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Neutrosophic Marshall Olkin Extended Burr-XII Distribution: Theoretical Framework and Applications with Multiple Survival Time Data Sets

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Abstract

In the analysis of complex data sets, selecting an appropriate distribution is crucial for real-life applications. Common probability distributions often fail to provide adequate results when dealing with imprecise, uncertain, or vague data. To address these complexities and achieve more accurate results, a neutrosophic probability distribution called the neutrosophic Marshall-Olkin extended Burr-XII distribution has been developed. This study aims to introduce a lifetime distribution capable of handling indeterminate data. Various properties of the proposed distribution are discussed. The maximum likelihood method, in terms of neutrosophic parameters, is utilized to estimate these parameters. A simulation study is conducted to validate the estimated neutrosophic parameters. Finally, two real-life data sets are analyzed to demonstrate the potential of the NMOE Burr-XII distribution, highlighting its superior efficiency and adaptability compared to classical distributions when dealing with indeterminate survival time data.

Key words: Neutrosophic statistics; Simulations; Burr-XII; Marshall-Olkin.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

The Burr-XII distribution is significant in lifetime and survival data analysis. Shao *et al.* (2004) investigated models for the extended three-parameter Burr type XII distribution and applied it to model severe events, such as flood frequency. Rodriguez (1977) examined the adaptability of the Burr type XII distribution, which has been widely used in various scientific fields, including actuarial science, forestry, ecotoxicology, dependability, and survival analysis. Marshall and Olkin (1997) introduced a parameter to create a new family of distributions that are more flexible and cover a broader range of behaviors than previous distributions, known as extended distributions. Al-Saiari *et al.* (2014) further extended this

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by adding one parameter to the Marshall-Olkin Extended (MOE) Burr-XII distribution, resulting in the Marshall Olkin extended Burr-XII distribution.

Neutrosophic statistics, initially introduced in 1995 and further developed by Smarandache (2014) explored the nature, origin, and application of neutralities. Neutrosophic logic is a special form of fuzzy logic. The neutrosophic statistics is more efficient than the classical statistics and interval-statistics see Smarandache (2022). While classical statistics rely on definite data, neutrosophic statistics handle partial, imprecise, ambiguous, or indeterminate data. The two fields coincide when indeterminacy is zero Chen *et al.* (2017). Neutrosophic statistics provide more accurate results by differentiating between those who partially and fully belong to a dataset. When all data and inference techniques are determined, both classical and neutrosophic statistics occur simultaneously. Neutrosophic statistics (NS) offer several advantages over interval statistics. In probability distributions, NS employs thick functions, formed by the intersections of curves, which may or may not be depicted as intervals Smarandache (2014). The neutrosophic probability distribution (NPD) for an event (x) comprises three curves: $NPD(x) = [T(x), I(x), F(x)]$, where $T(x)$ represents the probability of event E occurring, $I(x)$ denotes the indeterminate probability of E occurring or not, and $F(x)$ signifies the probability of E not occurring. These functions T(x),I(x), and F(x) can take on classical or neutrosophic (unclear, approximate, thick) forms depending on the specific application, and their sum ranges from 0 to 3 Smarandache (2013). Many researchers have developed neutrosophic probability distributions. For example, Fawzi *et al.* (2019) introduced the neutrosophic Weibull distribution and its related family, including the neutrosophic Weibull, Neutrosophic Rayleigh, neutrosophic inverse Weibull, and neutrosophic three- and six-parameter Weibull, as well as the Neutrosophic beta distribution. Rao (2023) developed the neutrosophic Log-logistic distribution, while Khan *et al.* (2021b) introduced the neutrosophic Gamma Distribution. Duan *et al.* (2021) presented the neutrosophic exponential distribution, and Khan *et al.* (2021a) proposed the Neutrosophic Beta distribution. Albassam *et al.* (2023) explored some basic properties of the neutrosophic Weibull Distribution with applications to wind speed in uncertain environments. Nayana *et al.* (2022) proposed the DUS Neutrosophic Weibull Distribution, and Eassa *et al.* (2023) introduced the neutrosophic generalized Pareto Distribution, modeling it on public debt in Egypt. Khan *et al.* (2021c) developed the Neutrosophic Rayleigh model for indeterminate data and also created V charts, neutrosophic run length, and Neutrosophic power curves for the proposed model. Sherwani *et al.* (2021) introduced new entropy measures for the Weibull Distribution under neutrosophic data, and Granados *et al.* (2022) applied both continuous and discrete probability distributions to Neutrosophic data. According to Granados *et al.* (2022), fuzzy logic is a special case of Neutrosophic logic, which generalizes fuzzy logic.

The article is structured as follows: Section 2 outlines the development of the novel Neutrosophic Marshall extended Burr-XII distribution, including graphical representation. Sections 3 and 4 discuss various properties of the proposed density. Section 5 focuses on the estimation of unknown parameters and simulation studies. Section 6 presents applications of the proposed model. Section 7 provides a discussion on these applications, and Section 8 offers concluding remarks.

2. Development of neutrosophic Marshall Olkin extended Burr-XII distribution

In this section, we will introduce the neutrosophic Marshall Olkin Extended Burr-XII distribution.

2.1. Marshall Olkin extended Burr-XII model

Burr type distributions are extensively used in life data and survival analysis. Adding more parameters to the Burr-XII distribution enhances its flexibility and appeal. Consequently, this study selects the Marshall-Olkin extended Burr-XII model for the development of a Neutrosophic model. The cumulative distribution function (CDF) and the probability density function (PDF) of the Marshall-Olkin Extended Burr-XII (MOE Burr-XII) distribution are as follows.

$$F(x;\alpha,\beta,\gamma) = \frac{1-\left(1+x^{\beta}\right)^{-\gamma}}{1-(1-\alpha)\left(1+x^{\beta}\right)^{-\gamma}}, x,\alpha,\beta,\gamma > 0 \tag{1}$$

and

$$f\left(x;\alpha,\beta,\gamma\right) = \frac{\alpha\beta\gamma x^{\beta-1}\left(1+x^{\beta}\right)^{-\gamma-1}}{\left[1-(1-\alpha)\left(1+x^{\beta}\right)^{-\gamma}\right]^{2}}, x,\alpha,\beta,\gamma > 0 \tag{2}$$

2.2. Neutrosophic random variable

Rao *et al.* (2023) discussed the extension of classical statistics called neutrosophic statistics. In classical statistics, we work with specific or predefined values. In contrast, neutrosophic statistics involves selecting values or data from a population within an unpredictable environment. For instance, when recording the temperature of a place, we might not be able to capture a precise value, such as 35°C. Instead, the value could have an uncertainty range, like 35°C to 38°C. The information in this context can be confusing, inaccurate, doubtful, partial, or even unknown.

