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Controllability and Observability of Fuzzy Matrix Lyapunov Discrete Dynamical System

L. N. Charyulu Rompicharla[1*], Venkata Sundaranand Putcha[2], and G. V. S. R. Deekshitulu[3]

[1] *Department of Mathematics, V.R.Siddhartha Engineering College, Kanuru, Vijayawada, A.P., India.*

[*] *Research Scholar, Jawaharlal Nehru Technological University Kakinada, A.P., India.*

[2] *Department of Mathematics, Rayalaseema University, Kurnool, A.P., India.*

[3] *Department of Mathematics, JNTU College of Engineering, Kakinada, A.P., India.*

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Abstract

This paper deals with the controllability and observability of the fuzzy matrix Lyapunov discrete dynamical system. The considered fuzzy system is vectorised by using Kronecker product. The resulting vector system is converted to matrix Lyapunov difference inclusion. For the considered fuzzy system, a symmetric controllability matrix is constructed and derived fuzzy control. A sufficient condition for complete controllability of the fuzzy matrix Lyapunov discrete dynamical system is established by fuzzy rule based approach. Center average defuzzifier approach is used to establish the sufficient conditions for the complete observability of the fuzzy matrix Lyapunov discrete dynamical system. A numerical example is presented to illustrate the theories established, results proved and formulae derived.

Key words: Lyapunov systems; Fuzzy discrete dynamical systems; Fuzzy rule; Controllability; Observability; Defuzzifier.

AMS Subject Classifications: 93B05, 93C55, 93C42, 93B07

1. Introduction

Real world systems represented by mathematical models require the knowledge of exact parameter model values. Many mathematical models do exhibit some degree of uncertainty because of the limitations in obtaining the exact values of the model parameters. This will naturally inspire scientists and engineers to construct models with uncertain parameters and uncertain initial conditions. This uncertainty cannot be ignored or neglected because of its influence on the model predictions. One of the important ways of incorporating the uncertainty or vagueness is by fuzzy dynamical modeling. The fundamental prerequisites for the design process are the controllability and the observability. The controllability conditions guarantee for the existence of control which will steer the state from the initial point to the

Corresponding Author: L.N.Charyulu Rompicharla
Email: narayanarompicharla@gmail.com

desired final point. So these two metrics are mandatory to test the possibility and feasibility of achieving the design requirements for the system of consideration. A simple criteria for the controllability and observability for the fuzzy dynamical systems similar to that of deterministic dynamical systems cannot be found because of the vagueness and uncertainty involved in the systems as well as initial condition. So the controllability and observability in fuzzy sense are to be explored. In the fuzzy case, the controllability cannot be characterized by finding a suitable control which can transfer the system from the initial state to any desired final state in a finite time interval since finite number of options emerge because coefficients, variables in the system and initial conditions are fuzzy, not deterministic. Cai and Tang (2000), Ding and Kandel (2000a), Farinwata and Vachtsevanos (1993) have studied the controllability of fuzzy systems.

Mastiani and Effati (2018) have investigated the controllability and the observability property of two systems that one of them has fuzzy variables and the other one has fuzzy coefficients and fuzzy variables(fully fuzzy system) by normalizing the fuzzy matrices. Gabr (2015) studied impact of propagation of fuzziness in the coefficients of dynamical systems in modeling, analysis, and design of automatic control systems. Difference equations describe the observed evolution phenomena in a better manner when compared to that of differential equations. Lyapunov matrix systems appear in determining the stability of the autonomous systems by the second method of Lyapunov without finding the solution of the system and also in the minimization of quadratic cost functionals in optimal control problems. Matrix Lyapunov systems have extensive applications in control theory, digital computers, optimal filters, population dynamics, differential games, power systems, signal processing and boundary value problems. Putcha *et al.* (2012) established variation of parameters formula for the matrix fuzzy dynamical systems and studied the controllability and observability of the fuzzy discrete dynamical systems by the defuzzifier approach. Anand and Murty (2005), Murty *et al.* (1997) studied the controllability and observability of the continuous and discrete dynamical systems. It is very important to study the controllability and observability of the mathematical models represented by fuzzy difference equations governing the ambiguity in dynamics which is not probabilistic. In general the problem of steering an initial state of a system to a desired final state in R^n become a problem of steering a fuzzy state to another fuzzy-state in E^s. Many of the physical applications may not have the exact information about their deterministic dynamics which is prerequisite to construct a dynamical system. The importance of control theory in mathematics and its occurrence in several problems such as mechanics, electromagnetic theory, thermodynamics, and artificial satellites are well known. In general, fuzzy systems are classified into 3 categories, (i) Pure fuzzy systems (ii) T-S fuzzy systems, and (iii) Fuzzy logic systems, using fuzzifiers and defuzzifiers. In this paper, we use fuzzy matrix Lyapunov discrete dynamical system to describe fuzzy logic system and establish sufficient conditions for controllability and observability of first order fuzzy matrix Lyapunov discrete dynamical system S_1 modeled by

$$\Delta T(n) = A(n)T(n) + T(n)A^T(n) + A(n)T(n)A^T(n) + F(n)U(n), T(n_0) = T_0, n > 0 \quad (1)$$

$$Y(n) = C(n)T(n) + D(n)U(n) \quad (2)$$

where $U(n)$ is an $m \times s$ fuzzy input matrix called fuzzy control and $Y(n)$ is an $r \times s$ fuzzy output matrix. Here $T(n), A(n), F(n), C(n)$ and $D(n)$ are matrices of order $s \times s, s \times s, s \times m, r \times s$ and $r \times m$ whose elements are continuous functions of n on $J = [0, N] \subset R(N > 0)$. Barnett (1975) studied the problem of controllability and observability for a system of ordinary differential equations. Anand and Murty (2005) established necessary and sufficient conditions for the controllability and observability of continuous matrix Lyapunov systems. Using fuzzy control, a complex system can be decomposed into several subsystems according to the expertise of human ability to understand the system and using the human control strategy represented by a simple control law. The popular fuzzy controllers in the literature are Mamdani fuzzy controllers and Takagi-Sugeno(TS) fuzzy controllers . The main difference between them is that the Mamdani fuzzy controllers use fuzzy sets whereas the (TS) fuzzy controllers use linear functions, to represent the fuzzy rules. The accessibility and the controllability properties of TS fuzzy logic control systems are studied by Biglarbegian *et al.* (2012) by using differential geometric and Lie-algebraic techniques. Ding and Kandel (2000b,a), established that the observability is a characteristic of the system to estimate the range of the fuzzy initial state with to the knowledge of the fuzzy input and the fuzzy output in a finite time interval for the fuzzy dynamical system with the fuzzy initial state. In the works of Takagi and Sugeno (1985), Johansen *et al.* (2000), Sugeno (1999) a crisp analytical function is used instead of a membership function in a fuzzy model. In recent years many authors Alwadie *et al.* (2003); Ying (1999, 2006); Ding *et al.* (2003, 1999) are studying TS fuzzy controllers, because of their ability to model real world problems. Anand and Murty (2005); Murty *et al.* (1995) established conditions for the controllability and observability of Liapunov type matrix difference system. Murty *et al.* (2008) presented criteria for the existence and uniqueness of solution to Kronecker product initial value problem associated with general first order matrix difference system. Murty *et al.* (2009) studied qualitative properties of general first order matrix difference systems. We obtain a unique solution of the system (1), when $U(n)$ is a crisp continuous matrix. We use fuzzy matrix discrete system to describe fuzzy logic system and extend some of the results of Ding and Kandel (2000a,b) developed for continuous case to that of discrete case by vectorizing the fuzzy matrix discrete system. We obtain sufficient conditions for controllability and observability of the system (1) satisfying the initial condition. The fundamental results established in Murty *et al.* (1995), Rompicharla *et al.* (2019, 2020) have in-fact motivated us to develop our results on fuzzy matrix Lyapunov discrete dynamical systems.

The paper is organized as follows. Section 2 presents basic definitions and results required to understand the paper. Section 3 is concerned with the formation of fuzzy matrix Lyapunov discrete dynamical systems. Sufficient conditions for the controllability and observability of fuzzy matrix Lyapunov discrete dynamical systems are presented in Section 4 and Section 5 respectively. Section 6 presents a numerical example.

