and λ are

$$\hat{\alpha} \pm z_{\eta/2}\sqrt{Variance(\hat{\alpha})},$$
$$\hat{\beta} \pm z_{\eta/2}\sqrt{Variance(\hat{\beta})},$$

and

$$\hat{\lambda} \pm z_{\eta/2}\sqrt{Variance(\hat{\lambda})},$$

respectively.

6.2. Bootstrap confidence interval

The bootstrap method is a powerful statistical technique used for estimating the sampling distribution of a statistic by resampling with a replacement from the observed data. Let $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ be the MLEs of parameters α, β and λ. Here we discussed the bootstrap percentile (Boot-p) confidence interval.

To do that, we need to generate a number (let B) of independent bootstrap samples from $u_1, u_2, ..., u_l$, and it is denoted as $u^*_{i1}, u^*_{i2}, ..., u^*_{il}$, for $i = 1, 2, ..., B$. Then, for each bootstrap sample, we calculated the MLEs of α, β, and λ, and the bootstrap MLEs are denoted as $\hat{\alpha}^*, \hat{\beta}^*$, and $\hat{\lambda}^*$, respectively.

Boot-p method

Let $\hat{Q}_1, \hat{Q}_2$, and $\hat{Q}_3$ be the CDF of $\hat{\alpha}^*, \hat{\beta}^*, \hat{\lambda}^*$ respectively. Then $(1-\eta)\%$ percentile confidence intervals are

$$\left(\hat{Q}_1^{-1}(\eta/2), \hat{Q}_1^{-1}(1-\eta/2)\right),$$
$$\left(\hat{Q}_2^{-1}(\eta/2), \hat{Q}_2^{-1}(1-\eta/2)\right),$$
and
$$\left(\hat{Q}_3^{-1}(\eta/2), \hat{Q}_3^{-1}(1-\eta/2)\right)$$

respectively.

7. Simulation study

In this section, the simulation study is used to examine the performance of estimators of PGDUS-IK(α, β, λ) distribution parameters.

By using the quantile function, a random sample of the PGDUS-IK(α, β, λ) distribution can be simulated by using

$$U = \left(1 - \left(\log(1 + j^{\frac{1}{\lambda}}(e-1))\right)^{\frac{1}{\beta}}\right)^{-\frac{1}{\alpha}} - 1, \qquad 0 < j < 1$$

where j from $U(0,1)$.

Here, differnt values of the sample size, $l = 50, 100, 200, 300,$ and 400 are considered and replicated 1000 times. The performance of MLE, MPS, and Bayes estimators of each parameter is examined using their biases and MSE values(see Table 1). Bayes estimators are obtained only by using informative prior gamma under the squared error loss function. It is observed that, biases and MSE values decreases to zero as sample size l increases.

8. Application

This section compares the PGDUS-IK distribution to DUS-IK distribution, IK distribution, and Weibull distribution. For that, we are using a vinyl chloride data obtained from clean upgrading monitoring wells in mg/L by Bhaumik *et al.* (2009) (Table 2).

A number of factors, including the p-value, log-likelihood value, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Kolmogrov-Smirnov (K-S) statistic can be applied to compare statistical models in order to evaluate which one has a better relative goodness-of-fit with the data. Lower K-S statistic, AIC, and BIC values indicate greater correspondence between the observed data and the model. Additionally, higher p-values and log-likelihood values indicate a stronger fit between the model and the observed data. If a single criterion consistently favors one model over another, that model is likely the better choice.

Based on the table values (see Table 3), compared to the other distributions described, PGDUS-IK(α, β, λ) possesses the lowest AIC, BIC and KS-statistic values moreover

Table 1: Simulation study for MLE, MPS and Bayes estimation for the values $\alpha = 0.4, \beta = 0.8, \lambda = 0.3$

method	l	bias($\hat{\alpha}$)	bias($\hat{\beta}$)	bias($\hat{\lambda}$)	MSE($\hat{\alpha}$)	MSE($\hat{\beta}$)	MSE($\hat{\lambda}$)
MLE	50	0.03124	0.26774	0.32454	0.01357	0.53269	4.10082
	100	0.01458	0.16442	0.14788	0.00618	0.31273	0.67138
	200	0.00782	0.06756	0.15049	0.00285	0.18506	0.43009
	300	0.00497	0.04771	0.08236	0.00187	0.12485	0.14573
	400	0.00307	0.03249	0.07471	0.00136	0.10573	0.13432
MPS	50	0.12090	1.08723	0.00376	0.03668	2.46970	0.47450
	100	0.06039	0.73472	-0.02447	0.01127	1.16509	0.13256
	200	0.03159	0.45171	-0.01279	0.00437	0.56686	0.07379
	300	0.01998	0.36702	-0.01325	0.00256	0.39957	0.04617
	400	0.01542	0.28923	-0.00465	0.00178	0.30120	0.03975
Bayesian	50	9.37736	2.68987	2.7458	0.00936	7.23541	7.53963
	100	0.03769	2.20796	0.49767	0.007187	4.87509	0.24768
	200	0.03386	1.03768	0.65245	0.00413	1.07678	0.42570
	300	0.03319	0.9113	0.6275	0.002381	0.83049	0.39383
	400	0.021983	0.2	-0.69736	0.00156	0.096756	0.48632

Table 2: Vinyl Chloride data

5.1	1.2	1.3	0.6	0.5	2.4
0.5	1.1	8.0	0.8	0.4	0.4
0.6	0.9	0.4	2.0	0.5	1.2
5.3	3.2	2.7	2.9	2.5	0.2
2.3	1.0	0.2	0.1	0.1	1.8
0.9	2.0	4.0	6.8		

a high log-likelihood value and p-value by the MLE, MPS, and Bayesian methods. We can therefore conclude that the PGDUS-IK distribution performs better than the given existing distribution for modeling a parallel system.

In Table 4, the estimated parameter values (based on the ML method) along with their 95% confidence interval, based on 1000 bootstrap samples for vinyl chloride data (see Table 2), are given.

9. Stress-Strength reliability(SSR)

Single-component SSR

Let U indicate the strength of a component or system that is subjected to a random stress, V. The system's functioning is then defined by stress-strength reliability. If U and V are distributed as PGDUS-IK$(\alpha, \beta, \lambda_1)$ and PGDUS-IK$(\alpha, \beta, \lambda_2)$, respectively, then stress-strength reliability is defined as

$$R = P(V < U) = \int_0^\infty q_U(u) Q_V(u) du$$

$$= \frac{\alpha\beta\lambda_1}{(e-1)^{\lambda_1+\lambda_2}} \int_0^\infty (1+u)^{-(\alpha+1)}\Big(1-(1+u)^{-\alpha}\Big)^{\beta-1} e^{(1-(1+u)^{-\alpha})^\beta}\left(e^{(1-(1+u)^{-\alpha})^\beta}-1\right)^{\lambda_1+\lambda_2-1} du.$$

Table 3: Data Analysis

Distribution		Estimates	KS Statistic	Log(L)	p-value	AIC	BIC
PGDUS IK	MLE	2.0103 5.9354 0.3584	0.0884	-55.4280	0.953	114.856	117.9088
	MPS	1.7439 2.1148 0.9072	0.1229	-56.1242	0.6834	116.2484	119.3011
	Bayesian	2.0004 4.2964 0.5078	0.08707	-55.5094	0.9588	115.0187	118.0715
DUS-IK	MLE	1.9467 1.8296	0.0892	-55.5702	0.9497	115.1403	118.193
	MPS	1.7365 1.8928	0.1244	-56.5598	0.6692	117.1196	120.1723
	Bayesian	2.2306 2.8658	0.1569	-57.1064	0.3725	118.2127	121.2654
IK	MLE	1.7409 2.1059	0.0966	-55.7707	0.909	115.5414	118.5941
	MPS	1.5286 2.1388	0.1136	-59.4084	0.7729	122.8169	125.8696
	Bayesian	1.9060 2.9559	0.1409	-57.00978	0.5095	118.0194	121.0721
Weibull	MLE	1.0102 1.8879	0.0918	-55.4496	0.9366	114.8992	117.952
	MPS	1.1075 2.2840	0.1735	-105.4977	0.2577	214.9953	218.0481
	Bayesian	0.8033 1.5418	0.16938	-56.9383	0.2835	117.8766	120.9294

Take $a = \left(e^{(1-(1+u)^{-\alpha})^{\beta}} - 1\right)^{\lambda_1+\lambda_2}$, hence the stress-strength reliability becomes

$$R = \frac{\lambda_1}{\lambda_1 + \lambda_2}, \qquad \lambda_1 > 0, \quad \lambda_2 > 0. \tag{46}$$

To evaluate the reliability value, we need to estimate the parameters first.