Assuming the neutrosophic random variable $X_N = X_L + I_N X_L$, where $I_N \in [I_L, I_U]$ wherever $I_N X_L$ is the indeterminate and $I_N \in [I_L, I_U]$ is the indeterminacy. It is importance to notice that the neutrosophic random variable is the extension of the classical random variable specifically when $I_L = 0$ the neutrosophic random variable converts into classical random variable. According to his, the properties of the expectation of the neutrosophic random variable $X_N = X_L + I_N X_L = (1+I_N)X_L$ is defined as:

Aslam and Albassam (2024) explored the mean properties of the neutrosophic random variable $X_N = X_L + X_L I_N$, defined as:

1. $E\left(X_N\right) = E\left(X_L + X_L I_N\right) = \left(1+I_N\right)E\left(X_L\right) = \left(1+I_N\right)\mu$
2. $E\left(X_N + t\right) = E\left[\left(X_L + X_L I_N\right) + t\right] = \left(1+I_N\right)\mu + t$ here t is a constant.
3. $E\left(sX_N + t\right) = E\left[s\left(X_L + X_L I_N\right) + t\right] = s\left(1+I_N\right)\mu + t$ here s and t are constant.
4. $E\left(X_N + Y_N\right) = \left(1+I_N\right)\mu_X + \left(1+I_N\right)\mu_Y$

Now, the variance properties of the neutrosophic random variables are as follows:

1. $V(X_N) = V(X_L + X_L I_N) = (1 + I_N)^2 V(X_L) = (1 + I_N)^2 \sigma^2$
2. $V(tX_N) = t^2 V(X_L + X_L I_N) = t^2 (1 + I_N)^2 \sigma^2$
3. $V(X_N + Y_N) = (1 + I_N)^2 \sigma_X^2 + (1 + I_N)^2 \sigma_Y^2 + 2I_N Cov(X_N, Y_N)$
4. V $(sX_N + tY_N) = s^2 (1 + I_N)^2 \sigma_X^2 + t^2 (1 + I_N)^2 \sigma_Y^2 + 2stI_N Cov(X_N, Y_N)$
5. If we have two independent variables, X_N and Y_N:
   $V(X_N + Y_N) = (1 + I_N)^2 \sigma_X^2 + (1 + I_N)^2 \sigma_Y^2$

Let suppose the random variable X arose from the Marshall Olkin extended Burr-XII distribution with the CDF and PDF given in equations 1 and 2, we consider that the neutrosophic statistical number N, and $I_N \in [I_L, I_U]$ is an interval of indeterminacy. If the neutrosophic variable $X_N = X_L + I_N X_L$, generates the neutrosophic values of data. According to this, the neutrosophic variable is defined as: $X_N = X_L + I_N X_L = (1 + I_N)X_L$ here indeterminate and determined parts are described by X_L and $I_N X_L$ respectively.

If the random variable in terms of neutrosophic statistic $X_N \in (1+I_N)X_L$ follows the Marshall Olkin Extended Burr-XII (NMOE Burr-XII) then by using the equations 1 and 2, the PDF and CDF of the neutrosophic Marshall Olkin Extended Burr-XII (NMO Burr-XII) distribution are developed as given below.

$$f_N(x_N; \alpha, \beta, \gamma) = \frac{\alpha\beta\gamma(1 + I_N)\left[(1 + I_N)x_L\right]^{\beta-1}\left[1 + \{(1 + I_N)x_L\}^\beta\right]^{-\gamma-1}}{\left[1 - (1 - \alpha)\left[1 + \{(1 + I_N)x_L\}^\beta\right]^{-\gamma}\right]^2}, x_N, \alpha, \beta, \gamma > 0 \tag{3}$$

Similarly, the CDF of the NMOE Burr-XII distribution is,

$$F_N(x_N; \alpha, \beta, \gamma) = \int_0^x f_N(x_N; \alpha, \beta, \gamma)\, dx$$

$$F_N(x_N; \alpha, \beta, \gamma) = \frac{\left[1 - \left[1 + \{(1 + I_N)x_L\}^\beta\right]^{-\gamma}\right]}{\left[1 - (1 - \alpha)\left[1 + \{(1 + I_N)x_L\}^\beta\right]^{-\gamma}\right]} \tag{4}$$

Special cases of NMOE Burr-XII distribution.

1. For $\alpha = 1$, the NMOE Burr-XII becomes Neutrosophic Burr-XII distribution.
2. For $\beta = 1$, NMOE Burr-XII becomes the Neutrosophic Marshal Olkin Extended Lomax distribution.

To prove that equation 3 is density and equation 4 is CDF, the following theorems are given.