2. Preliminaries

In this section basic definitions of Kronecker product, properties of vectorization, α-level set, fundamental matrix solutions of homogeneous and non-homogeneous matrix Lyapunov discrete dynamical systems and corresponding initial value problems are presented. Let $\left(N_{n_0}^{+}\right) = \{n_0, n_0 \pm 1, ..., n_0 \pm k, ...\}$ where n_0 is an integer number.
Let $P_k\left(N_{n_0}^{+}\right)^s$ denotes the family of all nonempty compact convex subsets of $\left(N_{n_0}^{+}\right)^{s\times s}$.
Define the addition and scalar multiplication in $P_k\left(N_{n_0}^{+}\right)^s$ as usual. Rådström (1952) states that $P_k\left(N_{n_0}^{+}\right)^s$ is a commutative semi group under addition, which satisfies the cancellation law. Moreover, if $\alpha, \beta \in (N_{n_0}^{+})$ and $A, B \in P_k\left(N_{n_0}^{+}\right)^s$, then $\alpha(A + B) = \alpha A + \alpha B, \alpha(\beta A) =$

$(\alpha\beta)A, 1.A = A$, and if $\alpha, \beta \geq 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. The distance between A and B is defined by Hausdroff metric $d(A,B) = inf\{\epsilon : A \subset N(B,\epsilon), B \subset N(A,\epsilon)\}$, where $N(A,\epsilon) = \{x \in \left(N_{n_0}^+\right)^s : \|x-y\| < \epsilon$, for some $y \in A\}$.

Definition 1: A set valued function $F : J \to P_k\left(N_{n_0}^+\right)^s$, where $J = [0, N] \subset R(N > 0)$ is said to be measurable if it satisfies any one of the following equivalent conditions:

(1) for all $u \in (N_{n_0}^+)^s, n \to d_{F(n)}(u) = inf_{v \in F(n)} \parallel u - v \parallel$ is measurable,

(2) Gr $F = \{(t,u) \in J \times (N_{n_0}^+)^s : u \in F(n)\} \in \sum \times \beta(N_{n_0}^+)^s$, where $\sum, \beta(N_{n_0}^+)^s$ are Borel σ - field of J and $(N_{n_0}^+)^s$, respectively (Graph measurability),

(3) there exists a sequence $\{f_n(.)\}_{n\geq 1}$ of measurable functions such that $F(n) = \overline{\{f_n(.)\}_{n\geq 1}}$, for all $n \in J$ (Castaing's representation).

We denote by S_F^1 the set of all selections of $F(.)$ that belong to the Lebesgue Bochner space $L^1_{(N_{n_0}^+)^s}(J)$, that is, $S_F^1 = \left\{ f(.) \in L^1_{(N_{n_0}^+)^s}(J) : f(n) \in F(n) \text{ almost every where (a.e)} \right\}$. We present the Aumann's integral as follows: $\int_J F(t)dt = \left\{ \int_J f(t)dt, f(.) \in S_F^1 \right\}$. We say that $F : J \to P_k\left(N_{n_0}^+\right)^s$ is integrably bounded if it is measurable and there exists a function $h : J \to (N_{n_0}^+), h \in L^1_{(N_{n_0}^+)^s}(J)$, such that $\parallel u \parallel \leq h(t), u \in F(t)$. We know that if F is a closed valued measurable multifunction, then $\int_J F(t)dt$ is convex in $(N_{n_0}^+)^s$. Furthermore, if F is integrably bounded, then $\int_J F(t)dt$ is compact in $(N_{n_0}^+)^s$.
Let $E^s = \{u : \left(N_{n_0}^+\right)^s \to [0,1]/u$ satisfies the following $\}$;

(1) u is normal, that is, there exists an $n_0 \in \left(N_{n_0}^+\right)^{s\times s}$ such that $u(n_0) = 1$;

(2) u is fuzzy convex, that is, for $x, y \in \left(N_{n_0}^+\right)^s$ and $0 \leq \lambda \leq 1$, $u(\lambda x + (1-\lambda)y) \geq min[u(x), u(y)]$;

(3) u is upper semicontinuous;

(4) $[u]^0 = \overline{\{x \in \left(N_{n_0}^+\right)^s / u(x) > 0\}}$ is compact.

For $0 < \alpha \leq 1$, the $0 < \alpha \leq 1$, the α- level set is denoted and defined by $[u]^\alpha = \{x \in (N_{n_0}^+ \; u(x) \geq \alpha\}$. Then, from $(1)-(4)$ above, it follows that $[u]^\alpha \in P_k(N_{n_0}^+)$ for all $0 \leq \alpha \leq 1$. Define $D : E^n \times E^n \to [0,\infty]$ by $D(u,v) = sup\{d([u]^\alpha, [v]^\alpha)/\alpha \in [0,1]\}$, where d is the Hausdroff metric defined in $P_k\left(N_{n_0}^+\right)^s$. It is easy to show that D is a metric in E^s and using results of Rådström (1952), we see that (E^s, D) is a complete metric space, but not locally compact. Moreover, the distance D verifies that $D(u+w, v+w) = D(u,v), u, v \in E^s, D(\lambda u, \lambda v) = |\lambda| D(u,v), u, v \in E^s, \lambda \in R$, $D(u+w, v+z) \leq D(u,v) + D(w,z), u, v, w, z \in E^s$. We note that (E^s, D) is not a vector space. But it can be embedded isomorphically as a cone in Banach space (Rådström (1952)). Regarding fundamentals of differentiability and integrability of fuzzy functions, we refer to Kaleva (1987) and Lakshmikantham and Mohapatra (2003).

Definition 2: Let $A \in \mathbb{C}^{r\times s}(\mathbb{R}^{r\times s})$ and $B \in \mathbb{C}^{p\times q}(\mathbb{R}^{p\times q})$. Then Kronecker product of A and B is written as $A \otimes B$ and is defined as a partitioned matrix $\begin{bmatrix} a_{11}B & a_{12}B & ...a_{1s}B \\ a_{21}B & a_{22}B & ...a_{2s}B \\ ... & ... & ... \\ a_{r1}B & a_{r2}B & a_{rs}B \end{bmatrix}$ which is an $rp \times sq$ matrix and is in $\mathbb{C}^{rp\times sq}(\mathbb{R}^{rp\times sq})$.
The Kronecker product has the following properties.

(1) $(A \otimes B)^* = A^* \otimes B^*$

(2) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

(3) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.
This rule holds, provided the dimensions of the matrices are such that expressions are defined.

(4) $||\, A \otimes B \,|| = \|A\| \, \|B\|$, where $(\|A\| = max_{i,j}\,|a_{ij}|)$

(5) $(A + B) \otimes C = (A \otimes C) + (B \otimes C)$.
Vectorization of matrix A is denoted by VEC $(A) = \widehat{A}$ and defined as follows.

Definition 3: Let $A = [a_{ij}] \in \mathbb{C}^{r\times s}(\mathbb{R}^{r\times s})$, we denote VEC$(A) = \widehat{A} = [A_{.1}, A_{.2}, ..., A_{.s}]^T$ where $A_{.j} = [a_{1j}, a_{2j}, ..., a_{rj}]^T, (1 \le j \le s)$, where X is a matrix of size $s \times s$.
Vectorization has the following properties.
1. VEC $(AXB) = (B^* \otimes A)$ VEC X.
2. If A and B are square matrices of order s, then
VEC $(AX) = (I_s \otimes A)$ VEC X;
VEC $(XB) = (B^* \otimes I_s)$ VEC X.

Theorem 1: [Ralescu (1979); Murty and Kumar (2008)]
If $u \in E^s$ then

(1) $[u]^\alpha \in P_k\left(N_{n_0}^+\right)^{s\times s}$ for all $0 \le \alpha \le 1$;

(2) $[u]^{\alpha_2} \subset [u]^{\alpha_1}$ for all $0 \le \alpha_1 \le \alpha_2 \le 1$;

(3) α_k is non decreasing sequence converging to $\alpha > 0$, then $[u]^\alpha = \cap_{k\ge1}[u]^{\alpha_k}$. Conversely, if $\{A^\alpha : 0 \le \alpha \le 1\}$ is a family of subsets of $\left(N_{n_0}^+\right)^{s\times s}$ satisfying $(1) - (3)$, then there exists a $u \in E^s$ such that $[u]^\alpha = A^\alpha$ for $0 < \alpha \le 1$ and $[u]^0 = \overline{U_{0\le\alpha\le1}A^\alpha} \subset A^0$.