Multi-component SSR

Let's consider a system comprising identical d components, which operates successfully if at least c $(1 \leq c \leq d)$ of these components survive a shared random stress. This

Table 4: Estimate value and 95 % bootstrap CI of Vinyl Chloride data

Method		α	β	λ
MLE	Estimate	2.0103428	5.9354142	0.3584253
	CI	(1.623967,2.791699)	(1.981512,23.06395)	(0.12504,1.286034)

situation is called multi-component systems. Bhattacharya and Johnson (1974) first studied the multi-component stress-strength reliability system and defined the reliability of a multi-component stress-strength model as

$$R_{c,d} = Pr\{\text{at least } \ c \ \text{ of } \ \text{ the } \ (U_1, U_2, ..., U_d) \ \text{ exceed } \ V\}. \tag{47}$$

Let $U = (U_1, U_2, ..., U_d)$ be random strength variables from PGDUS-IK$(\alpha, \beta, \lambda_1)$ with CDF $H(u)$, and V be the random stress variable from PGDUS-IK$(\alpha, \beta, \lambda_2)$ with CDF $Q(v)$. Then the reliability of a multi-component stress-strength model defined by Bhattacharya and Johnson (1974) is given as

$$\begin{aligned}
R_{c,d} &= \sum_{i=c}^{d} \binom{d}{i} \int_{-\infty}^{\infty} (1 - H(u))^i (H(u))^{d-i} dQ(u) \\
&= \sum_{i=c}^{d} \binom{d}{i} \int_{0}^{\infty} \frac{\alpha\beta\lambda_2}{(e-1)^{\lambda_2}} (1+u)^{-(\alpha+1)} \Big(1 - (1+u)^{-\alpha}\Big)^{\beta-1} e^{(1-(1+u)^{-\alpha})^{\beta}} \\
&\quad \left(e^{(1-(1+u)^{-\alpha})^{\beta}} - 1\right)^{\lambda_2 - 1} \left(\left(\frac{e^{(1-(1+u)^{-\alpha})^{\beta}} - 1}{e-1}\right)^{\lambda_1}\right)^{d-i} \left(1 - \left(\frac{e^{(1-(1+u)^{-\alpha})^{\beta}} - 1}{e-1}\right)^{\lambda_1}\right)^{i} du \\
&= \sum_{i=c}^{d} \binom{d}{i} \int_{0}^{\infty} \frac{\alpha\beta\lambda_2}{(e-1)^{\lambda_2}} (1+u)^{-(\alpha+1)} \Big(1 - (1+u)^{-\alpha}\Big)^{\beta-1} e^{(1-(1+u)^{-\alpha})^{\beta}} \\
&\quad \left(e^{(1-(1+u)^{-\alpha})^{\beta}} - 1\right)^{\lambda_1(d-i)+\lambda_2-1} \left((e-1)^{\lambda_1} - \left(e^{(1-(1+u)^{-\alpha})^{\beta}} - 1\right)^{\lambda_1}\right)^{i} du \\
&= \sum_{i=c}^{d} \sum_{p=0}^{i} \binom{d}{i} \binom{i}{p} \frac{(-1)^p \alpha\beta\lambda_2}{(e-1)^{\lambda_1(d+p-i)+\lambda_2}} \\
&\quad \int_{0}^{\infty} (1+u)^{-(\alpha+1)} \Big(1 - (1+u)^{-\alpha}\Big)^{\beta-1} e^{(1-(1+u)^{-\alpha})^{\beta}} \left(e^{(1-(1+u)^{-\alpha})^{\beta}} - 1\right)^{\lambda_1(d+p-i)+\lambda_2-1} du \\
&= \sum_{i=c}^{d} \sum_{p=0}^{i} \binom{d}{i} \binom{i}{p} \frac{(-1)^p \lambda_2}{\lambda_1(d+p-i)+\lambda_2}.
\end{aligned}$$

That is,

$$R_{c,d} = \sum_{i=c}^{d} \sum_{p=0}^{i} \binom{d}{i} \binom{i}{p} \frac{(-1)^p \lambda_2}{\lambda_1(d+p-i)+\lambda_2}, \qquad \lambda_1 > 0, \quad \lambda_2 > 0. \tag{48}$$

Suppose $U = (U_1, U_2, ..., U_d)$ are parallelly connected, then $c = 1$ and $R_{c,d}$ will become

$$R_{1,d} = \sum_{i=1}^{d} \sum_{p=0}^{i} \binom{d}{i} \binom{i}{p} \frac{(-1)^p \lambda_2}{\lambda_1(d+p-i)+\lambda_2}, \qquad \lambda_1 > 0, \quad \lambda_2 > 0.$$

Similarly, when $U = (U_1, U_2, ..., U_d)$ are connected in series, so $c = d$ and

$$R_{d,d} = \sum_{p=0}^{d} \binom{d}{p} \frac{(-1)^p \lambda_2}{\lambda_1 p + \lambda_2}, \qquad \lambda_1 > 0, \quad \lambda_2 > 0.$$

9.1. Estimation of reliability

To obtain the estimates of both single-component SSR and multi-component SSR, we need to get the respective parameter estimates. Hence, here we are using the ML method. Let $U = (U_1 < U_2 < ... < U_l)$ and $V = (V_1 < V_2 < ... < V_z)$ be the random samples from PGDUS-IK$(\alpha, \beta, \lambda_1)$ and PGDUS-IK$(\alpha, \beta, \lambda_2)$, respectively.

9.1.1. Estimation of R

The likelihood function for the observed samples for $\underline{\theta} = (\alpha, \beta, \lambda_1, \lambda_2)$ can be written as follows:

$$\begin{aligned}
&LF(u, v, \underline{\theta})\\
&= \prod_{i=1}^{l} \left(\frac{\alpha\beta\lambda_1}{(e-1)^{\lambda_1}} (1+u_i)^{-(\alpha+1)} (1-(1+u_i)^{-\alpha})^{\beta-1} e^{(1-(1+u_i)^{-\alpha})^{\beta}} (e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1)^{\lambda_1 - 1} \right)\\
&\quad \prod_{j=1}^{z} \left(\frac{\alpha\beta\lambda_2}{(e-1)^{\lambda_2}} (1+v_j)^{-(\alpha+1)} (1-(1+v_j)^{-\alpha})^{\beta-1} e^{(1-(1+v_j)^{-\alpha})^{\beta}} (e^{(1-(1+v_j)^{-\alpha})^{\beta}} - 1)^{\lambda_2 - 1} \right)\\
&= \left(\frac{\alpha\beta\lambda_1}{(e-1)^{\lambda_1}} \right)^{l} \prod_{i=1}^{l} (1+u_i)^{-(\alpha+1)} (1-(1+u_i)^{-\alpha})^{\beta-1} e^{(1-(1+u_i)^{-\alpha})^{\beta}} (e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1)^{\lambda_1 - 1}\\
&\quad \left(\frac{\alpha\beta\lambda_2}{(e-1)^{\lambda_2}} \right)^{z} \prod_{j=1}^{z} (1+v_j)^{-(\alpha+1)} (1-(1+v_j)^{-\alpha})^{\beta-1} e^{(1-(1+v_j)^{-\alpha})^{\beta}} (e^{(1-(1+v_j)^{-\alpha})^{\beta}} - 1)^{\lambda_2 - 1}.
\end{aligned}$$

Then,

$$\begin{aligned}
\log LF = {} & (l+z)\Big(\log\alpha + \log\beta\Big) + l\log\lambda_1 + z\log\lambda_2 - (l\lambda_1 + z\lambda_2)\log(e-1)\\
& - (\alpha+1)\Bigg(\sum_{i=1}^{l} \log(1+u_i) + \sum_{j=1}^{z} \log(1+v_j) \Bigg) + (\beta-1)\Bigg(\sum_{i=1}^{l} \log(1-(1+u_i)^{-\alpha})\\
& + \sum_{j=1}^{z} \log(1-(1+v_j)^{-\alpha}) \Bigg) + \sum_{i=1}^{l} (1-(1+u_i)^{-\alpha})^{\beta} + \sum_{j=1}^{z} (1-(1+v_j)^{-\alpha})^{\beta}\\
& + (\lambda_1 - 1)\sum_{i=1}^{l} \log\left(e^{(1-(1+u_i)^{-\alpha})^{\beta}} - 1 \right) + (\lambda_2 - 1)\sum_{j=1}^{z} \log\left(e^{(1-(1+v_j)^{-\alpha})^{\beta}} - 1 \right).
\end{aligned}$$