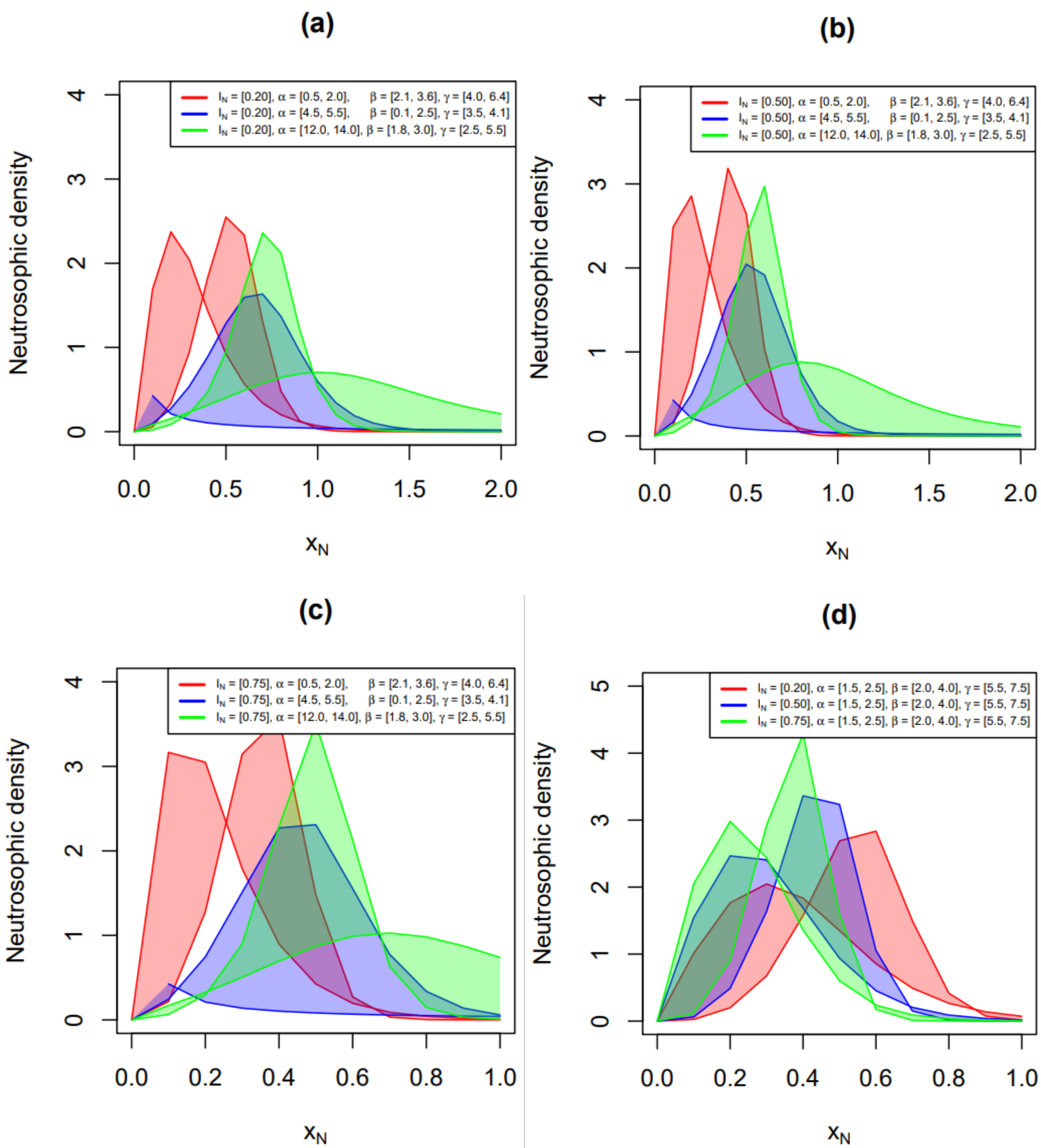

Figure 1: Density plots for the NMOE burr-XII distribution for different values of I_N, and parameters

Theorem 1: Consider $X_N \in (1 + I_N)X_L$ here indeterminate and determined parts are described by X_L and $I_N X_L$ respectively; suppose X_N follows the function given in equation 3 is a valid density function.

Proof: The random variable X follows the NMOE Burr-XII distribution in equation 3 then

$$\int_0^\infty \frac{\alpha\beta\gamma\left[(1+I_N)\,x_L\right]^{\beta-1}\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma-1}}{\left[1-(1-\alpha)\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma}\right]^{2}}\,(1+I_N)\;dx = 1$$

Let $\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma}=u$, and after some simplifications we get the

$$\alpha\int_0^1 \frac{1}{\left[1-(1-\alpha)u\right]^2}\,du=1$$

Again transform $[1-(1-\alpha)u]=z$, and simplifying it we get,

$$\frac{\alpha}{1-\alpha}\int_\alpha^1 \frac{1}{z^2}\,dz=1$$

The above integral is equal to one. Hence it is proved that equation 3 is a valid density function.

Theorem 2: Let the random variable $X_N \in (1+I_N)X_L$ here indeterminate and determined parts are described by X_L and $I_N X_L$ follows the NMOE Burr-XII distribution then the CDF given in equation 4 is a valid distribution function.

Proof: consider the random variable $X_N \in (1+I_N)X_L$ follows the CDF given in equation 4 then, it is proved that:

$$F(0)=0$$
$$F(\infty)=\infty$$

Hence the equation 4 is a valid distribution function. The graphical representation of the NMOE Burr-XII distribution is displayed below for different values of the parameters and varying I_N, Here, β & γ are the shape parameters, while α is the scale parameter. Figure 1 illustrates that the density is clearly unimodal.

3. Neutrosophic reliability measures

In this section, we develop several properties related to lifetime analysis, including survival analysis and the hazard function. The survival function is defined as the probability that an event or observation in survival data occurs after a specified time point. The survival function for the NMOE Burr-XII distribution is given as follows.

$$S_N\left(x_N;\alpha,\beta,\gamma\right)=\frac{\alpha\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma}}{\left[1-(1-\alpha)\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma}\right]} \tag{5}$$

The hazard rate function is a fundamental concept in survival analysis, which examines time-to-event data. The hazard rate function (HRF) for the NMOE Burr-XII distribution is derived as follows.

$$h_N\left(x_N;\alpha,\beta,\gamma\right)=\frac{\beta\gamma\,(1+I_N)\,\{(1+I_N)\,x_L\}^{\beta-1}}{\left[1-(1-\alpha)\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma}\right]\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]} \tag{6}$$

Figure 2 presents the HRF shapes with various values of parameters and with different I_N. HRF of the NMOE Burr-XII distribution exhibits monotone increasing trend.

The cumulative hazard rate function for the NMOE Burr-XII distribution is

$$H\left(x_N,\alpha,\beta,\gamma\right)=-ln\left[\frac{\alpha\left[1+\left\{\left(1+I_N\right)x_L\right\}^{\beta}\right]^{-\gamma}}{\left[1-\left(1-\alpha\right)\left[1+\left\{\left(1+I_N\right)x_L\right\}^{\beta}\right]^{-\gamma}\right]}\right]$$

The reversed hazard rate function for the NMOE Burr-XII distribution is

$$r\left(x_N,\alpha,\beta,\gamma\right)=\frac{\alpha\beta\gamma\left(1+I_N\right)\left[\left(1+I_N\right)x_L\right]^{\beta-1}\left[1+\left\{\left(1+I_N\right)x_L\right\}^{\beta}\right]^{-\gamma-1}}{\left[1-\left(1-\alpha\right)\left[1+\left\{\left(1+I_N\right)x_L\right\}^{\beta}\right]^{-\gamma}\right]\left[1-\left[1+\left\{\left(1+I_N\right)x_L\right\}^{\beta}\right]^{-\gamma}\right]}$$

In the context of neutrosophic reliability measures, censoring can be accommodated by incorporating neutrosophic sets to handle the uncertainty and indeterminacy associated with censored data. This approach allows for a more flexible representation of reliability metrics, where traditional binary logic (failure or survival) is extended to include degrees of membership, indeterminacy, and non-membership, thus providing a nuanced way to account for incomplete information due to censoring.