Theorem 2: [Sundaranand Putcha (2014); Rompicharla *et al.* (2019)]
Let $\phi(n, 0)$ and $\phi^*(n, 0)$ be the fundamental matrix solutions of
$\Delta T(n) = A(n)T(n)$ and $\Delta T(n) = T(n)A^T(n)$.
Then the matrix $\phi(n, n_0)C\phi^*(n, n_0)$ (where C is a constant square matrix of size s) be the fundamental matrix for the system

$$\Delta T(n) = A(n)T(n) + T(n)A^T(n) + A(n)T(n)A^T(n), T(n_0) = I_s. \tag{3}$$

The matrix $(\phi^*(n,n_0)\otimes\phi(n,n_0))$ is the fundamental matrix of

$$\Delta\widehat{T}(n)=((A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n)))\widehat{T}(n),\widehat{T}(n_0)=\widehat{T_0} \quad (4)$$

and the solution of (4) is $\widehat{T}(n)=(\phi^*(n,0)\otimes\phi(n,0)\widehat{T_0}$.

Theorem 3: [Putcha and Prathyusha (2019)]
Let $\phi(n,n_0)C\phi^*(n,n_0)$ be the fundamental matrix for the system (3). Then the unique solution of the initial value problem

$$\Delta\widehat{T}(n)=[(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}(n),\widehat{T}(n_0)=\widehat{T}_0 \quad (5)$$

is given by

$$\widehat{T}(n)=(\phi^*(n,0)\otimes\phi(n,0))\widehat{T}_0+\sum_{j=0}^{n-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))\widehat{U}(j).$$

3. Inclusion approach to Fuzzy Matrix Lyapunov discrete dynamical system

This section presents a method of the conversion of fuzzy matrix Lyapunov discrete dynamical system to a matrix Lyapunov difference inclusion. Thus the solution of a fuzzy matrix Lyapunov discrete dynamical system can be expressed as the solution set of the corresponding matrix Lyapunov difference inclusion.
Let $u_i(n)\in E^1, n\in J, i=1,2,..,s^2$, and define

$$\begin{aligned}\widehat{U}(n)=(u_1(n),u_2(n),...,u_{s^2}(n))&=u_1(n)\times u_2(n)\times...\times u_{s^2}(n)\\&=\{(u_1^\alpha(n),u_2^\alpha(n),...,u_{s^2}^\alpha(n):\alpha\in[0,1]\}\\&=\{(\tilde{u}_1(n),\tilde{u}_2(n),...,\tilde{u}_{s^2}(n):\tilde{u}_i(n)\in u_i^\alpha(n),\alpha\in[0,1]\},\end{aligned} \quad (6)$$

where $u_i^\alpha(n)$ is the α-level set of $u_i(n)$. From the above definition of $\widehat{U}(n)$ and Theorem 1, it can be easily seen that $\widehat{U}(n)\in E^{s^2}$. We now show that the following system S_2 defined by

$$\Delta\widehat{T}(n)=((A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n)))\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}(n),\widehat{T}(0)=\widehat{T}_0 \quad (7)$$

and

$$\widehat{Y}(n)=((I_s\otimes C(n)))\widehat{T}(n)+(I_s\otimes D(n))\widehat{U}(n) \quad (8)$$

determines a fuzzy system by using the fuzzy control $\widehat{U}(n)$. Assume that $\widehat{U}(n)$ is continuous in E^{s^2}. Then the set $\widehat{U}^\alpha=u_1(n)\times u_2(n)\times,...,\times u_{s^2}(n)$ is a convex and compact set in $\left(N_{n_0}^+\right)^{s^2}$. For any positive number N, consider the following inclusions

$$\Delta\widehat{T}(n)\in[(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}^\alpha(n),n\in[0,N], \quad (9)$$

$$\widehat{T}(n_0)\in\widehat{T}_0. \quad (10)$$

Let $\widehat{T}^\alpha$ be the solution of (9) satisfying (10)

Lemma 1: $[\widehat{T}(n)]^\alpha \in P_k(N_{n_0}^+)^{s^2}$, for every $0 \leq \alpha \leq 1, n \in [0, N]$.

Proof: We can observe that $\widehat{T}^\alpha$ is non empty since $\widehat{U}^\alpha(n)$ has measurable selection.
By choosing

$$K = max_{n\in[0,N]} \parallel \phi(n, n_0) \parallel, L = max_{n\in[0,N]} \parallel \phi^*(n, n_0)) \parallel,$$

$$M = max\{\parallel u(n) \parallel : u(n) \in \widehat{U}^\alpha(n), n \in [0, N]\},$$
$$T = max_{n\in[0,N]} \parallel F(n) \parallel, J = max_{n\in[0,N]} \parallel I_s \parallel = 1.$$

If for any $\widehat{T} \in \widehat{T}^\alpha$, there exists a control $u(n) \in \widehat{U}^\alpha(n)$ such that

$$\widehat{T}(n) = (\phi^*(n, n_0) \otimes \phi(n, n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1} (\phi^*(n, j+1) \otimes \phi(n, j+1))(I_s \otimes F(j))U(j). \quad (11)$$

By taking norm on both sides of the equation (11), we get

$$\parallel \widehat{T}(n) \parallel \leq KL \parallel \widehat{T}_0 \parallel + KLTMN.$$

Hence $\widehat{T}^\alpha$ is bounded.
For any $n_1, n_2 \in [0, N]$, consider,

$$\widehat{T}(n_1) - \widehat{T}(n_2) = (\phi^*(n, n_1) \otimes \phi(n, n_1))\widehat{T}_0 + \sum_{j=n_0}^{n_1-1} (\phi^*(n_1, j+1) \otimes \phi(n_1, j+1))(I_s \otimes F(j))u(j) -$$

$$(\phi^*(n, n_2) \otimes \phi(n, n_2))\widehat{T}_0 - \sum_{j=n_0}^{n_2-1} (\phi^*(n_2, j+1) \otimes \phi(n_2, j+1))(I_s \otimes F(j))u(j)$$

Therefore

$$\parallel \widehat{T}(n_1) - \widehat{T}(n_2) \parallel \leq \parallel (\phi^*(n, n_1) \otimes \phi(n, n_1)) - (\phi^*(n, n_2) \otimes \phi(n, n_2)) \parallel \parallel \widehat{T}_0 \parallel +$$

$$\sum_{j=n_2-1}^{n_1-1} \parallel (\phi^*(n_1, j+1) \otimes \phi(n_1, j+1))(I_s \otimes F(j))u(j) \parallel +$$

$$\sum_{j=n_0}^{n_2-1} \parallel [(\phi^*(n_1, j+1) \otimes \phi(n_1, j+1)) - (\phi^*(n_2, j+1) \otimes \phi(n_2, j+1))](I_s \otimes F(j))u(j) \parallel$$

$$\leq \parallel (\phi^*(n, n_1) \otimes \phi(n, n_1)) - (\phi^*(n, n_2) \otimes \phi(n, n_2)) \parallel \parallel \widehat{T}_0 \parallel + KLTM \mid n_1 - n_2 \mid +$$

$$MT \sum_{j=n_0}^{N-1} \parallel (\phi^*(n_1, j+1) \otimes \phi(n_1, j+1) - (\phi^*(n_2, j+1) \otimes \phi(n_2, j+1)) \parallel .$$

Since $(\phi^*(n, n_0))$ and $(\phi(n, n_0))$ are both uniformly continuous on $[0, N]$,
$\widehat{T}$ is equicontinous. Thus, $\widehat{T}^\alpha$ is relatively compact.
Let $\widehat{T}_k \in \widehat{T}^\alpha$ and $\widehat{T}_k \to \widehat{T}$. For each $\widehat{T}_k$, there is a $u_k \in \widehat{U}^\alpha(n)$ such that

$$\widehat{T}_k(n) = ((\phi^*(n, n_0) \otimes \phi(n, n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1} (\phi^*(n, j+1) \otimes \phi(n, j+1))(I_s \otimes F(j))U_k(j). \quad (12)$$

Since $u_k \in \widehat{U}^\alpha(n)$ is closed, then there is a subsequence $< u_{k_i} >$ of $< u_k >$ converging weakly to $u \in \widehat{U}^\alpha(n)$. From Mazur's theorem Conway and Voglmeir (2016), there exists a sequence of numbers $\lambda_i > 0$, $\Sigma\lambda_i = 1$ such that $\Sigma\lambda_i u_{k_i}$ converges strongly to u. Thus from (12) we have

$$\Sigma\lambda_i \widehat{T}_{K_i}(n) = \sum \lambda_i((\phi^*(n,n_0)\otimes\phi(n,n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))\Sigma\lambda_i u_{k_i}(j). \tag{13}$$