Compute the partial derivatives of the $logLF$ with respect to the parameters α, β, λ_1, and λ_2, respectively. That is,

$$\begin{aligned}
\frac{\partial \log LF}{\partial \alpha} = {} & \frac{l+z}{\alpha} - \Bigg(\sum_{i=1}^{l} \log(1+u_i) + \sum_{j=1}^{z} \log(1+v_j) \Bigg)\\
& + (\beta-1)\Bigg(\sum_{i=1}^{l} \frac{(1+u_i)^{-\alpha}\log(1+u_i)}{(1-(1+u_i)^{-\alpha})} + \sum_{j=1}^{z} \frac{(1+v_j)^{-\alpha}\log(1+v_j)}{(1-(1+v_j)^{-\alpha})} \Bigg)\\
& + \beta\Bigg(\sum_{i=1}^{l} (1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}\log(1+u_i)
\end{aligned} \tag{49}$$

$$+\sum_{j=1}^{z}(1+v_j)^{-\alpha}(1-(1+v_j)^{-\alpha})^{\beta-1}\log(1+v_j)$$
$$+(\lambda_1-1)\sum_{i=1}^{l}\frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}\log(1+u_i)e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}$$
$$+(\lambda_2-1)\sum_{j=1}^{z}\frac{(1+v_j)^{-\alpha}(1-(1+v_j)^{-\alpha})^{\beta-1}\log(1+v_j)e^{(1-(1+v_j)^{-\alpha})^{\beta}}}{e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1}\Bigg),$$

$$\frac{\partial\log LF}{\partial\beta}=\frac{l+z}{\beta}+\left(\sum_{i=1}^{l}\log(1-(1+u_i)^{-\alpha})+\sum_{j=1}^{z}\log(1-(1+v_j)^{-\alpha})\right)$$
$$+\textstyle\sum_{i=1}^{l}(1-(1+u_i)^{-\alpha})^{\beta}\log(1-(1+u_i)^{-\alpha})+\sum_{j=1}^{z}(1-(1+v_j)^{-\alpha})^{\beta}\log(1-(1+v_j)^{-\alpha})$$
$$+(\lambda_1-1)\left(\sum_{i=1}^{l}\frac{(1-(1+u_i)^{-\alpha})^{\beta}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log(1-(1+u_i)^{-\alpha})}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}\right)$$
$$+(\lambda_2-1)\left(\sum_{j=1}^{z}\frac{(1-(1+v_j)^{-\alpha})^{\beta}e^{(1-(1+v_j)^{-\alpha})^{\beta}}\log(1-(1+v_j)^{-\alpha})}{e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1}\right),$$
$$\frac{\partial\log LF}{\partial\lambda_1}=\frac{l}{\lambda_1}-l\log(e-1)+\sum_{i=1}^{l}\log\left(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1\right),$$
$$\frac{\partial\log LF}{\partial\lambda_2}=\frac{z}{\lambda_2}-z\log(e-1)+\sum_{j=1}^{z}\log\left(e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1\right).$$

Then, the MLEs of $\alpha,\ \beta,\ \lambda_1$, and λ_2 can be determined by solving the following equations::

$$\frac{\partial\log LF}{\partial\alpha}=0,\frac{\partial\log LF}{\partial\beta}=0,\frac{\partial\log LF}{\partial\lambda_1}=0,\frac{\partial\log LF}{\partial\lambda_2}=0.$$

Substituting the estimated values for α and β, we get the MLE of λ_1 and λ_2 as

$$\hat{\lambda_1}=\frac{l}{l\log(e-1)-\sum_{i=1}^{l}(\log(e^{(1-(1+u_i)^{-\hat{\alpha}})^{\hat{\beta}}})-1)},$$

$$\hat{\lambda_2}=\frac{z}{z\log(e-1)-\sum_{j=1}^{z}(\log(e^{(1-(1+v_j)^{-\hat{\alpha}})^{\hat{\beta}}})-1)}.$$

Hence, the MLE of R will be

$$\hat{R}=\frac{\hat{\lambda_1}}{\hat{\lambda_1}+\hat{\lambda_2}},\qquad \lambda_1>0\quad \lambda_2>0. \tag{50}$$

9.1.2. Estimation of $R_{c,d}$

To compute the MLE of multi-component reliability, $R_{c,d}$, assume that $U_{i1},U_{i2},...,U_{id}$ and $V_i,\ i=1,2,...,l$ denote the observed data obtained using PGDUS-IK(α,β,λ_1) with pdf

$h(u)$ and PGDUS-IK$(\alpha, \beta, \lambda_2)$ with pdf $q(v)$, respectively. The likelihood function can be defined as:

$$
\begin{aligned}
LF_{c,d}(u, v, \underline{\theta}) &= \prod_{i=1}^{l} \Big(\prod_{j=1}^{d} h(u_{ij}) \Big) q(v_i) \\
&= \frac{(\alpha\beta)^{l(d+1)} \lambda_1^{ld} \lambda_2^{l}}{(e-1)^{l(d\lambda_1+\lambda_2)}} \prod_{i=1}^{l} \Bigg(\prod_{j=1}^{d} (1+u_{ij})^{-(\alpha+1)} (1-(1+u_{ij})^{-\alpha})^{\beta-1} e^{(1-(1+u_{ij})^{-\alpha})^{\beta}} \\
&\quad \Big(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}} - 1 \Big)^{\lambda_1 - 1} \Bigg) (1+v_i)^{-(\alpha+1)} (1-(1+v_i)^{-\alpha})^{\beta-1} e^{(1-(1+v_i)^{-\alpha})^{\beta}} \\
&\quad \Big(e^{(1-(1+v_i)^{-\alpha})^{\beta}} - 1 \Big)^{\lambda_2 - 1}.
\end{aligned}
$$

The logarithm likelihood function will be

$$
\begin{aligned}
&\log LF_{c,d} \\
&= l(d+1)\Big(\log\alpha + \log\beta \Big) + ld \log\lambda_1 + l\log\lambda_2 - l(d\lambda_1+\lambda_2)\log(e-1) \\
&- (\alpha+1)\Bigg(\sum_{i=1}^{l}\sum_{j=1}^{d} \log(1+u_{ij}) + \sum_{i=1}^{l} \log(1+v_i) \Bigg) + (\beta-1)\Bigg(\sum_{i=1}^{l}\sum_{j=1}^{d} \log(1-(1+u_{ij})^{-\alpha}) \\
&+ \sum_{i=1}^{l} \log(1-(1+v_i)^{-\alpha}) \Bigg) + \sum_{i=1}^{l}\sum_{j=1}^{d} (1-(1+u_{ij})^{-\alpha})^{\beta} + \sum_{i=1}^{d} (1-(1+v_i)^{-\alpha})^{\beta} \\
&+ (\lambda_1 - 1)\sum_{i=1}^{l}\sum_{j=1}^{d} \log\Big(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}} - 1 \Big) + (\lambda_2 - 1)\sum_{i=1}^{l} \log\Big(e^{(1-(1+v_i)^{-\alpha})^{\beta}} - 1 \Big).
\end{aligned}
$$

Consider the partial derivative of the $logLF_{c,d}$ with respect to the parameters and solving them by equating to zero, we can obtain the MLEs of the unknown parameters α, β, λ_1 and λ_2, respectively. That is,

$$
\frac{\partial \log LF_{c,d}}{\partial \alpha} = 0, \frac{\partial \log LF_{c,d}}{\partial \beta} = 0, \frac{\partial \log LF_{c,d}}{\partial \lambda_1} = 0, \frac{\partial \log LF_{c,d}}{\partial \lambda_2} = 0
$$

where

$$
\begin{aligned}
\frac{\partial \log LF_{c,d}}{\partial \alpha} &= \frac{l(d+1)}{\alpha} - \Bigg(\sum_{i=1}^{l}\sum_{j=1}^{d} \log(1+u_{ij}) + \sum_{i=1}^{l} \log(1+v_i) \Bigg) \\
&\quad + (\beta-1)\Bigg(\sum_{i=1}^{l}\sum_{j=1}^{d} \frac{(1+u_{ij})^{-\alpha}\log(1+u_{ij})}{(1-(1+u_{ij})^{-\alpha})} + \sum_{i=1}^{l} \frac{(1+v_i)^{-\alpha}\log(1+v_i)}{(1-(1+v_i)^{-\alpha})} \Bigg) \\
&\quad + \beta\Bigg(\sum_{i=1}^{l}\sum_{j=1}^{d} (1+u_{ij})^{-\alpha}(1-(1+u_{ij})^{-\alpha})^{\beta-1}\log(1+u_{ij}) \\
&\qquad + \sum_{i=1}^{l} (1+v_i)^{-\alpha}(1-(1+v_i)^{-\alpha})^{\beta-1}\log(1+v_i)
\end{aligned}
\tag{51}
$$

$$+ (\lambda_1 - 1)\sum_{i=1}^{l}\sum_{j=1}^{d}\frac{(1+u_{ij})^{-\alpha}(1-(1+u_{ij})^{-\alpha})^{\beta-1}\log(1+u_{ij})e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}}{e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1}$$
$$+ (\lambda_2 - 1)\sum_{i=1}^{l}\frac{(1+v_i)^{-\alpha}(1-(1+v_i)^{-\alpha})^{\beta-1}\log(1+v_i)e^{(1-(1+v_i)^{-\alpha})^{\beta}}}{e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1}\Bigg).$$