4. Some statistical properties of neutrosophic Marshall Olkin extended Burr-XII distribution

This section explores various statistical properties of the NMOE Burr-XII distribution, including the mean, variance, quantile function, skewness, and kurtosis.
The mean of the neutrosophic MOE Burr-XII distribution is derived as

$$\mu_N=E\left[\left(1+I_N\right)X_L\right]=\left(1+I_N\right)E\left(X_L\right)\tag{7}$$

Where,

$$E\left(X_L\right)=E\left(X\right)=\int_0^{\infty}x\frac{\alpha\beta\gamma x^{\beta-1}\left(1+x^{\beta}\right)^{-\gamma-1}}{\left[1-\left(1-\alpha\right)\left(1+x^{\beta}\right)^{-\gamma}\right]^2}dx$$

The above expression does not have a closed form, so we can determine its numerical values by substituting the parameter values.

Similarly, the variance of the neutrosophic MOE Burr-XII distribution is obtained as

$$\sigma^2=Var\left[\left(1+I_N\right)X_L\right]=\left(1+I_N\right)^2Var\left(X_L\right)\tag{8}$$

The variance also does not have a closed form. Therefore, we can determine its numerical values by substituting the parameter values.

Another important statistical property of the NMOE Burr-XII distribution is the quantile function, which is crucial for the Monte Carlo simulation approach. This function is also useful for generating random numbers from the probability distribution model. The quantile function of the NMOE Burr-XII distribution is derived as follows.

$$Q_N(p)=F_N^{-1}\left(X_p\right)$$

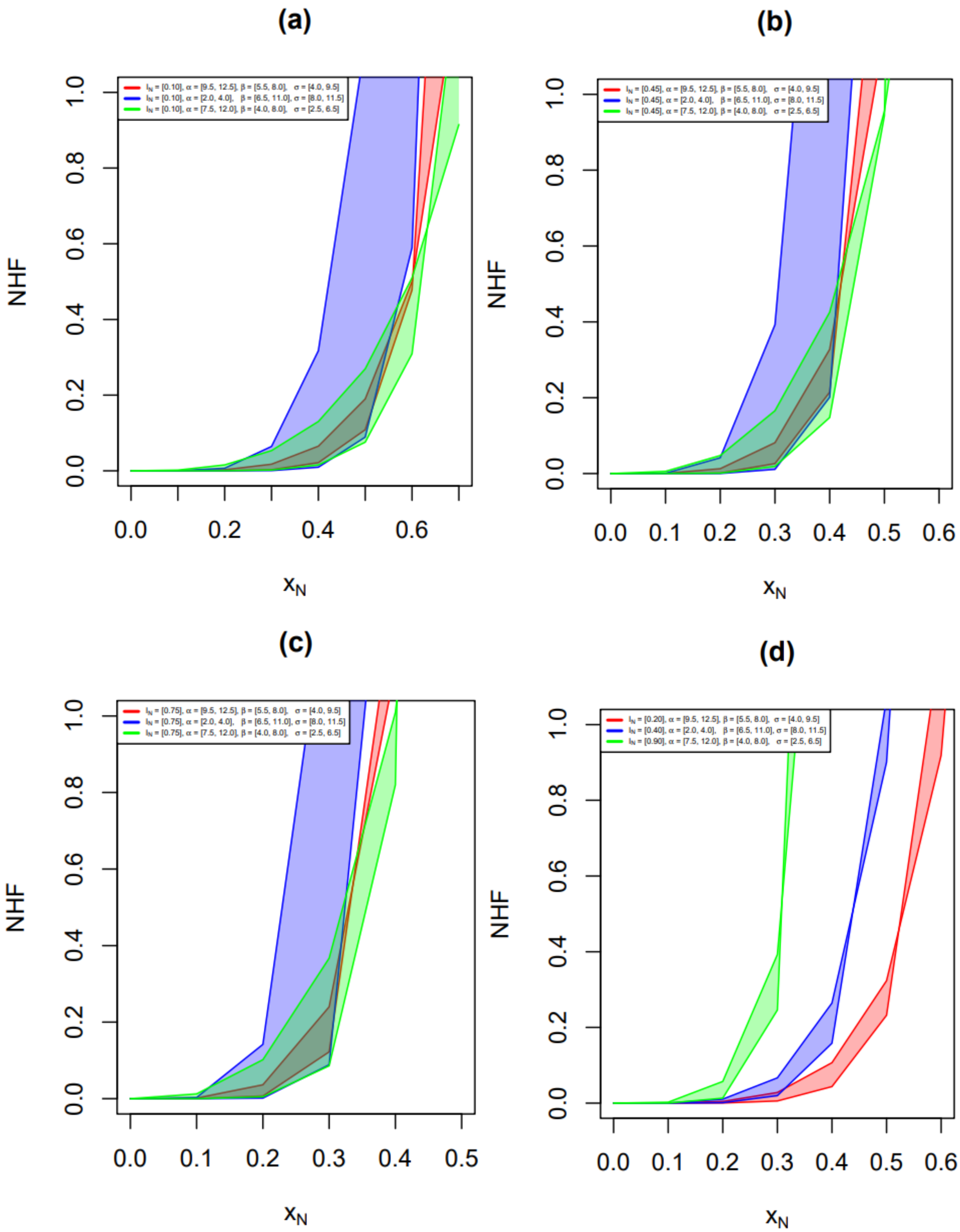

Figure 2: HRF plots for the NMOE Burr-XII distribution for different values of I_N, and parameters

$$Q_N(p) = \frac{\left[\left(\frac{1-p}{1-p(1-\alpha)}\right)^{-\frac{1}{\gamma}} - 1\right]^{\frac{1}{\beta}}}{(1+I_N)} \tag{9}$$

The median, first quartile, third quartile and Inter quartile range (IQR) for proposed distribution are calculated as $Median = Q_N(0.5)$, First quartile $= Q_N(0.25)$, Third quartile$=$ $Q_N(0.75)$ and $IQR = Q_N(0.75) - Q_N(0.25)$.