As $i \to \infty$ from equation (13) and Fatuou's lemma, it follows that

$$\widehat{T}(n) = (\phi^*(n,n_0)\otimes\phi(n,n_0))\widehat{T}_0 + \sum(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))u(j).$$

Thus $\widehat{T}(n) \in \widehat{T}^\alpha$, and hence $\widehat{T}^\alpha$ is closed.
Let $\widehat{T}_1, \widehat{T}_2 \in \widehat{T}^\alpha$, then there exists $u_1, u_2 \in \widehat{U}^\alpha(n)$ such that

$$\Delta\widehat{T}_1(n) = [(A(n)\otimes A(n)) + (A(n)\otimes I_s) + (I_s\otimes A(n))]\widehat{T}_1(n) + (I_s\otimes F(n))u_1(n),$$

$$\Delta\widehat{T}_2(n) = [(A(n)\otimes A(n)) + (A(n)\otimes I_s) + (I_s\otimes A(n))]\widehat{T}_2(n) + (I_s\otimes F(n))u_2(n).$$

Let $\widehat{T}(n) = \lambda\widehat{T}_1(n) + (1-\lambda)\widehat{T}_2(n), 0 \le \lambda \le 1$ then

$$\begin{aligned}
&\Delta\widehat{T}(n)\\
&\quad = \lambda\Delta\widehat{T}_1(n) + (1-\lambda)\Delta\widehat{T}_2(n)\\
&\quad = \lambda((A(n)\otimes A(n)) + (A(n)\otimes I_s) + (I_s\otimes A(n)))\widehat{T}_1(n)\\
&\qquad + (I_s\otimes F(n))u_1(n) + (1-\lambda)(((A(n)\otimes A(n)) + (A(n)\otimes I_s)\\
&\qquad + (I_s\otimes A(n)))\widehat{T}_2(n) + (I_s\otimes F(n))u_2(n)\\
&\quad = [(I_s\otimes A(n)) + (A(n)\otimes I_s) + (A(n)\otimes A(n))][\lambda\widehat{T}_1(n)\\
&\quad + (1-\lambda)\widehat{T}_2(n)] + (I_s\otimes F(n))[\lambda u_1(n) + (1-\lambda)u_2(n)]
\end{aligned}$$

Since $\widehat{U}^\alpha(n)$ is convex, $\lambda u_1(n) + (1-\lambda)u_2(n) \in \widehat{U}^\alpha(n)$, we have

$$\Delta\widehat{T}(n) \in (I_s\otimes A(n) + A(n)\otimes I_s + A(n)\otimes A(n))\widehat{T}(n) + (I_s\otimes F(n))\widehat{U}^\alpha(n),$$

i.e.,$\widehat{T} \in \widehat{T}^\alpha$. Thus $\widehat{T}^\alpha$ is convex. Therefore $\widehat{T}^\alpha$ is non empty, compact and convex in $\mathbb{C}[[0,N],(N_{n_0}^+)^{s^2}]$. Thus, from Arzela-Ascoli theorem, it follows that $[\widehat{T}(n)]^\alpha$ is convex in $(N_{n_0}^+)^{s^2}$, for every $n \in [0,N]$. Therefore $[\widehat{T}(n)]^\alpha \in P_k((N_{n_0}^+)^{s^2})$ for every $0 \le \alpha \le 1, n \in [0,N]$. □

Lemma 2: $[\widehat{T}(n)]^{\alpha_2} \subset [\widehat{T}(n)]^{\alpha_1}$, for all $0 \le \alpha_1 \le \alpha_2 \le 1$.

Proof: Let $0 \le \alpha_1 \le \alpha_2 \le 1$. Since $\widehat{U}^{\alpha_2}(n)$ is contained in $\widehat{U}^{\alpha_1}(n)$, it follows that

$$\widehat{U}^{\alpha_2}(n) = u_1^{\alpha_2}(n)\times u_2^{\alpha_2}(n)\times ...\times u_{s^2}^{\alpha_2}(n) \subset u_1^{\alpha_1}(n)\times u_2^{\alpha_1}(n)\times ...\times u_{s^2}^{\alpha_1}(n) = \widehat{U}^{\alpha_1}(n)$$

and also the following inclusions:

$$\begin{aligned}
\Delta\widehat{T}(n) &\in [(A(n)\otimes A(n)) + (A(n)\otimes I_s) + (I_s\otimes A(n))]\widehat{T}(n) + (I_s\otimes F(n))\widehat{U}^{\alpha_2}(n)\\
&\subset [(A(n)\otimes A(n)) + (A(n)\otimes I_s) + (I_s\otimes A(n))]\widehat{T}(n) + (I_s\otimes F(n))\widehat{U}^{\alpha_1}(n)
\end{aligned} \tag{14}$$

Consider the following inclusions:

$$\Delta\widehat{T}(n) \in [(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}^{\alpha_2}(n), n \in [0, N] \quad (15)$$

$$\Delta\widehat{T}(n) \in [(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}^{\alpha_1}(n), n \in [0, N] \quad (16)$$

Let $\widehat{T}^{\alpha_2}$ and $\widehat{T}^{\alpha_1}$ be the solution sets of (15) and (16) respectively. Clearly the solution of (15) satisfies the following inclusion:

$$\widehat{T}(n) \in (\phi^*(n,n_0)\otimes\phi(n,n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))S^1_{\widehat{U}^{\alpha_2}(j+1)}$$

$$\subset (\phi^*(n,n_0)\otimes\phi(n,n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))S^1_{\widehat{U}^{\alpha_1}(j+1)}. \quad (17)$$

Thus $\widehat{T}^{\alpha_2} \subset \widehat{T}^{\alpha_1}$. And hence $\widehat{T}^{\alpha_2}(n) \subset \widehat{T}^{\alpha_1}(n)$ □

Lemma 3: If $< \alpha_k >$ is nondecreasing sequence converging to $\alpha > 0$ then $\widehat{T}^{\alpha}(n) = \cap_{k\geqslant 1}\widehat{T}^{\alpha_k}(n)$.

Proof: Let

$$\widehat{U}^{\alpha_k}(n) = u_1^{\alpha_k}\times u_2^{\alpha_k}\times,...,\times u_{s^2}^{\alpha_k}, \widehat{U}^{\alpha}(n) = u_1^{\alpha}\times u_2^{\alpha}\times,...,u_{s^2}^{\alpha}$$

and consider the inclusions

$$\Delta\widehat{T}(n) \in [(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}^{\alpha_k}(n) \quad (18)$$

$$\Delta\widehat{T}(n) \in [(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}^{\alpha}(n) \quad (19)$$

Let $\widehat{T}^{\alpha_k}$ and $\widehat{T}^{\alpha}$ be the solution sets of 18 and 19 respectively. Since $u_i(n)$ is a fuzzy set and from Theorem 1, we have

$$u_i^{\alpha} = \cap_{k\geq 1}u_i^{\alpha_k}, \quad (20)$$

we consider

$$\widehat{U}^{\alpha}(n) = u_1^{\alpha}\times u_2^{\alpha}\times,...,\times u_{s^2}^{\alpha} = \cap_{k\geq 1}u_1^{\alpha_k}\times\cap_{k\geq 1}u_2^{\alpha_k}\times,...,\cap_{k\geq 1}u_{s^2}^{\alpha_k} = \cap_{k\geq 1}\widehat{U}^{\alpha_k}(n) \quad (21)$$

and then $S^1_{\widehat{U}^{\alpha}(n)} = S^1_{\cap_{k\geqslant 1}\widehat{U}^{\alpha_k}(n)}$.
Therefore

$$\Delta\widehat{T}(n) \in [(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}^{\alpha}(n)$$

$$= [(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\cap_{k\geq 1}\widehat{U}^{\alpha_k}(n)$$

$$\subset [(A(n)\otimes A(n))+(A(n)\otimes I_s)+(I_s\otimes A(n))]\widehat{T}(n)+(I_s\otimes F(n))\widehat{U}^{\alpha_k}(n), k = 1,2,3,... \quad (22)$$

Thus we have $\widehat{T}^{\alpha} \subset \widehat{T}^{\alpha_k}, k = 1,2,3...$, which implies that

$$\widehat{T}^{\alpha} \subset \cap_{k\geq 1}\widehat{T}^{\alpha_k}. \quad (23)$$