$$\begin{aligned}\frac{\partial \log LF_{c,d}}{\partial \beta} &= \frac{l(d+1)}{\beta} + \left(\sum_{i=1}^{l}\sum_{j=1}^{d}\log(1-(1+u_{ij})^{-\alpha}) + \sum_{i=1}^{l}\log(1-(1+v_i)^{-\alpha})\right)\\ &+ \sum_{i=1}^{l}\sum_{j=1}^{d}(1-(1+u_{ij})^{-\alpha})^{\beta}\log(1-(1+u_{ij})^{-\alpha})\\ &+ \sum_{i=1}^{l}(1-(1+v_i)^{-\alpha})^{\beta}\log(1-(1+v_i)^{-\alpha})\\ &+ (\lambda_1-1)\left(\sum_{i=1}^{l}\sum_{j=1}^{d}\frac{(1-(1+u_{ij})^{-\alpha})^{\beta}e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}\log(1-(1+u_{ij})^{-\alpha})}{e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1}\right)\\ &+ (\lambda_2-1)\left(\sum_{i=1}^{l}\frac{(1-(1+v_i)^{-\alpha})^{\beta}e^{(1-(1+v_i)^{-\alpha})^{\beta}}\log(1-(1+v_i)^{-\alpha})}{e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1}\right).\\ \frac{\partial \log LF_{c,d}}{\partial \lambda_1} &= \frac{ld}{\lambda_1} - ld\log(e-1) + \sum_{i=1}^{l}\sum_{j=1}^{d}\log\left(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1\right).\\ \frac{\partial \log LF_{c,d}}{\partial \lambda_2} &= \frac{l}{\lambda_2} - l\log(e-1) + \sum_{i=1}^{l}\log\left(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1\right).\end{aligned}$$

By substituting the MLEs of α and β, the MLEs of λ_1 and λ_2 will be in the form

$$\hat{\lambda_1} = \frac{ld}{ld\log(e-1) + \sum_{i=1}^{l}\sum_{j=1}^{d}\log\left(e^{(1-(1+u_{ij})^{-\hat{\alpha}})^{\hat{\beta}}-1}\right)},$$

and

$$\hat{\lambda_2} = \frac{l}{l\log(e-1) + \sum_{i=1}^{l}\log\left(e^{(1-(1+v_i)^{-\hat{\alpha}})^{\hat{\beta}}-1}\right)}.$$

Hence, the MLE of stress-strength reliability of the multi-component system will be

$$\hat{R_{c,d}} = \sum_{i=c}^{d}\sum_{p=0}^{i}\binom{d}{i}\binom{i}{p}\frac{(-1)^p\hat{\lambda_2}}{\hat{\lambda_1}(d+p-i)+\hat{\lambda_2}}, \qquad \lambda_1 > 0, \quad \lambda_2 > 0.$$

9.2. Asymptotic distribution

This section discusses the asymptotic distribution of R and $R_{c,d}$ by using their MLEs.

9.2.1. Asymptotic distribution of R

The asymptotic distribution of MLE of R is normal with mean zero and variance-covariance matrix $I^{-1}(\underline{\theta})$. That is,

$$\sqrt{l+o}(\hat{R}-R) \rightarrow N(0, G^T I^{-1}(\underline{\theta}) G) \tag{52}$$

where G^T - $\left(\frac{\partial R}{\partial \alpha}, \frac{\partial R}{\partial \beta}, \frac{\partial R}{\partial \lambda_1}, \frac{\partial R}{\partial \lambda_2}\right)$

$I^{-1}(\underline{\theta})$ - inverse of Fisher information matrix for unknown parameters

$$I(\underline{\theta}) = E\begin{bmatrix} \frac{-\partial^2 \log LF}{\partial \alpha^2} & \frac{-\partial^2 \log LF}{\partial \alpha \partial \beta} & \frac{-\partial^2 \log LF}{\partial \alpha \partial \lambda_1} & \frac{-\partial^2 \log LF}{\partial \alpha \partial \lambda_2} \\ \frac{-\partial^2 \log LF}{\partial \alpha \partial \beta} & \frac{-\partial^2 \log LF}{\partial \beta^2} & \frac{-\partial^2 \log LF}{\partial \beta \partial \lambda_1} & \frac{-\partial^2 \log LF}{\partial \beta \partial \lambda_2} \\ \frac{-\partial^2 \log LF}{\partial \alpha \partial \lambda_1} & \frac{-\partial^2 \log LF}{\partial \beta \partial \lambda_1} & \frac{-\partial^2 \log LF}{\partial \lambda_1^2} & \frac{-\partial^2 \log LF}{\partial \lambda_1 \partial \lambda_2} \\ \frac{-\partial^2 \log LF}{\partial \alpha \partial \lambda_2} & \frac{-\partial^2 \log LF}{\partial \beta \partial \lambda_2} & \frac{-\partial^2 \log LF}{\partial \lambda_1 \partial \lambda_2} & \frac{-\partial^2 \log LF}{\partial \lambda_2^2} \end{bmatrix}$$

Second order partial derivative of the log-likelihood function with respect to each parameters $\alpha,\ \beta,\ \lambda_1,$ and λ_2 are briefly derived and given in the appendix. Due to the complexity of the expectations, an approximate estimation of the variance-covariance matrix of $(\alpha, \beta, \lambda_1, \lambda_2)$ is $I^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda_1}, \hat{\lambda_2})$, where $\hat{\alpha},\ \hat{\beta},\ \hat{\lambda_1},$ and $\hat{\lambda_2}$ are the estimates of the respective parameters. From Eq.(45), we can obtain the approximate estimate of the variance of $\hat{R}$ as

$$Vari\hat{a}nce(\hat{R}) \simeq G^T I^{-1} G.$$

Thus,

$$\frac{(\hat{R}-R)}{\sqrt{Vari\hat{a}nce(\hat{R})}} \sim N(0,1).$$

This yields the asymptotic $100(1-\eta)\%$ confidence interval for R as

$$\hat{R} \pm Z_{\eta/2}\sqrt{Vari\hat{a}nce(\hat{R})}$$

where $\hat{R}$ is the MLE of R and $Z_{\eta/2}$ is the upper $(\eta/2)^{th}$ quantile of the standard Normal distribution.

9.2.2. Asymptotic distribution of $R_{c,d}$

Similarly, for large sample size, the asymptotic distribution of MLE of $R_{c,d}$ is given by

$$\sqrt{l+ld}(\hat{R}_{c,d} - R_{c,d}) \rightarrow N(0, G^T I^{-1} G)$$

where G^T - $\left(\frac{\partial R_{c,d}}{\partial \alpha}, \frac{\partial R_{c,d}}{\partial \beta}, \frac{\partial R_{c,d}}{\partial \lambda_1}, \frac{\partial R_{c,d}}{\partial \lambda_2}\right)$

and I^{-1} - variance-covariance matrix or inverse of the Fisher information matrix and is given by

$$I^{-1} = E\begin{bmatrix} \frac{-\partial^2 \log LF_{c,d}}{\partial \alpha^2} & \frac{-\partial^2 \log LF_{c,d}}{\partial \alpha \partial \beta} & \frac{-\partial^2 \log LF_{c,d}}{\partial \alpha \partial \lambda_1} & \frac{-\partial^2 \log LF_{c,d}}{\partial \alpha \partial \lambda_2} \\ \frac{-\partial^2 \log LF_{c,d}}{\partial \alpha \partial \beta} & \frac{-\partial^2 \log LF_{c,d}}{\partial \beta^2} & \frac{-\partial^2 \log LF_{c,d}}{\partial \beta \partial \lambda_1} & \frac{-\partial^2 \log LF_{c,d}}{\partial \beta \partial \lambda_2} \\ \frac{-\partial^2 \log LF_{c,d}}{\partial \alpha \partial \lambda_1} & \frac{-\partial^2 \log LF_{c,d}}{\partial \beta \partial \lambda_1} & \frac{-\partial^2 \log LF_{c,d}}{\partial \lambda_1^2} & \frac{-\partial^2 \log LF_{c,d}}{\partial \lambda_1 \partial \lambda_2} \\ \frac{-\partial^2 \log LF_{c,d}}{\partial \alpha \partial \lambda_2} & \frac{-\partial^2 \log LF_{c,d}}{\partial \beta \partial \lambda_2} & \frac{-\partial^2 \log LF_{c,d}}{\partial \lambda_1 \partial \lambda_2} & \frac{-\partial^2 \log LF_{c,d}}{\partial \lambda_2^2} \end{bmatrix}^{-1}$$