Neutrosophic Measure of Skewness and Kurtosis based on the Quantile function for NMOE Burr-XII distribution are given as follows,

$$SK_N = \frac{Q_N(6/8) - 2Q_N(4/8) + Q_N(2/8)}{Q_N(6/8) - Q_N(2/8)} \tag{10}$$

and

$$K_N = \frac{Q_N(7/8) - Q_N(5/8) + Q_N(3/8) - Q_N(1/8)}{Q_N(6/8) - Q_N(2/8)} \tag{11}$$

5. Parameter estimation

In this section, we discuss the estimation of unknown parameters for the NMOE Burr-XII distribution using the method of maximum likelihood estimator (MLE).

Maximum likelihood estimation method

Given the observed data, this method is used to find the parametric values of the proposed distribution. Suppose that $(1+I_N)X_N1, (1+I_N)X_N2, \ldots, (1+I_N)X_Nn$, be a neutrosophic random samples of NMOE Burr-XII distribution then log-likelihood function is derived as:

The loglikelihood function is:

$$\begin{aligned} l\left(\alpha,\beta,\gamma\right) = log(\alpha) + log(\beta) + log(\gamma) + log(1+I_N) + (\beta-1)\sum_{i=1}^{n} log(1+I_N)x_i \\ -(\gamma+1)log\sum_{i=1}^{n}\left[1+\{(1+I_N)x\}^{\beta}\right] - 2log\sum_{i=1}^{n}\left[1-(1-\alpha)\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma}\right] \end{aligned} \tag{12}$$

To find the values of parameters, obtain the derivative of the above expression with respect to α, β and γ.

$$\frac{\partial l}{\partial \alpha} = \frac{1}{\alpha} - \frac{\left[1+\{(1+I_N)x\}^{\beta}\right]^{-\gamma}}{\left[1-(1-\alpha)\left[1+\{(1+I_N)\,x_L\}^{\beta}\right]^{-\gamma}\right]} \tag{13}$$

$$\begin{aligned} \frac{\partial l}{\partial \beta} = \frac{1}{\beta} + log\{(1+I_N)x\} - \frac{(\gamma+1)\{(1+I_N)x\}^{\beta} log\{(1+I_N)x\}}{[1+\{(1+I_N)x\}^{\beta}]} \\ - \frac{2\gamma log\{(1+I_N)x+1\}\left[-\left[-(\alpha-1)\{(1+I_N)x+1\}^{\beta}\right]^{-\gamma}\right]}{1-(1-\alpha)\left[1+\{(1+I_N)x\}^{\beta}\right]^{-\gamma}} \end{aligned} \tag{14}$$

$$\frac{\partial l}{\partial \gamma} = \frac{1}{\gamma} - log\{1+\{(1+I_N)x\}^{\beta}\} - \frac{2\left[1+\{(1+I_N)x\}^{\beta}\right]^{-\gamma} log\left[1+\{(1+I_N)x\}^{\beta}\right]}{1-(1-\alpha)\left[1+\{(1+I_N)x\}^{\beta}\right]^{-\gamma}} \tag{15}$$

5.1. Simulation study

In this section, we conduct a Monte Carlo simulation study to evaluate the performance of the estimated parameters for the NMOE Burr-XII distribution. We assess the performance of the neutrosophic Maximum Likelihood estimator using the neutrosophic average biased (AB_N) and the neutrosophic root mean square error (RMSE).

$$AB_N = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}_{N_i} - \theta_N \right)$$

and

$$RMSE_N = \frac{1}{N} \sum_{i=1}^{N} \left(\hat{\theta}_{N_i} - \theta_N \right)^2$$

In R software, a Monte Carlo simulation with varying sample sizes and fixed values of the Neutrosophic parameters $\alpha = [0.01, 0.2]$, $\beta = [2.0, 2.7]$ and $\gamma = [1.2, 1.8]$ is conducted. The NMOE Burr-XII is used to build an imprecise dataset with $\alpha = [0.01, 0.2]$, $\beta = [2.0, 2.7]$ and $\gamma = [1.2, 1.8]$, and simulation is replicated $N = 10000$ times with sample sizes of $n = 50, 100, 300, 500$, respectively. The performance of the neutrosophic Maximum Likelihood estimators is then computed and shown in Tables 1, 2, 3 and 4. In the tables from 1-4, it is observed as the sample size increases the MSE, MRE and bias is decreasing for all parameters. Moreover, comparing estimated results when the I_N has been calculated from respective parameters *i.e.* $I_{N_{parameters}}$ with when $I_N = 0$. Then the MSE, Bias and MRE for I_{N_α} given in table 1, I_{N_β} given in table 2, and I_{N_γ} given in table 3, are less as compared to when $I_N = 0$, given in table 4.

Table 1: Parameter's bias, average bias, mean square error (MSE), and mean relative error (MRE) for $\alpha_N = [0.01, 0.09], \beta_N = [0.5, 1.5], \gamma_N = [0.05, 1.5]$ and $I_{N_\alpha} = 0.89$ calculated from α_N

Sizes	MLE Estimates	$\alpha_N = [0.01, 0.09]$	$\beta_N = [0.5, 1.5]$	$\gamma_N = [0.05, 1.5]$
50	Bias	[0.1641, 1.2994]	[0.4957, 1.4747]	[0.2716, 2.5005]
	Average Bias	[0.1566, 1.2434]	[0.0487,0.1905]	[0.2339, 1.4319]
	MSE	[20.8938, 196.1348]	[0.0038, 0.0635]	[0.2364, 5.1005]
	MRE	[15.6634, 17.8155]	[0.0974, 0.1270]	[4.6771, 0.9546]
100	Bias	[0.0358, 0.5668]	[0.4973, 1.4839]	[0.1589, 2.0133]
	Average Bias	[0.0285, 0.5085]	[0.0339, 0.1324]	[0.1218, 0.9201]
	MSE	[0.0036, 89.8824]	[0.0018, 0.0301]	[0.0517, 1.9413]
	MRE	[2.8496, 5.6503]	[0.0678, 0.0882]	[2.4361, 0.6134]
300	Bias	[0.0170, 0.1080]	[0.4996,1.4960]	[0.0824, 1.6468]
	Average Bias	[0.0098, 0.0427]	[0.0189, 0.0731]	[0.0459, 0.4588]
	MSE	[0.0003, 0.0040]	[0.0006, 0.0085]	[0.0055, 0.3739]
	MRE	[0.9829, 0.4742]	[0.0379, 0.0487]	[0.9182, 0.3058]
500	Bias	[0.0137, 0.1006]	[0.4998, 1.4971]	[0.0672, 1.5890]
	Average Bias	[0.0064, 0.0313]	[0.0149, 0.0559]	[0.0302, 0.3473]
	MSE	[0.0001, 0.0019]	[0.0004, 0.0050]	[0.0021, 0.2086]
	MRE	[0.6396, 0.3487]	[0.0297, 0.0373]	[0.6055, 0.2315]