Let $\widehat{T}$ be the solution set of the inclusion (18) for $k \geq 1$. Then

$$\widehat{T}(n) \in (\phi^*(n,n_0) \otimes \phi(n,n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1} (\phi^*(n,j+1) \otimes \phi(n,j+1))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha_k}}(n). \quad (24)$$

It follows that

$$\widehat{T}(n) \in (\phi^*(n,n_0) \otimes \phi(n,n_0))\widehat{T}_0 + \cap_{k\geq 1} \sum_{j=n_0}^{n-1} (\phi^*(n,j+1) \otimes \phi(n,j+1))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha k}(n)}$$
$$\subset (\phi^*(n,n_0) \otimes \phi(n,n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1} (\phi^*(n,j+1) \otimes \phi(n,j+1))(I_s \otimes F(j))S^I_{\cap k\geq 1 \widehat{U}^{\alpha_k}}$$
$$= (\phi^*(n,n_0) \otimes \phi(n,n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1} (\phi^*(n,j+1) \otimes \phi(n,j+1))(I_s \otimes F(j))S^1_{\widehat{U}^{\alpha_k}}.$$

This implies that $\widehat{T} \in \widehat{T}^{\alpha}$. Therefore,

$$\cap_{k\geq 1}\widehat{T}^{\alpha_k} \subset \widehat{T}^{\alpha}. \quad (25)$$

From (23) and (25), we have $\widehat{T}^{\alpha} = \cap_{k\geq 1}\widehat{T}^{\alpha_k}$ and hence, $\widehat{T}^{\alpha}(n) = \cap_{k\geq 1}\widehat{T}^{\alpha_k}(n)$.

□

The following theorem establishes the equivalence of fuzzy matrix Lyapunov discrete dynamical system with that of matrix Lyapunov difference inclusion and presents the solution set.

Theorem 4: The system (7) and (8) is a fuzzy matrix Lyapunov discrete dynamical system, and it can be expressed as

$$\Delta\widehat{T}(n) = [(A(n) \otimes A(n)) + (A(n) \otimes I_s) + (I_s \otimes A(n))]\widehat{T}(n) + (I_s \otimes F(n))\widehat{U}(n), \widehat{T}(n_0) = \{\widehat{T}_0\}; \quad (26)$$

$$\widehat{Y}(n) = (I_s \otimes C(n))\widehat{T}(n) + (I_s \otimes D(n))\widehat{U}(n). \quad (27)$$

The solution set of fuzzy matrix Lyapunov discrete dynamical system (26) and (27) is given by

$$\widehat{T}(n) \in (\phi^*(n,n_0) \otimes \phi(n,n_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1} (\phi^*(n,j+1) \otimes \phi(n,j+1))(I_s \otimes F(j))\widehat{U}(j). \quad (28)$$

Proof: Proof follows from the Lemmas 1,2,3 and Theorem 1 since there exists $\widehat{T}(n) \in E^{s^2}$ on $[0,N]$ such that $\widehat{T}^{\alpha}(n)$ is a solution set to the difference inclusions (9) and (10).

□

The following corollary is the input characterization of the solution set of the initial value problem associated with the non homogeneous matrix Lyapunov discrete dynamical system.corollary

Corollary 1: If the input is in the form $\widehat{U}(n) = \widetilde{u}_1(n) \times \widetilde{u}_2(n) \times ... \times \widetilde{u}_i(n) \times ... \times \widetilde{u}_{s^2}(n)$ where $\widetilde{u}_k(n) \in R^1, k \neq i$ are crisp numbers, then the ith component of the solution set of (5) is a fuzzy set in E^1.

The following definitions fuzzy controllability, fuzzy observability, α-level sets and product of fuzzy matrix with α-level sets are essential for exploring the controllability and observability of the fuzzy matrix Lyapunov discrete dynamical system.

Definition 4: The fuzzy system given by equations (26)-(27) is said to be completely controllable if for any initial state $\widehat{T}(n_0) = \widehat{T}_0$ and any given final state $\widehat{T}_f$ there exists a finite time $n_1 > 0$ and a control $\widehat{U}(n)$, $0 \leq n \leq n_1$ such that $\widehat{T}(n_1) = \widehat{T}_f$.

Definition 5: The fuzzy system given by equations (26)-(27) is said to be completely observable over the interval $[0, N]$ if the knowledge of rule base of input $\widehat{U}$ and output $\widehat{Y}$ over $[0, N]$ suffices to determine a rule base of initial state $\widehat{T}_0$. Let $u_i^l, y_i^l, i = 1, 2, .., s^2, l = 1, 2, ..., m$, be fuzzy sets in E^l. We assume that the rule base for the input and output is given by

$$R^l : \text{If } \ \tilde{u}_1(n) \ \text{ is in } \ u_1^l(n), \tilde{u}_2(n) \ \text{ is in } \ u_2^l(n), ..., \tilde{u}_{s^2}(n) \ \text{ is in } \ u_{s^2}^l(n),$$

$$\text{Then } \ \tilde{y}_1(n) \ \text{ is in } \ y_1^l(n), \tilde{y}_2(n) \ \text{ is in } \ y_2^l(n), ..., \tilde{y}_{s^2}(n) \ \text{ is in } \ y_{s^2}^l(n), l = 1, 2, ..., m \tag{29}$$

and the output can be expressed as a function of input by the equation

$$\widehat{Y}(n) = (I_s \otimes C(n))\widehat{T}(n) + (I_s \otimes D(n))\widehat{U}(n).$$

Definition 6: Let $x, y \in E^{s^2}$ and $x = x_1 \times x_2 \times ... \times x_{s^2}$ and $y = y_1 \times y_2 \times ... \times y_{s^2}$, $x_i, y_i \in E^1, i = 1, 2, ..., s^2$.

If $y = z+x$, then $z = y-x$ which is defined by $[z]^\alpha = [y-x]^\alpha = [y]^\alpha - [x]^\alpha = \begin{bmatrix} [y_1]^\alpha - [x_1]^\alpha \\ ... \\ [y_{s^2}]^\alpha - [x_{s^2}]^\alpha \end{bmatrix}$

If $y = w - x$, then $w = y + x$ which is defined by

$$[w]^\alpha = [y + x]^\alpha = [y]^\alpha + [x]^\alpha = \begin{bmatrix} [y_1]^\alpha + [x_1]^\alpha \\ ... \\ [y_{s^2}]^\alpha + [x_{s^2}]^\alpha \end{bmatrix}.$$

Definition 7: Let $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s^2} \\ c_{21} & c_{22} & \cdots & c_{2s^2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{s^21} & c_{s^22} & \cdots & c_{s^2s^2} \end{bmatrix}$ be an $s^2 \times s^2$ matrix, $p = p_1 \times p_2 \times ... \times p_{s^2}$, let $p_i \in E^1, i = 1, 2, ..., s^2$, be a fuzzy set in E^{s^2}, and let $[p_i]^\alpha$ be α- level sets of p_i, define the product Cp of C and p as

$$[Cp]^\alpha = C[p]^\alpha = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1s^2} \\ c_{21} & c_{22} & \cdots & c_{2s^2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{s^21} & c_{s^22} & \cdots & c_{s^2s^2} \end{bmatrix} \begin{bmatrix} [p_1]^\alpha \\ [p_2]^\alpha \\ \cdots \\ [p_{s^2}]^\alpha \end{bmatrix} = \begin{bmatrix} c_{11}[p_1]^\alpha + ... + c_{1s^2}[p_{s^2}]^\alpha \\ c_{21}[p_1]^\alpha + ... + c_{2s^2}[p_{s^2}]^\alpha \\ \cdots \\ c_{s^21}[p_1]^\alpha + ... + c_{s^2s^2}[p_{s^2}]^\alpha \end{bmatrix}$$

4. Controllability of Fuzzy Matrix Lyapunov discrete dynamical system

A sufficient condition for controllability of fuzzy matrix Lyapunov discrete dynamical system is derived by fuzzy rule based approach via corresponding Lyapunov difference inclusion.