The second order partial derivative of log-likelihood function with respect to the parameters $\alpha,\ \beta,\ \lambda_1,$ and λ_2 are derived and given in the appendix. An approximate estimation of the variance-covariance matrix, $I^{-1}(\hat{\alpha},\hat{\beta},\hat{\lambda_1},\hat{\lambda_2})$, of the parameters can be obtained by replacing the values with the estimate values of $\alpha,\ \beta,\ \lambda_1$ and λ_2, respectively. Hence we may estimate the variance of $\hat{R}_{c,d}$ as

$$\begin{aligned} Vari\hat{a}nce(\hat{R}_{c,d}) &\simeq G^T I^{-1} G \\ &= Variance(\hat{\lambda_1})\Big(\frac{\partial R_{c,d}}{\partial \lambda_1}\Big)^2 + Variance(\hat{\lambda_2})\Big(\frac{\partial R_{c,d}}{\partial \lambda_2}\Big)^2 + 2\frac{\partial R_{c,d}}{\partial \lambda_1}\frac{\partial R_{c,d}}{\partial \lambda_2} I_{12}^{-1} \quad (53) \end{aligned}$$

where

$$Variance(\hat{\lambda_1}) = \left(E\Big(-\frac{\partial^2 \log LF_{c,d}}{\partial \hat{\lambda_1}^2}\Big)\right)^{-1}$$
$$Variance(\hat{\lambda_2}) = \left(E\Big(-\frac{\partial^2 \log LF_{c,d}}{\partial \hat{\lambda_2}^2}\Big)\right)^{-1}$$

and

$$I_{12}^{-1} = \left(E\Big(-\frac{\partial^2 \log LF_{c,d}}{\partial \hat{\lambda_1}\partial \hat{\lambda_2}}\Big)\right)^{-1}.$$

for large sample size,

$$\frac{(\hat{R}_{c,d} - R_{c,d})}{\sqrt{Vari\hat{a}nce(\hat{R}_{c,d})}} \sim N(0,1)$$

and the asymptotic $100(1-\eta)\%$ confidence interval for $R_{c,d}$ is given by

$$\hat{R}_{c,d} \pm Z_{\eta/2}\sqrt{Vari\hat{a}nce(\hat{R_{c,d}})}$$

where $\hat{R}_{c,d}$ is the MLE of $R_{c,d}$ and $Z_{\eta/2}$ is the upper $(\eta/2)^{th}$ quantile of the standard Normal distribution.

9.3. Simulation study

Here, a simulation study is carried out to compare the performance of MLEs of R and $R_{c,d}$ in terms of their biases and MSEs. Here we use the parameter values $(\alpha,\beta,\lambda_1,\lambda_2) = (2.5, 0.5, 6, 5)$, then the theoretical value of R is 0.5454545.Additionally, we calculate the confidence intervals using the ML method.The simulation results of R are given in Table 5. For $R_{c,d}$, take the values $(c,d) = \{(2,4),(3,6),(1,3)\}$ in each sample size (l,z). The simulation results of $R_{c,d}$ are reported in Table 6.

From the simulation results of both R and $R_{c,d}$, it is noted that as the sample size (l,z) increases, the biases and MSE values decrease. For the single-component stress-strength model, we considered $(l,z) = \{(10,10),(30,30),(50,50),(100,100)\}$. In the case $R_{c,d}$, we are considering another combination of sample size (l,z) as given in Table 6. The theoretical values of $R_{c,d}$ for different values of $(c,d) = \{(2,4),(3,6),(1,3)\}$ are 0.6476762, 0.6228523, and 0.7826087, respectively.

Table 5: Simulation study for Stress-Strength reliability for parameter values $\alpha = 2.5, \beta = 0.5, \lambda_1 = 6, \lambda_2 = 5, R = 0.5454545$

(l,z)	$\hat{R}$	Bias	MSE	95% ACI
(10,10)	0.5477	0.0022	0.0144	(0.54750, 0.54793)
(30,30)	0.5467	0.0012	0.0041	(0.54564, 0.54772)
(50,50)	0.5451	-0.0004	0.0026	(0.52439, 0.56579)
(100,100)	0.5453	-0.0002	0.0013	(0.48744, 0.60314)

Table 6: Simulation study for Multi-component Stress-Strength reliability for parameter values $\alpha = 2.5, \beta = 0.5, \lambda_1 = 6, \lambda_2 = 5$

$R_{2,4} = 0.6476762, R_{3,6} = 0.6228523, R_{1,3} = 0.7826087$

(c,d)	l	$\hat{R_{cd}}$	Bias	MSE	95% ACI
(2,4)	10	0.6824	0.0347	0.0012	(0.68195, 0.68270)
	30	0.6767	0.0289	0.0008	(0.66641,0.68694)
	40	0.6542	0.0065	4.1882e-05	(0.65342,0.65488)
	80	0.6296	-0.0179	0.0003	(0.62722,0.63228)
	500	0.6447	-0.0029	8.9723e-06	(0.61204,0.6773)
(3,6)	10	0.5876	0.0353	0.0013	(0.57622, 0.59892)
	20	0.6552	0.03231	0.0010	(0.58033,0.72999)
	50	0.6413	0.0184	0.0003	(0.63725,0.64531)
	100	0.6366	0.0138	0.0002	(0.53606,0.73716)
	500	0.6337	0.0108	0.0001	(0.62503,0.64230)
(1,3)	10	0.8313	0.0487	0.0024	(0.27759,1.38508)
	40	0.7972	0.01457	0.0002	(0.72687, 0.86749)
	80	0.7860	0.00342	1.1702e-05	(0.78464,0.78856)
	100	0.7729	0.0022	4.6651e-06	(0.75395,0.79199)
	500	0.7846	0.00191	3.7655e-06	(0.75172,0.81736)

9.4. Data analysis

In this section, we analyze two real datasets introduced by Badar and Priest (1982) to illustrate the use of our proposed estimation method. The first data set (denoted by U) is strength measured in GPA for single carbon fibers tested under tension at a gauge length of 20mm. The second one (denoted by V) is the strength measured in GPA for single carbon fiber tested under tension at a gauge of 10 mm.

The PGDUS-IK(α, β, λ) model fits both data sets. The estimated values of the parameters are obtained. Log-likelihood values, KS values with corresponding p-values, CVM values with corresponding p-values, AIC, and BIC values for both datasets are given in the table. The estimated value for reliability is obtained as 0.2127864.

In the case of the multi-component stress-strength model, the same data set fits with the model for each value of $(c, d) = \{(1, 3), (2, 4), (3, 6)\}$. The parameter estimators,reliability estimate value, K-S values with p-value, and CVM values with p-value are given in Table 8.

Table 7: Stress-Strength Data analysis

Estimates		K-S(p-value)	CVM(p-value)	Log-Likelihood	AIC	BIC
$\hat{\alpha} = 7.47$ $\hat{\beta}$ =2476.96	X	0.13565 (0.1966)	0.36097 (0.09158)	-47.36073	98.72146	103.0077
$\hat{\lambda}_1$=1.36468 $\hat{\lambda}_2$=5.04871	Y	0.087442 (0.7211)	0.07484 (0.724)	-135.015	274.03	278.3136

Table 8: Data Analysis of Multi-Component SSR Model

(c,d)	$\hat{R}_{c,d}$	Estimates	U		V	
			K-S (p)	CVM (p)	K-S (p)	CVM (p)
(1,3)	0.45163	7.3788 2124.06 1.4377 5.2369	0.13587 (0.1952)	0.35368 (0.09586)	0.08338 (0.7735)	0.07191 (0.7417)
(2,4)	0.23708	7.3625 2072.29 1.4474 5.2571	0.13651 (0.1909)	0.35254 (0.09655)	0.08263 (0.7829)	0.07148 (0.7443)
(3,6)	0.18989	7.3441 2017.05 1.4574 5.2762	0.13725 (0.1862)	0.35133 (0.09729)	0.08178 (0.7935)	0.07104 (0.747)

10. Summary

The present paper proposes a new lifetime distribution, called the PGDUS-IK distribution, with parameters α, β, and λ, respectively, by using the PGDUS transformation on the IK (α, β) distribution for modeling a parallel system. The statistical properties, including moments, moment generating function, characteristic function, cumulant generating function, quantile function, order statistics, and entropy, are derived. Also, the expected additional lifetime given that the system has survived until a time t is defined in terms of its mean residual life function. Then we move on to the topic estimation of unknown parameters α, β, and λ of the proposed distribution. In this paper, we consider different types of estimation methods, such as the MLE method, the method of maximum product spacing estimation, the method of moment estimation, the method of least squares estimation, and bayesian analysis, respectively. The confidence interval is a range of values that describes the uncertainty around an estimate. For PGDUS-IK(α, β, λ), asymptotic confidence interval and bootstrap confidence interval are obtained. Simulation of data from the proposed distribution is obtained by three different methods: MLE, MPS, and Bayesian. Table 1 shows that, biases and MSEs for the parameters α, β, and λ decrease with increasing sample size. A dataset of vinyl chloride data obtained from clean upgrading and monitoring wells is used for the data analysis. It can be concluded that the proposed PGDUS-IK is effective in providing a better fit of data when compared with other competing distributions, such as the DUS-IK, IK, and Weibull distributions. Stress-strength reliability for single-component and multi-component models is discussed. Reliability estimates in both models are obtained from the parameter estimate values. The asymptotic distributions of single-component stress-strength reliability and multi-component stress-strength reliability are derived. As the sample size increases, the biases and MSEs of the simulated estimator of reliability in both models decrease. Both the single-component SSR model and the multi-component SSR model are applied to real data obtained from Badar and Priest (1982) and show that both models fit the data.