Table 2: Parameter's bias, average bias, mean square error (MSE), and mean relative error (MRE) for $\alpha_N = [0.01, 0.09], \beta_N = [0.5, 1.5], \gamma_N = [0.05, 1.5]$ and $I_{N_\beta} = 0.67$ calculated from β_N

Sizes	MLE Estimates	$\alpha_N = [0.01, 0.09]$	$\beta_N = [0.5, 1.5]$	$\gamma_N = [0.05, 1.5]$
50	Bias	[0.1553, 1.4602]	[0.4958, 1.4721]	[0.2834, 2.5290]
	Average Bias	[0.1477, 1.4042]	[0.0501, 0.1899]	[0.2453, 1.4662]
	MSE	[30.9636, 232.3428]	[0.0041, 0.0641]	[0.2568, 5.3880]
	MRE	[14.7735, 15.6019]	[0.1001, 0.1266]	[4.9058, 0.9774]
100	Bias	[0.0350, 0.3235]	[0.4979, 1.4878]	[0.1562, 1.9594]
	Average Bias	[0.0278, 0.2666]	[0.0342, 0.1303]	[0.1196, 0.8842]
	MSE	[0.0033, 30.7377]	[0.0019, 0.0280]	[0.0483, 1.7041]
	MRE	[2.7784, 2.9620]	[0.0683, 0.0868]	[2.3922, 0.5895]
300	Bias	[0.0167, 0.1083]	[0.4999, 1.4970]	[0.0809, 1.6536]
	Average Bias	[0.0095, 0.0429]	[0.0192, 0.0725]	[0.0448, 0.4618]
	MSE	[0.0003, 0.0041]	[0.0006, 0.0085]	[0.0054, 0.3837]
	MRE	[0.9519, 0.4764]	[0.0383, 0.0484]	[0.8950, 0.3079]
500	Bias	[0.0139, 0.1009]	[0.4997, 1.4974]	[0.0682, 1.5950]
	Average Bias	[0.0066, 0.0316]	[0.0147, 0.0562]	[0.0314, 0.3505]
	MSE	[0.0001, 0.0019]	[0.0003, 0.0051]	[0.0023, 0.2099]
	MRE	[0.6631, 0.3513]	[0.0293, 0.0375]	[0.6272, 0.2336]

Table 3: Parameter's bias, average bias, mean square error (MSE), and mean relative error (MRE) for $\alpha_N = [0.01, 0.09], \beta_N = [0.5, 1.5], \gamma_N = [0.05, 1.5]$ and $I_{N_\gamma} = 0.97$ calculated from γ_N

Sizes	MLE Estimates	$\alpha_N = [0.01, 0.09]$	$\beta_N = [0.5, 1.5]$	$\gamma_N = [0.05, 1.5]$
50	Bias	[0.1257, 1.4512]	[0.4954, 1.4742]	[0.2921, 2.5106]
	Average Bias	[0.1183, 1.3951]	[0.0507, 0.1901]	[0.2544, 1.4363]
	MSE	[6.8381, 231.8247]	[0.0041, 0.0638]	[0.2779, 5.1529]
	MRE	[11.8248, 15.5015]	[0.1014, 0.1267]	[5.0875, 0.9576]
100	Bias	[0.0349, 0.4358]	[0.4978, 1.4863]	[0.1563, 1.9758]
	Average Bias	[0.0276, 0.3778]	[0.0343, 0.1311]	[0.1194, 0.8831]
	MSE	[0.0031, 60.1300]	[0.0019, 0.0288]	[0.0485, 1.7537]
	MRE	[2.7619, 4.1978]	[0.0687, 0.0874]	[2.3882, 0.5888]
300	Bias	[0.0170, 0.1085]	[0.4994, 1.4952]	[0.0821, 1.6575]
	Average Bias	[0.0098, 0.0430]	[0.0187, 0.0728]	[0.0459, 0.4647]
	MSE	[0.0003, 0.0040]	[0.0006, 0.0085]	[0.0055, 0.3847]
	MRE	[0.9831, 0.4776]	[0.0374, 0.0485]	[0.9187, 0.3098]
500	Bias	[0.0138, 0.0996]	[0.4999, 1.4989]	[0.0678, 1.5792]
	Average Bias	[0.0065, 0.0313]	[0.0148, 0.0563]	[0.0307, 0.3494]
	MSE	[0.0001, 0.0019]	[0.0003, 0.0050]	[0.0022, 0.2066]
	MRE	[0.6473, 0.3478]	[0.0295, 0.0375]	[0.6144, 0.2329]

Table 4: Parameter's bias, average bias, mean square error (MSE), and mean relative rrror (MRE) for $\alpha_N = [0.01], \beta_N = [0.5], \gamma_N = [0.05]$ and $I_N = 0$

Sizes	MLE Estimates	$\alpha_N = [0.01]$	$\beta_N = [0.5]$	$\gamma_N = [0.05]$
50	Bias	0.1923	0.4954	0.2899
	Average Bias	0.1849	0.0503	0.2523
	MSE	27.206	0.0041	0.2861
	MRE	18.4847	0.1007	5.0461
100	Bias	0.0348	0.4978	0.1552
	Average Bias	0.0275	0.0344	0.1186
	MSE	0.0031	0.0019	0.0469
	MRE	2.7534	0.0687	2.3711
300	Bias	0.0171	0.4994	0.0828
	Average Bias	0.0099	0.0189	0.0465
	MSE	0.0003	0.0006	0.0056
	MRE	0.9923	0.0377	0.9292
500	Bias	0.0140	0.4998	0.0688
	Average Bias	0.0067	0.0147	0.0319
	MSE	0.0001	0.0003	0.0023
	MRE	0.6712	0.0295	0.6376