Theorem 5: The fuzzy system (26)-(27) is completely controllable if the $s^2 \times s^2$ symmetric controllable matrix

$$W(n_0,N)=\sum_{j=n_0}^{N-1}[(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))(I_s\otimes F(j))^*(\phi^*(N,j+1)\otimes\phi(N,j+1))^*] \tag{30}$$

(Where $*$ represents the conjugate transpose) is nonsingular. Furthermore, the fuzzy control $\widehat{U}(n)$ which transfers the state of the system from $\widehat{T}(0)=\widehat{T}_0$ to a fuzzy state

$$\widehat{T}(N)=\widehat{T}_f=(t_{f_1},t_{f_2},...,t_{f_{s^2}}) \tag{31}$$

can be modified by the following fuzzy rule base:

$$R:\text{IF}\quad \tilde{t_1}\quad \text{is in}\quad t_{f_1},t_{f_2},...,\tilde{t}_{f_{s^2}}\quad \text{is in}\quad t_{f_{s^2}}\text{THEN}\quad \tilde{u}_1\quad \text{is in}\quad u_1...\tilde{u}_{s^2}\quad \text{is in}\quad u_{s^2} \tag{32}$$

where

$$\begin{aligned}(\tilde{u}_1(n),\tilde{u}_2(n),...,\tilde{u}_{s^2}(n)) = & \\ &\frac{1}{N}(I_s\otimes F(n))^{-1}(\phi^*(N,j+1)\otimes\phi(N,j+1)))^{-1}\\ &\times(\tilde{t_1}(N),\tilde{t_2}(N),...,t_{f_i},..,\tilde{t}_{s^2}(N))\\ &-(I_s\otimes F(n))^*(\phi^*(N,j+1)\otimes\phi(N,j+1))^*\\ &W^{-1}(n_0,N)(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}(0), i=1,2,..,s^2.\end{aligned}$$

Proof: Suppose that the symmetric controllability matrix $W(n_0,N)$ is nonsingular. Therefore $W^{-1}(n_0,N)$ exists. By multiplying equation (30) on both sides by $W^{-1}(n_0,N)(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0$, we get

$$(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0=\sum_{j=0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))$$

$$\times(I_s\otimes F(j))^*(\phi^*(N,j+1)\otimes\phi(N,j+1)^*)W^{-1}(0,N)(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0. \tag{33}$$

Now our problem is to find the control $\widehat{U}(n)$ such that

$$\widehat{T}(N)=\widehat{T}_f=(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0+\sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))\widehat{U}(j). \tag{34}$$

Since $\widehat{T}$ is fuzzy and $\widehat{U}(n)$ must be fuzzy, otherwise the left side of equation (34) cannot be equal to the crisp right side. Now $\widehat{T}_f$ can be written as

$$\begin{aligned}\widehat{T}_f=\frac{1}{N}\sum_{j=n_0}^{N-1}\widehat{T}_f=&\frac{1}{N}\sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))\\ &\times(I_s\otimes F(j))^{-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))^{-1}\widehat{T}_f.\end{aligned} \tag{35}$$

From (34) and (35) we have

$$\frac{1}{N}\sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))\times(I_s\otimes F(j))^{-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))^{-1}\widehat{T}_f$$

$$= (\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0 + \sum_{j=0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))\widehat{U}(j). \quad (36)$$

From (33) and (35) it follows that

$$\frac{1}{N}\sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))(I_s\otimes F(j))^{-1}$$

$$(\phi^*(N,j+1)\otimes\phi(N,j+1))^{-1}\widehat{T}_f = \sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))$$

$$(I_s\otimes F(j))^*(\phi^*(N,j+1)\otimes\phi(N,j+1))^*\times W^{-1}(n_0,N)(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0$$

$$+\sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))\widehat{U}(j) \quad (37)$$

i.e.,

$$\sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))(I_s\otimes F(j))\widehat{U}(j) = \sum_{j=n_0}^{N-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))$$

$$(I_s\otimes F(j))\{\frac{1}{N}(I_s\otimes F(j))^{-1}(\phi^*(N,j+1)\otimes\phi(N,j+1))^{-1}\widehat{T}_f -$$

$$(I_s\otimes F(j))^*(\phi^*(N,j+1)\otimes\phi(N,j+1))^*W^{-1}(n_0,N)(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0\}. \quad (38)$$

Now $\widehat{U}(N)$ can be expressed as

$$\widehat{U}(N) = \frac{1}{N}(I_s\otimes F(n))^{-1}(\phi^*(N,n+1)\otimes\phi(N,n+1))^{-1}\widehat{T}_f -$$

$$(I_s\otimes F(n))^*(\phi^*(N,n+1)\otimes\phi(N,n+1))^*\times W^{-1}(n_0,N)(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0\}. \quad (39)$$

Now we have the following two possible cases for (39)

Case(i)
When $\widehat{T}(N) = \widehat{T}_f = (\widetilde{t}_1(N), \widetilde{t}_2(N), ..., \widetilde{t}_{s^2}(N))$ is a crisp point, equation (39) gives corresponding control $\widehat{U}(n)$ and is given by $\widehat{U}(n) = (\widetilde{u}_1, \widetilde{u}_2, ..., \widetilde{u}_{s^2})$.

Case(ii)
When $\widehat{T}(N) = (\widetilde{t}_1(N), \widetilde{t}_2(N), ..., t_{f_i}, ..., \widetilde{t}_{s^2}(N))$, equation (39) gives the corresponding control $\widehat{U}(n)$ and is given by $\widehat{U}(n) = (\widetilde{u}_1, \widetilde{u}_2, ..., u_i, ..., \widetilde{u}_{s^2})$ in which the component of $\widehat{U}(n)$ is a fuzzy set in E^1.
Clearly $\widetilde{u}_i(n)$ is in $u_i(n)$, $\mu_{t_{f_i}}(\widetilde{t}_i(N))$ gives the grade of the membership of $\widetilde{t}_i(N)$ in t_{f_i}.
Hence fuzzy rule base for the control $\widehat{U}$ given by equations (31) and (32) follows. □

Note: The converse of the above theorem need not be true. Since fuzzy rule base cannot imply the non singularity of the controllability matrix $W(n_0, N)$ given by (30). It follows that the condition in the above theorem is only sufficient condition but not necessary.

5. Observability of Fuzzy Matrix Lyapunov discrete dynamical system

A sufficient condition for observability of fuzzy matrix Lyapunov discrete dynamical system is constructed by center average defuzzifier approach via corresponding Lyapunov difference inclusion.

Theorem 6: Assume that the fuzzy rule base (29) holds, then the fuzzy system (26)-(27) is completely observable over the interval $[0, N]$ and $(I_s \otimes C(N))(\phi^*(N, N_0) \otimes \phi(N, N_0))$ is nonsingular. Furthermore, if

$$\widehat{T}_0 = (\tilde{t}_0^1, \tilde{t}_0^2, ..., \tilde{t}_0^{s^2}) \tag{40}$$

then one has the following rule base for the initial value $\widehat{T_0}$,

$$R^l : \text{If } \ \tilde{u}_1(N) \in u_1^l(N), ..., \tilde{u}_{s^2}(N) \in u_{s^2}^l(N) \ \text{ and } \ \tilde{y}_1(N) \in y_1^l(N), ..., \tilde{y}_{s^2}(N) \in y_{s^2}^l(N)$$

$$\text{Then } \ \tilde{t}_0^1 \ \text{ is in } \ t_0^l(1), ..., \tilde{t}_0^{s^2}(n) \ \text{ is in } \ t_0^l(S^2), l = 1, 2, .., m. \tag{41}$$

where

$$t_0^l(i) = [(I_S \otimes C(N))(\phi^*(N, N_0) \otimes \phi(N, N_0))]^{-1}\{V_i^l(N) - (I_S \otimes D(N))\widehat{U}(N) -$$

$$(I_s \otimes C(N)) \sum_{j=n_0}^{N-1} (\phi^*(N, j+1) \otimes \phi(N, j+1))(I_s \otimes F(j))H_i^l(j)\}, \tag{42}$$

$$\widehat{T_0} = ((I_s \otimes C(N))(\phi^*(N, N_0) \otimes \phi(N, N_0))^{-1}\{\tilde{y}(N) - (I_s \otimes D(N))\tilde{U}(N) -$$

$$(I_s \otimes C(N)) \times \sum_{j=n_0}^{N-1} (I_s \otimes \phi(N - j - 1))(I_s \otimes F(j))\tilde{U}(j), \tag{43}$$

$$H_i^l(n) = \tilde{u}_1(n) \times \tilde{u}_2(n) \times, ..., u_i^l(n) ... \times \tilde{u}_{s^2}(n), \tag{44}$$

$$V_i^l(n) = \tilde{y}_1(n) \times \tilde{y}_2(n) \times, ..., y_i^l(n) ... \times \tilde{y}_{s^2}(n), i = 1, 2, .., s^2; l = 1, 2, .., m. \tag{45}$$