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Appendix

$$\log LF(x) = l\Big(\log\alpha + \log\beta + \log\lambda - \lambda\log(e-1)\Big) - (\alpha+1)\sum_{i=1}^{l}\log(1+u_i) + (\beta-1)\sum_{i=1}^{l}\log(1-(1+u_i)^{-\alpha}) + \sum_{i=1}^{l}(1-(1+u)^{-\alpha})^{\beta} + (\lambda-1)\sum_{i=1}^{l}\log\left(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1\right).$$

$$\frac{\partial^2\log LF}{\partial\alpha^2} = \frac{-l}{\alpha^2} + (\beta-1)\sum_{i=1}^{l}\left(\frac{(1+u_i)^{-\alpha-1}(1-(1+u_i)^{-\alpha}+(1+u_i)\log^2(1+u_i))}{(1-(1+u_i)^{-\alpha})^2}\right) + \beta\sum_{i=1}^{l}(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-2}\log^2(1+u_i)(\beta(1+u_i)^{-\alpha}-1) + \beta^2(\lambda-1)\sum_{i=1}^{l}\Bigg(\frac{(1+u_i)^{-2\alpha}(1-(1+u_i)^{-\alpha})^{\beta-2}\log^2(1+u_i)e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2} \left(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-(1-(1+u_i)^{-\alpha})^{\beta-1}-1\right)\Bigg) - \beta(\lambda-1)\frac{\sum_{i=1}^{l}(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-2}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log^2(1+u_i)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)}$$

$$\frac{\partial^2\log LF}{\partial\beta^2} = \frac{-l}{\beta^2} + \sum_{i=1}^{l}(1-(1+u_i)^{-\alpha})^{\beta}\log^2(1-(1+u_i)^{-\alpha}) + \Bigg((\lambda-1)\sum_{i=1}^{l}\frac{(1-(1+u_i)^{-\alpha})^{\beta}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log^2(1-(1+u_i)^{-\alpha})}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2} \left(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-(1-(1+u_i)^{-\alpha})^{\beta}-1\right)\Bigg)$$

$$\frac{\partial^2\log LF}{\partial\alpha\partial\beta} = \sum_{i=1}^{l}\frac{(1+u_i)^{-\alpha}\log(1+u_i)}{1-(1+u_i)^{-\alpha}} + \beta\sum_{i=1}^{l}(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}\log(1+u_i)(\log(1-(1+u_i)^{-\alpha})+1) - \beta(\lambda-1)\sum_{i=1}^{l}\Bigg(\frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{2\beta-1}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log(1+u_i)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2} \log(1-(1+u_i)^{-\alpha})\Bigg) + (\lambda-1)\sum_{i=1}^{l}\Bigg(\frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1} (1+\beta(1-(1+u_i)^{-\alpha})\log(1+u_i)\log(1-(1+u_i)^{-\alpha})\Bigg)$$

$$\frac{\partial^2\log LF}{\partial\lambda^2} = \frac{-l}{\lambda^2}$$

$$\frac{\partial^2 \log LF}{\partial\alpha\partial\lambda} = \beta\sum_{i=1}^{l}\frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}e^{(1-(1+u_i)^{-\alpha})^\beta}\log(1+u_i)}{e^{(1-(1+u_i)^{-\alpha})^\beta}-1}$$

$$\frac{\partial^2 \log LF}{\partial\beta\partial\lambda} = \sum_{i=1}^{l}\frac{(1-(1+u_i)^{-\alpha})^{\beta}e^{(1-(1+u_i)^{-\alpha})^\beta}\log(1-(1+u_i)^{-\alpha})}{e^{(1-(1+u_i)^{-\alpha})^\beta}-1}$$

$$M_{11} = \frac{\partial M}{\partial^2\alpha}, \quad M_{22} = \frac{\partial M}{\partial^2\beta}, \quad M_{33} = \frac{\partial M}{\partial^2\lambda}$$

$$M_{12} = M_{21} = \frac{\partial^2 M}{\partial\alpha\partial\beta}, \quad M_{13} = M_{31} = \frac{\partial^2 M}{\partial\alpha\partial\lambda}, \quad M_{23} = M_{32} = \frac{\partial^2 M}{\partial\beta\partial\lambda}$$

$$M_{111} = \beta\sum\log^2(1+u_i)\Bigg(\frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-2}e^{(1-(1+u_i)^{-\alpha})^\beta}\Big(\beta(1+u_i)^{-\alpha}-1\Big)}{e^{(1-(1+u_i)^{-\alpha})^\beta}-1} - \frac{\beta(1+u_i)^{-2\alpha}(1-(1+u_i)^{-\alpha})^{2(\beta-1)}e^{(1-(1+u_i)^{-\alpha})^\beta}}{(e^{(1-(1+u_i)^{-\alpha})^\beta}-1)^2}\Bigg)$$

$$\begin{aligned}
M_{122} &= M_{221} = M_{212}\\
&= \sum(1+u_i)^{-\alpha}\log(1+u_i)(1-(1+u_i)^{-\alpha})^{\beta-1}\log(1-(1+u_i)^{-\alpha})\\
&\quad \Big(2+\log(1-(1+u_i)^{-\alpha})\Big) + (\lambda-1)\sum(1+u_i)^{-\alpha}\log(1+u_i)\log(1-(1+u_i)^{-\alpha})\\
&\quad \left(\frac{(1-(1+u_i)^{-\alpha})^{\beta-1}e^{(1-(1+u_i)^{-\alpha})^\beta}(e^{(1-(1+u_i)^{-\alpha})^\beta}-1-(1-(1+u_i)^{-\alpha})^\beta)}{\Big(e^{(1-(1+u_i)^{-\alpha})^\beta}-1\Big)^2}\right)\\
&+(\lambda-1)\textstyle\sum(1+u_i)^{-\alpha}\log(1+u_i)\log(1-(1+u_i)^{-\alpha})(1-(1+u_i)^{-\alpha})^{\beta-1}e^{(1-(1+u_i)^{-\alpha})^\beta}\\
&\quad \left(\frac{\beta(1-(1+u_i)^{-\alpha})^\beta\log(1-(1+u_i)^{-\alpha})}{\Big(e^{(1-(1+u_i)^{-\alpha})^\beta}-1\Big)^4} + \frac{(1+(1-(1+u_i)^{-\alpha})^\beta)}{\Big(e^{(1-(1+u_i)^{-\alpha})^\beta}-1\Big)^4}\right.\\
&\quad \Big(1+\beta\log(1-(1+u_i)^{-\alpha})\Big(1+(1-(1+u_i)^{-\alpha})^\beta(1+e^{(1-(1+u_i)^{-\alpha})^\beta})\Big)\Big)\\
&\quad \left. - \frac{2(1-(1+u_i)^{-\alpha})^{2\beta-1}e^{2(1-(1+u_i)^{-\alpha})^\beta}(1+(1-(1+u_i)^{-\alpha})^\beta)log(1-(1+u_i)^{-\alpha})}{\Big(e^{(1-(1+u_i)^{-\alpha})^\beta}-1\Big)^2}\right)
\end{aligned}$$

$$\begin{aligned}
M_{112} &= M_{121} = M_{211} = -\sum\frac{(1+u_i)^{-\alpha}\log^2(1+u_i)}{(1-(1+u_i)^{-\alpha})^2} + \sum(1+u_i)^{-\alpha}\log^2(1+u_i)\\
&\quad (1-(1+u_i)^{-\alpha})^{\beta-2}\Big((\beta(1+u_i)^{-\alpha}-1)(1+\beta\log(1-(1+u_i)^{-\alpha})) + \beta(1+u_i)^{-\alpha}\Big)\\
&\quad - (\lambda-1)\sum(1+u_i)^{-\alpha}\log^2(1+u_i)(1-(1+u_i)^{-\alpha})^{\beta-2}e^{(1-(1+u_i)^{-\alpha})^\beta}\\
&\quad \left(\tfrac{(e^{(1-(1+u_i)^{-\alpha})^\beta}-1)(1+\beta\log(1-(1+u_i)^{-\alpha}))-\beta(1-(1+u_i)^{-\alpha})^\beta\log(1-(1+u_i)^{-\alpha})}{(e^{(1-(1+u_i)^{-\alpha})^\beta}-1)^2}\right)\\
&\quad + \beta(\lambda-1)\sum(1+u_i)^{-2\alpha}\log^2(1+u_i)\Bigg((1-(1+u_i)^{-\alpha})^\beta e^{(1-(1+u_i)^{-\alpha})^\beta}\\
&\quad \left(\tfrac{(e^{(1-(1+u_i)^{-\alpha})^\beta}-(1-(1+u_i)^{-\alpha})^{\beta-1}-1)(2+\beta\log(1-(1+u_i)^{-\alpha})(1+(1-(1+u_i)^{-\alpha})^\beta))}{(e^{(1-(1+u_i)^{-\alpha})^\beta}-1)^2}\right.
\end{aligned}$$