6. Applications

In this section, we apply the NMOE Burr-XII model to two real-world datasets characterized by uncertain or complex values. We aim to gauge the suitability of the NMOE Burr-XII model for such data. Various model selection methods are employed to assess the performance of the proposed distribution and compare it with other competing distributions to determine the best model. Two datasets are used in this study, that are Remission time dataset and Covid-19 dataset. The understudy datasets are presented in interval form, meaning they exhibit uncertainty in the upper bounds of their data values, rather than providing single, fixed values. This inherent uncertainty may result in insufficient information. To address this issue, the upper bounds in each dataset are calculated using the indeterminacy component I_N, thereby converting them into neutrosophic statistics. The values of I_N can be changed to $2\%, 5\%$, or 10% based on the desired degree of assurance or uncertainty. By immediately identifying and incorporating uncertainties into each dataset, this technique enables a more nuanced analysis and improves the comprehensiveness and utility of the data in medical research and decision-making across different investigations. In applications $I_N = 0.05$ is used to find the upper values of the datasets. A balance between being cautious and accommodating of data uncertainties is achieved by setting $I_N = 0.05$. It allows for considerable flexibility while maintaining a respectable degree of analytical precision.

Remission time dataset

The first data consists of a collection of 128 cancer patients' remission durations measured in months. After getting therapy, each value indicates how long a patient stayed in remission. When it comes to cancer therapy, remission is the time when the disease's symptoms and indicators are either minimal or nonexistent. This data has been taken from bladder cancer study reported by Lee and Wang (2003). The understudy data (remission time) is available in neutrosophic form. We took the lower limit of the data from the source and estimated the upper limit by applying an indeterminacy factor of $I_N = 0.05$. This same indeterminacy factor was then used to calculate the descriptive statistics shown in 7, estimate the values of neutrosophic parameters also in Table 8, and model the proposed density in Table 10 for the remission time data. [Note: this factor can be taken any other value.].

Table 5: Remission time Dataset

0.08	2.09	3.48	4.87	6.94	8.66	13.11	23.63	0.2
2.23	3.52	4.98	6.97	9.02	13.29	0.4	2.26	3.57
5.06	7.09	9.22	13.8	25.74	0.5	2.46	3.64	5.09
[**7.26, 8.2**]	9.47	14.24	25.82	0.51	2.54	3.7	5.17	7.28
9.74	14.76	[**5.3,7.1**]	0.81	2.62	3.82	5.32	7.32	10.06
[**12, 14.77**]	32.15	2.64	3.88	5.32	7.39	10.34	14.83	34.26
0.9	2.69	4.18	5.34	7.59	10.66	15.96	36.66	1.05
2.69	4.23	5.41	7.62	10.75	16.62	43.01	1.19	2.75
4.26	5.41	7.63	[**15, 17.2**]	46.12	1.26	2.83	4.33	5.49
7.66	11.25	17.14	[**75.02, 81**]	1.35	2.87	5.62	7.87	11.64
17.36	1.4	3.02	4.34	5.71	7.93	11.79	18.1	1.46
4.4	5.85	8.26	11.98	19.13	1.76	3.25	4.5	6.25
8.37	12.02	[**1.5, 3.2**]	3.31	4.51	6.54	[**7.5, 8.2**]	12.03	20.28
2.02	3.36	6.76	12.07	21.73	2.07	3.36	6.93	8.65
12.63	22.69	—	—	—	—	—	—	—

Covid-19 dataset

The data research by Almongy *et al.* (2021) describes the duration of relief in hours for 30 patients who received analgesic medication, likely as a part of a treatment for managing COVID-19-related symptoms. The data displays a range of response times, which suggests that patients in the research group had varying responses to the medicine. The relief time data is available from the source in neutrosophic form. We considered the lower limit of the data and calculated the upper limit by using an indeterminacy value of $I_N = 0.05$. This same indeterminacy value is later used to determine the descriptive statistics in Table 7, estimate the neutrosophic parameters in Table 9, and model the proposed density in Table 11 for the relief time data.

Table 6: Covid-19 Dataset

(14.918, 15.6639)	(10.056, 11.1888)	(12.274, 12.88770)	(10.289, 10.80345)
(10.832, 11.3736)	(7.099, 7.4539)	(5.928, 6.22440)	(13.211, 13.87155)
(7.968, 8.36640)	(7.584, 7.96320)	(5.555, 5.83275)	(6.027, 6.32835)
(4.097, 4.30185)	(3.611, 3.79155)	(4.960, 5.20800)	(7.498, 7.87290)
(6.940, 7.28700)	(5.307, 5.57235)	(5.048, 5.30040)	(2.857, 2.99985)
(2.254, 2.36670)	(5.431, 5.70255)	(4.462, 4.68510)	(3.883, 4.07715)
(3.461, 3.63405)	(3.647, 3.82935)	(1.974, 2.07270)	(1.273, 1.33665)
(1.416, 1.48680)	(4.235, 4.44675)	—	—

Table 7: Descriptive statistics for both data sets from proposed density

Descriptives	Remission time data	COVID-19
Mean	[0.9010, 0.8945]	[0.8162, 0.8205]
Variance	[0.0009, 0.0007]	[1.1e-05, 2.587e-06]
Median	[0.9004, 0.8941]	[0.8169, 0.8212]
First Quartile	[0.8821, 0.8785]	[0.8141, 0.8189]
Third Quartile	[0.9191, 0.9100]	[0.8193, 0.8230]
Skewness	[0.4324, 0.7092]	[-13.4441, -121.5781]
Kurtosis	[0.7801, 0.5726]	[177.7792, 941.5738]

Table 8: ML estimates and standard errors remission time dataset

NMOE Burr-XII	α	[63.6772, 58.4272]	[59.2463, 47.0918]
	β	[59.2463, 47.0918]	[0.955, 0.9926]
	γ	[0.955, 0.9926]	[0.3767, 0.3711]
N-Burr-III	θ	[1.033, 1.0232]	[0.0601, 0.0591]
	λ	[0.0601, 0.0591]	[4.3325, 4.5161]
Burr-XII	θ	[2.3454, 2.3303]	[0.355, 0.3518]
	λ	[0.355, 0.3518]	[0.2351, 0.235]

Table 9: ML estimates and standard errors for the COVID-19 (relief time) data

NMOE Burr-XII	α	[97.7882, 118.707]	[81.9116, 102.2903]
	β	[81.9116, 102.2903]	[25.5239, 23.8676]
	γ	[25.5239, 23.8676]	[30.2665, 86.2472]
N-Burr-III	θ	[1.6581, 1.6477]	[0.1983, 0.1974]
	λ	[0.1983, 0.1974]	[10.657, 11.3512]
Burr-XII	θ	[21.4342, 15.715]	[32.6983, 23.3943]
	λ	[32.6983, 23.3943]	[0.0283, 0.0375]

From Table 7, the following results information is obtained.