Proof: Consider the case when $l = 1$. Let

$$\tilde{u}(n) = (\tilde{u}_1(n), \tilde{u}_2(n), ..., \tilde{u}_{s^2}(n)), \tag{46}$$

$$\tilde{y}(n) = (\tilde{y}_1(n), \tilde{y}_2(n), ..., \tilde{y}_{s^2}(n)) \tag{47}$$

Let $\mu_{u_i^1(n)}(\tilde{u}_i(n))$ be the grade of the membership of $\tilde{u}_i(n)$ in $u_i^1(n)$, and let $\mu_{y_i^1(n)}(\tilde{y}_i(n))$ be the grade of membership of $\tilde{y}_i(n)$ in $y_i^1(n)$. Since $(I_s \otimes C(N))(\phi^*(N, N_0) \otimes \phi(N, N_0))$ is nonsingular and from (28) we have

$$\widehat{T}_0 = [(I_s \otimes C(N))(\phi^*(N, N_0) \otimes \phi(N, N_0))]^{-1}\{\tilde{y}(N) - (I_s \otimes D(N))\tilde{u}(N) -$$

$$(I_s \otimes C(N)) \sum_{j=n_0}^{N-1} (\phi^*(n, j+1) \otimes \phi(n, j+1))(I_s \otimes F(j))\tilde{u}(j)\} \tag{48}$$

When the input and output are both fuzzy sets it follows from equation 8 that

$$(I_s \otimes C(N))\widehat{T}(n) = \widehat{Y}(n) - (I_s \otimes D(N))\tilde{u}(N) \tag{49}$$

is a fuzzy set. From equation (28), we get

$$(I_s \otimes C(N))(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0 + \sum_{j=n_0}^{N-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))\widehat{U}(j)$$
$$= \widehat{Y}(n) - ((I_s\otimes D(N)))\widehat{U}(n). \tag{50}$$

Using Definition 6, it follows that

$$(I_s \otimes C(N))(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0 = \{\widehat{Y}(n) - (I_s\otimes D(N))\widehat{U}(n) -$$
$$(I_s\otimes C(N))\}\sum_{j=n_0}^{N-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))\widehat{U}(j). \tag{51}$$

Since $(I_s \otimes C(N))(\phi^*(N,N_0)\otimes\phi(N,N_0))$ is nonsingular, we have

$$\widehat{T}_0 = [(I_s \otimes C(N))(\phi^*(N,N_0)\otimes\phi(N,N_0)]^{-1}\{\widehat{Y}(N) - ((I_s\otimes D(N))\widehat{U}(N)) -$$
$$(I_s\otimes C(N))\times\sum_{j=n_0}^{N-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))\widehat{U}(j)\} \tag{52}$$

Now, the initial value $\widehat{T}_0$ should be a fuzzy set but not a crisp value. The following assumptions will enable us to determine each component of $\widehat{T}_0$

$$H_i^1(n) = \widetilde{u}_1(n)\times u_i(n+1)\times \ldots \times \widetilde{u}_{s^2}(n)$$
$$V_i^1(n) = \widetilde{y}_1(n)\times y_i(n+1)\times\ldots\times\widetilde{y}_{s^2}(n) \text{ where } \ i=1,2,\ldots,s^2 \tag{53}$$

From the Corollary 3, we know that the ith component of the set

$$(\phi^*(N,N_0)\otimes\phi(N,N_0))\widehat{T}_0 + \sum_{j=n_0}^{n-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))H_i^1(n) \tag{54}$$

is a fuzzy set in E^1. From the fact that the product of a square matrix of size s^2 and column vector whose elements are α- level sets defined on fuzzy set in E^{s^2} is again a fuzzy set in E^{s^2}, it follows that the product

$$(I_s\otimes C(N))\times\sum_{j=n_0}^{n-1}(\phi^*(n,j+1)\otimes\phi(n,j+1))(I_s\otimes F(j))H_i^1(n) \tag{55}$$

is a fuzzy set in E^{s^2}. Hence $\widehat{T}_o$ is a fuzzy set in E^{s^2} and the i^{th} component of it denoted by $t_0^1(i)$ is a fuzzy set in E^1. The grade of membership of $\widetilde{t}_0^i$ in $t_0^1(i)$ is defined by $\mu_{t_0^1(i)}(\widetilde{t}_0^i) = min\{\mu_{u_i^1(n)}(\widetilde{u}_i(n)), \mu_{y_i^1}(n)(\widetilde{y}_i(n))\}$. Now the initial value is determined by using the equations (41) to (45). In general, computation of $t_0^l(i)$ is very difficult, but to solve the real value problem the following approximation is chosen. Now we take the point $(\widetilde{t}_0^i, \mu_{t_0^l}(i)(\widetilde{t_0}^i))$ and the zero level set $[t_0^l(i)]^0$ to determine a triangle as the new fuzzy set $t_0^l(i)$. We can use the centre average defuzzifier

$$\widetilde{t}_0^i = \frac{\sum_{l=1}^m(\widetilde{t}_0^i)^l\mu_{t_0^l(i)}(\widetilde{t}_0^i)^l}{\sum_{l=1}^m\mu_{t_0^l(i)}(\widetilde{t}_0^i)^l} \tag{56}$$

to determine the initial value $\widehat{T}_0 = (\widetilde{t}_0^1, \widetilde{t}_0^2, \ldots, \widetilde{t}_0^{s^2})$. To obtain more accurate value for the initial state, more rule bases may be provided. □

6. Numerical example

In this section, a numerical example which verify and validate the established conditions of controllability and observability of fuzzy matrix Lyapunov discrete dynamical system is presented. Consider the fuzzy matrix Lyapunov discrete dynamical system (27) satisfying (28) with $A(n) = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$, $F(n) = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$, $C(n) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $D(n) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $N = 2$, $T(0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let the final state $\widehat{t_f} = (t_{f_1}, t_{f_2}, t_{f_3}, t_{f_4})$ in E^4, where $[\widehat{T_f}]^\alpha = ([t_{f_1}]^\alpha, [t_{f_2}]^\alpha, [t_{f_3}]^\alpha, [t_{f_4}]^\alpha)^T = [[\alpha-1, 1-\alpha], [\alpha-1, 1-\alpha], [0.1(\alpha-1), 0.1(1-\alpha)], [0.1(\alpha-1), 0.1(1-\alpha)]]^T$. Choose the points $\widetilde{t}_{f_1} = 0.5, \widetilde{t}_{f_2} = 0.25, \widetilde{t}_{f_3} = 0.05, \text{and } \widetilde{t}_{f_4} = 0.025$, which are in $t_{f_1}, t_{f_2}, t_{f_3}$, and t_{f_4} whose membership function values are $0.5, 0.75, 0.5$ and 0.75 respectively. The fundamental matrix of homogeneous discrete dynamical system $\Delta T(n) = A(n)T(n)$ is given by $\phi(n, n_0) = \begin{bmatrix} 1^{n-n_0} & 0 \\ 0 & (-2)^{n-n_0} \end{bmatrix}$. The $2^2 \times 2^2$ symmetric controllable matrix $W(0,2)$ obtained by equation (30) of Theorem 5 we get $W(0,2) = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 13 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 13 \end{bmatrix}$ is nonsingular. Thus from Theorem 5 the α- level fuzzy control $\widehat{U}(n)$ is computed by

$$\widehat{U}^\alpha(n) = \begin{bmatrix} (2)^{-n-1}[\alpha-1, 1-\alpha] \\ 3^{-n}(-2)^n[\alpha-1, 1-\alpha] \\ (2)^{-n-1}[0.1(\alpha-1), 0.1(1-\alpha)] \\ 3^{-n}(-2)^n[0.1(\alpha-1), 0.1(1-\alpha)] \end{bmatrix} - \begin{bmatrix} (2)^n 0.2 \\ (0.304)3^n(-2)^{-n+1} \\ (0.8)2^n \\ (1.1216)3^n(-2)^{-n+1} \end{bmatrix}.$$