$$+ \frac{(1-(1+u_i)^{-\alpha})^{\beta-3}((1-(1+u_i)^{-\alpha})e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)\log(1-(1+u_i)^{-\alpha})}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2}\Bigg)$$

$$- \frac{2\beta(1-(1+u_i)^{-\alpha})^{2\beta-2}e^{2(1-(1+u_i)^{-\alpha})^{\beta}}\log(1-(1+u_i)^{-\alpha})}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^3}\Bigg)$$

$$M_{123} = \sum(1+u_i)^{-\alpha}\log(1+u_i)(1-(1+u_i)^{-\alpha})^{\beta-1}e^{(1-(1+u_i)^{-\alpha})^{\beta}}$$
$$\left(\frac{1}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1} + \beta\frac{(1-(1+u_i)^{-\alpha})^{\beta-1}\log(1-(1+u_i)^{-\alpha})e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2}\right.$$
$$\left.\left(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1-(1-(1+u_i)^{-\alpha})^{\beta}\right)\right)$$

$$M_{231} = \sum\frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log(1+u_i)}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}$$
$$+ \beta\sum\log^2(1+u_i)(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-2}e^{(1-(1+u_i)^{-\alpha})^{\beta}}$$
$$\left(\frac{\left((\beta(1+u_i)^{-\alpha}-1)(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)-\beta(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta}\right)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2}\right)$$

$$M_{113} = M_{131} = M_{311} = \beta^2\sum(1+u_i)^{-2\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1}\log^2(1+u_i)e^{(1-(1+u_i)^{-\alpha})^{\beta}}$$
$$\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-(1-(1+u_i)^{-\alpha})^{\beta-1}-1}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2}$$
$$-\sum\frac{\beta(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-2}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log^2(1+u_i)}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}$$

$$M_{223} = M_{232} = M_{322} = \sum\Bigg(\log^2(1-(1+u_i)^{-\alpha})(1-(1+u_i)^{-\alpha})^{\beta}e^{(1-(1+u_i)^{-\alpha})^{\beta}}$$
$$\frac{\left(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1-(1-(1+u_i)^{-\alpha})^{\beta}\right)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2}\Bigg)$$

$$M_{331} = 0 = M_{332}$$

$$M_{222} = \frac{2l}{\beta^3}\sum(1-(1+u_i)^(-\alpha))^{\beta}\log^3(1-(1+u_i)^{-\alpha})$$
$$+(\lambda-1)\sum(1-(1+u_i)^{-\alpha})^{\beta}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log^3(1-(1+u_i)^{-\alpha})$$
$$\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}}(1+2(1-(1+u_i)^{-\alpha})^{\beta})-(1-(1+u_i)^{-\alpha})^{\beta}(3+(1-(1+u_i)^{-\alpha})^{\beta})-1}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2}$$
$$-2\sum(1-(1+u_i)^{-\alpha})^{2\beta}e^{2(1-(1+u_i)^{-\alpha})^{\beta}}\log^3(1-(1+u_i)^{-\alpha})$$
$$\frac{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-(1-(1+u_i)^{-\alpha})^{\beta}-1}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^3}$$

$$M_{333} = \frac{2l}{\lambda^3}$$

In the case stress-strength reliability,

$$\frac{\partial^2 \log LF}{\partial \lambda_1^2} = \frac{-l}{\lambda_1^2}$$

$$\frac{\partial^2 \log LF}{\partial \lambda_2^2} = \frac{-o}{\lambda_2^2}$$

$$\frac{\partial^2 \log LF}{\partial \lambda_1 \partial \lambda_2} = 0$$

$$\frac{\partial^2 \log LF}{\partial \beta \partial \lambda_1} = \sum_{i=1}^{l} \frac{(1-(1+u_i)^{-\alpha})^{\beta} e^{(1-(1+u_i)^{-\alpha})^{\beta}} \log(1-(1+u_i)^{-\alpha})}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}$$

$$\frac{\partial^2 \log LF}{\partial \beta \partial \lambda_2} = \sum_{j=1}^{o} \frac{(1-(1+v_j)^{-\alpha})^{\beta} e^{(1-(1+v_j)^{-\alpha})^{\beta}} \log(1-(1+v_j)^{-\alpha})}{e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1}$$

$$\frac{\partial^2 \log LF}{\partial \alpha \partial \lambda_1} = \beta \sum_{i=1}^{l} \frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-1} e^{(1-(1+u_i)^{-\alpha})^{\beta}} \log(1+u_i)}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}$$

$$\frac{\partial^2 \log LF}{\partial \alpha \partial \lambda_2} = \beta \sum_{j=1}^{o} \frac{(1+v_j)^{-\alpha}(1-(1+v_j)^{-\alpha})^{\beta-1} e^{(1-(1+v_j)^{-\alpha})^{\beta}} \log(1+v_j)}{e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1}$$

$$\frac{\partial^2 \log LF}{\partial \alpha \partial \beta} = \sum_{i=1}^{l} \frac{(1+u_i)^{-\alpha} \log(1+u_i)}{(1-(1+u_i)^{-\alpha})} + \sum_{j=1}^{o} \frac{(1+v_j)^{-\alpha} \log(1+v_j)}{(1-(1+v_j)^{-\alpha})}$$

$$\begin{aligned}
\frac{\partial^2 \log LF}{\partial \beta^2} &= -\frac{l+o}{\beta^2} + \sum_{i=1}^{l}(1-(1+u_i)^{-\alpha})^{\beta} \log^2(1-(1+u_i)^{-\alpha}) \\
&+ \quad \sum_{j=1}^{o}(1-(1+v_j)^{-\alpha})^{\beta} \log^2(1-(1+v_j)^{-\alpha}) \\
&+ (\lambda_1-1) \sum_{i=1}^{l} \Bigg(\frac{(1-(1+u_i)^{-\alpha})^{\beta} \log^2(1-(1+u_i)^{-\alpha}) e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1} \\
&- \frac{(1-(1+u_i)^{-\alpha})^{2\beta} \log^2(1-(1+u_i)^{-\alpha}) e^{(1-(1+u_i)^{-\alpha})^{\beta}}}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2} \Bigg) \\
&+ (\lambda_2-1) \sum_{j=1}^{o} \Bigg(\frac{(1-(1+v_j)^{-\alpha})^{\beta} \log^2(1-(1+v_j)^{-\alpha}) e^{(1-(1+v_j)^{-\alpha})^{\beta}}}{e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1}
\end{aligned}$$

$$- \frac{(1-(1+v_j)^{-\alpha})^{2\beta}\log^2(1-(1+v_j)^{-\alpha})e^{(1-(1+v_j)^{-\alpha})^{\beta}}}{(e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1)^2}\Bigg)$$

$$\frac{\partial^2 \log LF}{\partial\alpha^2} = -\frac{l+o}{\alpha^2} - (\beta-1)\Bigg(\sum_{i=1}^{l}\frac{(1+u_i)^{-\alpha}\log^2(1+u_i)}{(1-(1+u_i)^{-\alpha})^2} + \sum_{j=1}^{o}\frac{(1+v_j)^{-\alpha}\log^2(1+v_j)}{(1-(1+v_j)^{-\alpha})^2}\Bigg)$$

$$+ \quad \beta\Bigg(\sum_{i=1}^{l}(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{\beta-2}(\beta(1+u_i)^{-\alpha}-1)\log^2(1+u_i)$$

$$+ \quad \sum_{j=1}^{o}(1+v_j)^{-\alpha}(1-(1+v_j)^{-\alpha})^{\beta-2}(\beta(1+v_i)^{-\alpha}-1)\log^2(1+v_j)\Bigg)$$

$$+ \quad \beta(\lambda_1-1)\textstyle\sum_{i=1}^{l}\Bigg(-\beta\frac{(1+u_i)^{-2\alpha}(1-(1+u_i)^{-\alpha})^{2(\beta-1)}e^{(1-(1+u_i)^{-\alpha})^{\beta}}\log^2(1+u_i)}{(e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1)^2}$$

$$+ \quad \frac{(1+u_i)^{-\alpha}(1-(1+u_i)^{-\alpha})^{(\beta-2)}e^{(1-(1+u_i)^{-\alpha})^{\beta}}(\beta(1+u_i)^{-\alpha}-1)\log^2(1+u_i)}{e^{(1-(1+u_i)^{-\alpha})^{\beta}}-1}\Bigg)$$

$$+ \quad \beta(\lambda_2-1)\textstyle\sum_{j=1}^{o}\Bigg(-\beta\frac{(1+v_j)^{-2\alpha}(1-(1+v_j)^{-\alpha})^{2(\beta-1)}e^{(1-(1+v_j)^{-\alpha})^{\beta}}\log^2(1+v_j)}{(e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1)^2}$$

$$+ \quad \frac{(1+v_j)^{-\alpha}(1-(1+v_j)^{-\alpha})^{(\beta-2)}e^{(1-(1+v_j)^{-\alpha})^{\beta}}(\beta(1+v_j)^{-\alpha}-1)\log^2(1+v_j)}{e^{(1-(1+v_j)^{-\alpha})^{\beta}}-1}\Bigg)$$