- The average remission time (in months) for bladder cancer patients is between the interval by mean is [0.9010, 0.8945] and by median is [0.9004, 0.8941] with spread [0.0009, 0.0007]. From the skewness and kurtosis, it is seen that the remission time data is slightly positively skewed, and platykurtic. From the first quartile it is seen that 25% of the patients have less remission time by this interval [0.8821, 0.8785] and from third quartile it is seen that 75% of the patients have less remission time by this interval [0.9191, 0.9100].

- The average relief times for bladder cancer patients is between the interval by mean is [0.8162, 0.8205] and by median is [0.8169, 0.8212] with spread [1.1e-05, 2.587e-06]. From the skewness and kurtosis, it is seen that the relief times data is extremely negatively skewed, and leptokurtic. From the first quartile it is seen that 25% of the patients have relief times by this interval [0.8141, 0.8189] and from third quartile it is seen that 75% of the patients have less relief times by this interval [0.8193, 0.8230].

Model Selection Criteria with estimates for the remission time dataset is shown in table 8 and 10. Model Selection Criteria with estimates for the COVID-19 (relief time) data is shown in tables 9 and 11. The estimated values of the parameters in tables 8 and 9 are in interval form because the parameters are neutrosophic due the uncertainty in the data sets.

7. Comparative study

This section presents a comparative study of the proposed model using two real-life datasets. The comparison is conducted with the neutrosophic Burr-II and classical Burr-XII models.

In Tables 10 and 11, the proposed neutrosophic density NMOE-Burr-XII is modeled and compared for both datasets, with the indeterminacy component I_N set at 0.05. This value represents the uncertainty in the datasets. When I_N equals 0, the density is in its classical form, such as NMOE Burr-XII and N-Burr-III, with the test statistic criterion being the lower bound only. However, when I_N is 0.05 or any other value, the densities become neutrosophic, and the test statistic criterion is presented in interval form due to the neutrosophic nature of the data.

In table 10, the modeling of the proposed density on remission time for bladder cancer patients' data shows that NMOE Burr-XII distribution shows more flexibility over the neutrosophic Burr-III (N-Burr-III) and classical Burr-XII distributions due to the lowest values of AIC, BIC, CAID, HQIC KS test and larger p-value for the KS test. It is also observed that Burr-XII shows p-value as 0.000 which particularly shows its inadequacy for the neutrosophic data, while NMOE Burr-XII and N-Burr-III both fit the data, but NMOE Burr-XII provides very strong p-value which shows its superiority. Table 11 shows the modeling of the proposed model on the relief time for COVID-19 data, the results shows that the NMOE Burr-XII distribution shows more flexibility as compared to the N-Burr-III and classical Burr-XII distributions due to the lowest values of AIC, BIC, CAID, HQIC KS test and larger p-value for the KS test.

Furthermore, the proposed density demonstrates superior flexibility and provides evidence across all three datasets compared to the classical Burr-XII and even the N-Burr-III distribution. Importantly, it is observed that the classical Burr-XII distribution does not fit well on both neutrosophic datasets (remission time for bladder cancer and relief time for COVID-19 datasets), yielding a p-value of 0.000.

In conclusion, the neutrosophic Marshall-Olkin Extended Burr-XII distribution emerges as a valuable tool particularly in scenarios where data is indeterminate, contrasting with the classical Burr-XII distribution. Classical distributions are unsuitable for modeling indeterminate and ambiguous datasets. The two data examples discussed above fall under the neutrosophic setup because they deal with lifetime data that inherently includes elements of uncertainty and incomplete information, which are better handled using neutrosophic statistics. While classical methods rely on precise probabilities, neutrosophic methods provide a more comprehensive framework by incorporating indeterminacy and partial truth values, thus offering more robust and realistic estimates in the presence of real-world complexities. This allows for better decision-making and reliability assessments in environments where data is not perfectly exact or complete.

Table 10: Model selection criteria and parameter estimates for remission time data

Models	LL	AIC	BIC	CAIC	HQIC	K-S	p-value
NMOE Burr-XII	[-405.695, -407.067]	[817.390, 820.134]	[825.946, 828.690]	[817.583, 820.327]	[820.866, 823.610]	[0.033, 0.039]	[0.9989, 0.9904]
N-Burr-III	[-423.562, -431.383]	[851.123, 866.767]	[856.828, 872.471]	[851.219, 866.863]	[853.441, 869.084]	[0.107, 0.105]	[0.106, 0.121]
Burr-XII	449.983	903.9661	909.670	904.062	906.284	0.254	0.000

Table 11: Model selection criteria and parameter estimates for the COVID-19 (relief time) data

Models	LL	AIC	BIC	CAIC	HQIC	K-S	p-value
NMOE Burr-XII	[-77.063, -78.736]	[160.127, 163.472]	[164.330, 167.675]	[161.050, 164.395]	[161.472, 164.817]	[0.066, 0.064]	[0.9983, 0.9989]
N-Burr-III	[-80.267, -81.901]	[164.534, 167.803]	[167.337, 170.605]	[164.979, 168.247]	[165.431, 168.699]	[0.139, 0.139]	[0.563, 0.5616]
Burr-XII	-94.298	192.595	195.397	193.039	193.492	0.362	0.000

8. Conclusion

In this study, we introduce a novel model called the neutrosophic Marshall-Olkin Extended Burr-XII distribution. We demonstrate that this model is advantageous for analyzing survival and reliability datasets with indeterminacies compared to classical distributions. Various neutrosophic properties are explored, including the neutrosophic survival function, hazard function, mean, variance, mode, skewness, and kurtosis. The distribution exhibits left-skewed, right-skewed, and symmetric shapes. The hazard rate function displays a monotonically increasing trend. Parametric values are determined using the maximum likelihood method. A simulation study assesses the performance of estimators across small, medium, and large sample sizes, revealing a decrease in mean square error with increasing sample size. Additionally, the proposed NMOE Burr-XII distribution is applied to two real-life datasets with uncertain values, demonstrating its superior flexibility compared to classical distributions.

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Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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