The α- level sets of fuzzy input $\widehat{U}(n)$ and fuzzy output $\widehat{Y}(n)$ by Rule Base 1 are denoted by $[\widehat{U}^{(1)}]^\alpha$, $[\widehat{Y}^{(1)}]^\alpha$ and are given by
Rule Base 1:

$$[\widehat{U}^{(1)}]^\alpha = \begin{bmatrix} [0, -0.75(\alpha-1)] \\ [0.75(\alpha-1)+1, 1] \\ [0, -0.5(\alpha-1)] \\ [0.5(\alpha-1)+1, 1] \end{bmatrix} [\widehat{Y}^{(1)}]^\alpha = \begin{bmatrix} [0, -2(\alpha+1)] \\ [0.5\alpha+2.5, 3] \\ [0, -1.5(\alpha-1)] \\ [0.5(\alpha-1)+3, 3] \end{bmatrix}$$

The α- level sets of fuzzy input $\widehat{U}(n)$ and fuzzy output $\widehat{Y}(n)$ by Rule Base 2 are denoted by $[\widehat{U}^{(2)}]^\alpha$, $[\widehat{Y}^{(21)}]^\alpha$ and are given by.
Rule Base 2:

$$[\widehat{U}^{(2)}]^\alpha = \begin{bmatrix} [0, -0.8(\alpha-1)] \\ [0.8\alpha+0.2, 1] \\ [0, -0.5(\alpha-1)] \\ [0.5\alpha+0.5, 1] \end{bmatrix} [\widehat{Y}^{(2)}]^\alpha = \begin{bmatrix} [0, -1.5(\alpha-1)] \\ [\alpha+1, 2] \\ [0, -2.5(\alpha-1)] \\ [(2\alpha+1), 3] \end{bmatrix}.$$

From Rule Base 1, select

$$\widetilde{u}^1 = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) = (0.5, 0.85, 0.4, 0.75)$$

the values of the membership function of $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ and $\widetilde{u}_4$ are $\frac{1}{3}, 0.8, 0.2,$ and $\frac{1}{2}$, respectively.
Also

$$\widetilde{y}^1 = (\widetilde{y}^1, \widetilde{y}^2, \widetilde{y}^3, \widetilde{y}^4) = (1, 2.8, 0.5, 2.9)$$

the values of the membership function of the output $\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3,$ and $\widetilde{y}_4$, are $\frac{1}{2}, 0.6, \frac{2}{3}$ and 0.8 respectively.
From Rule Base 2, we select

$$\widetilde{u}^2 = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4) = (0.5, 0.8, 0.25, 0.75),$$

the values of the membership function of $\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3$ and $\widetilde{u}_4$ respectively are $\frac{1}{3}, 0.8, 0.2$ and $\frac{1}{2}$ respectively. Also

$$\widetilde{y}^2 = (\widetilde{y}^1, \widetilde{y}^2, \widetilde{y}^3, \widetilde{y}^4) = (1, 1.75, 2, 1.5)$$

the values of the membership function of $\widetilde{y}^1, \widetilde{y}^2, \widetilde{y}^3,$ and $\widetilde{y}^4$ are $\frac{1}{3}, \frac{3}{4}, 0.2$ and 0.25 respectively.

From Rule Base 1 and equation (43) we have $\widehat{T}_0 = \begin{bmatrix} [1.3] \\ [0.0375] \\ [1.7] \\ [-0.0625] \end{bmatrix}$. From Rule Base 1 and equation 42 we have $t_0^1(1) = \begin{bmatrix} [2.8; 2.2.5\alpha + 0.55] \\ [-0.2125; -0.5\alpha - 0.7125] \\ [2.9] \\ [-1.1875] \end{bmatrix}$. When $\alpha = 0$, we observed that $\widetilde{t}_0^1 = 1.3$ belong to the interval $[2.8, 0.55]$. We choose its function in t_0^1 as
$\mu_{t_0^1}(1) = min\{\mu_{u_1^1}(\widetilde{u}_1(n)), \mu_{y_1^1}(n)(\widetilde{y}_1(n))\} = min(\frac{1}{3}, \frac{1}{2}) = \frac{1}{3}$.
$t_0^1(2) = \begin{bmatrix} [0.5\alpha + 1; 1.5] \\ [0.1875(1-\alpha); 0] \\ [2.9] \\ [-1.175] \end{bmatrix}$. When $\alpha = 0$ we observed that $\widetilde{t}_0^2 = 0.0375$ belong to the interval $[0.1875; 0]$. We choose its membership grade in $t_0^1(2)$ as $\mu_{t_0^1(2)}\widetilde{t}_0^1 = min(0.8, 0.6) = 0.6$.
$t_0^1(3) = \begin{bmatrix} [1.3] \\ [0.0375] \\ [2.9] \\ [-13125, -0.375\alpha - 1.5625] \end{bmatrix}$. When $\alpha = 0$, we observed that $\widetilde{t}_0^3 = 1.7$ belong to the interval $[2.9; 0]$. We choose its membership grade in $t_0^1(3)$ as $\mu_{t_0^1}(3)\widetilde{t}_0^1 = min(0.2, \frac{2}{3}) = 0.2$.
$t_0^1(4) = \begin{bmatrix} [1.3] \\ [0.0375] \\ [0.5\alpha + 2.5; 3] \\ [-0.875\alpha - 0.75; -1.625] \end{bmatrix}$. When $\alpha = 0$, we observed that $\widetilde{t}_0^4 = -0.0625$ belong to the interval $[-0.75, -1.625]$. We choose its membership grade in $t_0^1(4)$ as $\mu_{t_0^1}(4)\widetilde{t}_0^1 = min(\frac{1}{2}, 0.8) = \frac{1}{2}$.

Similarly for Rule Base 2 using equation (43) we get $\widehat{T}_0 = \begin{bmatrix} [0.95] \\ [-0.125] \\ [0.75] \\ [0.3125] \end{bmatrix}$.

By using Rule Base 2 and equation (42) we get

$$t_0^2(1) = \begin{bmatrix} [1.75; 2.4\alpha - 0.25] \\ [-0.2; -0.375\alpha + 0.05] \\ [1.5] \\ [-0.8125] \end{bmatrix}, \mu_{t0}^2(1)\widetilde{t}_0^2 = min(\tfrac{3}{8}, \tfrac{1}{3}) = \tfrac{1}{3},$$

$$t_0^2(2) = \begin{bmatrix} [\alpha - 0.5; 0.5] \\ [0.2 - 0.2\alpha; 0] \\ [1.5] \\ [-0.2] \end{bmatrix}, \ \mu_{t0}^2(2)\widetilde{t}_0^2 = min(\tfrac{3}{4}, \tfrac{3}{4}) = \tfrac{3}{4},$$

$$t_0^2(3) = \begin{bmatrix} [0.25] \\ [0.05] \\ [1.5] \\ [-1.3125; -0.625\alpha - 0.6875] \end{bmatrix}, \mu_{t_0^2}(3)\widetilde{t}_0^2 = min(\tfrac{1}{2}, 0.2) = 0.2,$$

$$t_0^2(4) = \begin{bmatrix} [0.25] \\ [0.5] \\ [2\alpha + 1; 3] \\ [-0.875\alpha - 0.375; -1.25] \end{bmatrix}, \mu_{t0}^2(4)\widetilde{t}_0^2 = min(\tfrac{1}{2}, 0.25) = 0.25.$$

By using the center average defuzzifier given by equation (56) the initial value $\widehat{T}_0 = (\widetilde{t}_0^1, \widetilde{t}_0^2, \widetilde{t}_0^3, \widetilde{t}_0^4)$ is given by

$$\widetilde{t}_0^1 = \frac{[1.3 \times \frac{1}{3} + (0.95) \times \frac{1}{3}]}{\frac{1}{3} + \frac{1}{3}} = 2.55142, \widetilde{t}_0^2 = \frac{[0.0375 \times (0.6) + (-0.125) \times 0.75]}{0.6 + 0.75} = -0.0527,$$

$$\widetilde{t}_0^3 = \frac{[1.7 \times (0.2) + (0.75) \times (0.2)]}{0.2 + 0.2} = 1.225, \widetilde{t}_0^4 = \frac{[-0.0625 \times (0.5) + (0.3125) \times (0.25)]}{0.5 + 0.25} = 0.0625.$$

By considering more rule bases the accuracy of the initial state can be improved.

7. Conclusion

In this paper, by visualizing fuzzy matrix Lyapunov discrete dynamical system as a Lyapunov difference inclusion, sufficient conditions for the controllability and observability of the fuzzy matrix Lyapunov discrete dynamical system are constructed by following the fuzzy rule base. We have constructed the rule base for the initial value without the knowledge of the solution of the system. This approach is new for the Lyapunov discrete dynamical systems. The constructed example clearly demonstrates the established results.

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