In the case of multi-component stress-strength reliability, the log-likelihood function is given as

$$\begin{aligned}\log LF_{c,d} &= l(d+1)\Big(\log\alpha+\log\beta\Big) + ld\log\lambda_1 + l\log\lambda_2 - l(d\lambda_1+\lambda_2)\log(e-1)\\ &-(\alpha+1)\Big(\textstyle\sum_{i=1}^{l}\sum_{j=1}^{d}\log(1+u_{ij})+\sum_{i=1}^{l}\log(1+v_i)\Big)+(\beta-1)\Big(\textstyle\sum_{i=1}^{l}\sum_{j=1}^{d}\log(1-(1+u_{ij})^{-\alpha})\\ &+\sum_{i=1}^{l}\log(1-(1+v_i)^{-\alpha})\Big)+\sum_{i=1}^{l}\sum_{j=1}^{d}(1-(1+u_{ij})^{-\alpha})^{\beta}+\sum_{i=1}^{d}(1-(1+v_i)^{-\alpha})^{\beta}\\ &+(\lambda_1-1)\textstyle\sum_{i=1}^{l}\sum_{j=1}^{d}\log\Big(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1\Big)+(\lambda_2-1)\sum_{i=1}^{l}\log\Big(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1\Big).\end{aligned}$$

$$\frac{\partial^2\log LF_{c,d}}{\partial\lambda_1^2} = \frac{-ld}{\lambda_1^2}$$

$$\frac{\partial^2\log LF_{c,d}}{\partial\lambda_1\partial\lambda_2} = 0 = \frac{\partial^2\log LF_{c,d}}{\partial\lambda_2\partial\lambda_1}$$

$$\frac{\partial^2 \log LF_{c,d}}{\partial\alpha\partial\lambda_1} = \beta\sum_{i=1}^{l}\sum_{j=1}^{c}\frac{(1+u_{ij})^{-\alpha}(1-(1+u_{ij})^{-\alpha})^{\beta-1}e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}\log(1+u_{ij})}{e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1}$$

$$\frac{\partial^2 \log LF_{c,d}}{\partial\beta\partial\lambda_1} = \sum_{i=1}^{l}\sum_{j=1}^{c}\frac{(1-(1+u_{ij})^{-\alpha})^{\beta}e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}\log(1-(1+u_{ij})^{-\alpha})}{e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1}$$

$$\frac{\partial^2 \log LF_{c,d}}{\partial\lambda_2^2} = \frac{-l}{\lambda_2^2}$$

$$\frac{\partial^2 \log LF_{c,d}}{\partial\alpha\partial\lambda_2} = \beta\sum_{i=1}^{l}\frac{(1+v_i)^{-\alpha}(1-(1+v_i)^{-\alpha})^{\beta-1}e^{(1-(1+v_i)^{-\alpha})^{\beta}}\log(1+v_i)}{e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1}$$

$$\frac{\partial^2 \log LF_{c,d}}{\partial\beta\partial\lambda_2} = \sum_{i=1}^{l}\frac{(1-(1+v_i)^{-\alpha})^{\beta}e^{(1-(1+v_i)^{-\alpha})^{\beta}}\log(1-(1+v_i)^{-\alpha})}{e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1}$$

$$\begin{aligned}
\frac{\partial^2 \log LF_{c,d}}{\partial\alpha\partial\beta} &= \sum_{i=1}^{l}\sum_{j=1}^{c}\frac{(1+u_{ij})^{-\alpha}\log(1+u_{ij})}{1-(1+u_{ij})^{-\alpha}} + \sum_{i=1}^{l}\frac{(1+v_i)^{-\alpha}\log(1+v_i)}{1-(1+v_i)^{-\alpha}} \\
&+ \beta\Bigg(\sum_{i=1}^{l}\sum_{j=1}^{c}(1-(1+u_{ij})^{-\alpha})^{\beta-1}\log(1-(1+u_{ij})^{-\alpha})(1+u_{ij})^{-\alpha}\log(1+u_{ij}) \\
&+ \sum_{i=1}^{l}(1-(1+v_i)^{-\alpha})^{\beta-1}\log(1-(1+v_i)^{-\alpha})(1+v_i)^{-\alpha}\log(1+v_i)\Bigg) \\
&+ \sum_{i=1}^{l}\sum_{j=1}^{c}(1-(1+u_{ij})^{-\alpha})^{\beta-1}(1+u_{ij})^{-\alpha}\log(1+u_{ij}) \\
&+ \sum_{i=1}^{l}(1-(1+v_i)^{-\alpha})^{\beta-1}(1+v_i)^{-\alpha}\log(1+v_i) \\
&+ (\lambda_1-1)\sum_{i=1}^{l}\sum_{j=1}^{c}(1-(1+u_{ij})^{-\alpha})^{\beta-1}(1+u_{ij})^{-\alpha}\log(1+u_{ij})e^{(1-(1+u_{ij})^{-\alpha})^{\beta}} \\
&\left(\frac{(\beta\log(1-(1+u_{ij})^{-\alpha})+1)(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1)-\beta(1-(1+u_{ij})^{-\alpha})^{\beta}\log(1-(1+u_{ij})^{-\alpha})}{(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1)^2}\right) \\
&+ (\lambda_2-1)\sum_{i=1}^{l}(1-(1+v_i)^{-\alpha})^{\beta-1}(1+v_i)^{-\alpha}\log(1+v_i)e^{(1-(1+v_i)^{-\alpha})^{\beta}} \\
&\left(\frac{(\beta\log(1-(1+v_i)^{-\alpha})+1)(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1)-\beta(1-(1+v_i)^{-\alpha})^{\beta}\log(1-(1+v_i)^{-\alpha})}{(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1)^2}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \log LF_{c,d}}{\partial^2\beta} &= -\frac{l(d+1)}{\beta^2} + \sum_{i=1}^{l}\sum_{j=1}^{c}(1-(1+u_{ij})^{-\alpha})^{\beta}\log^2(1-(1+u_{ij})^{-\alpha}) \\
&+ \sum_{i=1}^{l}(1-(1+v_i)^{-\alpha})^{\beta}\log^2(1-(1+v_i)^{-\alpha}) + (\lambda_1-1)\sum_{i=1}^{l}\sum_{j=1}^{c}e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}
\end{aligned}$$

$$\frac{(1-(1+u_{ij})^{-\alpha})^{\beta}\log^{2}(1-(1+u_{ij})^{-\alpha})\Big(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1-(1-(1+u_{ij})^{-\alpha})^{\beta}\Big)}{(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1)^{2}}$$

$$+(\lambda_2-1)\sum_{i=1}^{l}e^{(1-(1+v_i)^{-\alpha})^{\beta}}(1-(1+v_i)^{-\alpha})^{\beta}\log^{2}(1-(1+v_i)^{-\alpha})$$

$$\left(\frac{e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1-(1-(1+v_i)^{-\alpha})^{\beta}}{(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1)^{2}}\right)$$

$$\frac{\partial^2\log LF_{c,d}}{\partial^2\alpha}=-\frac{l(d+1)}{\alpha^2}-(\beta-1)\left(\sum_{i=1}^{l}\sum_{j=1}^{c}\frac{(1+u_{ij})^{-\alpha}\log^2(1+u_{ij})}{(1-(1+u_{ij})^{-\alpha})^2}+\sum_{i=1}^{l}\frac{(1+v_i)^{-\alpha}\log^2(1+v_i)}{(1-(1+v_i)^{-\alpha})^2}\right)$$

$$+\beta\left(\sum_{i=1}^{l}\sum_{j=1}^{c}(1+u_{ij})^{-\alpha}\log^2(1+u_{ij})(1-(1+u_{ij})^{-\alpha})^{\beta-2}\Bigg((\beta(1+u_{ij})^{-\alpha}-1)\right.$$

$$\left.+\frac{e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}\Big((\beta(1+u_{ij})^{-\alpha}-1)(e^{(1-(1+u_{ij})^{-\alpha})^{\beta}}-1)-\beta(1+u_{ij})^{-\alpha}(1-(1+u_{ij})^{-\alpha})^{\beta}\Big)}{(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1)^2}\right)$$

$$+\sum_{i=1}^{l}(1+v_i)^{-\alpha}\log^2(1+v_i)(1-(1+v_i)^{-\alpha})^{\beta-2}\Bigg((\beta(1+u_{ij})^{-\alpha}-1)$$

$$\left.\left.+\frac{e^{(1-(1+v_i)^{-\alpha})^{\beta}}\Big((\beta(1+v_i)^{-\alpha}-1)(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1)-\beta(1+v_i)^{-\alpha}(1-(1+v_i)^{-\alpha})^{\beta}\Big)}{(e^{(1-(1+v_i)^{-\alpha})^{\beta}}-1)^2}\right)\right)$$