Statistics and Applications {ISSN 2454-7395 (online)}
Volume 24, No. 1, 2026 (New Series), pp 1–26
http://www.ssca.org.in/journal

Analysing the Power Gemeay Distribution: Properties and Diverse Applications

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Received: 19 January 2025; Revised: 27 April 2025; Accepted: 30 April 2025

Abstract

This study proposes a novel statistical distribution called the power Gemeay distribution by modifying the Gemeay distribution. Various statistical properties like hazard rate function and its graphics, moments and related measures, order statistics, and incomplete moments are derived. Many estimation methods like the maximum likelihood method, Anderson-Darling estimation, right-tail Anderson-Darling estimation, and more are used to estimate the parameters. A simulation study is carried out to evaluate the efficiency of the estimation techniques. Based on the analysis of two real data sets, the novel distribution is more suitable than other existing models.

Key words: Gemeay distribution; Entropy; Order statistics; Anderson-Darling estimation.

AMS Subject Classifications: 60G25, 63M10, 63M20, 62P99

The video recording of the paper made under the SSCA's Online Lecture series is available at the Youtube channel URL `https://youtu.be/tKBrcQECaac`.

1. Introduction

Numerous statistical distributions have been established in recent years. These models can be used to describe data from the real world. Researchers are focusing on creating novel models to advance lifetime data investigations. Developing new statistical distributions is one aspect of such studies. To make the corresponding models more versatile and responsive to various types of data, researchers frequently modify the existing distributions. As a result, numerous distribution families have emerged and are still developing.

Distributions such as Exponential, Lindley, and XLindley possess a singular parameter, necessitating enhanced adaptability to real-time data. It is common to modify the distribution by adding parameters to the existing one to improve adaptability.

When the global pandemic of COVID-19 struck, experts put efforts into creating a

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new model that would be adaptable to COVID-19 data. As part of this, several models have been produced and a number more are in the works. Different models can be used to assess different aspects, such as the number of patients with COVID-19 or patients who died from the disease. Almongy *et al.* (2021), Abu El Azm *et al.* (2021), Muse *et al.* (2021) *etc.* are some of them. Afify *et al.* (2022b) proposed a novel model Marshall-Olkin reduced Kies (MORKi) flexible for COVID-19 recovery data.

Gemeay *et al.* (2023a) introduced a general two-parameter distribution (GTPD), a family of two-parameter continuous distributions, to extend the flexibility to the models. A specific case of the distribution, the Gemeay distribution (GD), was also proposed, which was more adaptable for the COVID-19 data than other models such as the exponential distribution, Frechet distribution, and Lindley distribution.

This study focuses on modifying GD by introducing an additional parameter through power transformation and analysing the properties of the distribution. The power transformation technique plays a vital role in improving the adaptability of models to accommodate diverse data structures. The transformation facilitates enhanced performance in practical applications when contrasted with standard-based models. We utilised $X = T^{1/\alpha}$ as it is essential to incorporate an additional shape parameter into our proposed model since employing $X = T^{\alpha}$ will designate α as a power scaling parameter instead of a tail shaping parameter. Various estimation approaches are employed, and the model's flexibility is evaluated. The motivation behind the new distribution is to enhance the model's adaptability for fitting actual data sets, unlike existing models in the literature. The model will augment prior efforts in this field and enable the modelling of survival statistics.

The power transformations have served as a technique for generating a new class of distributions. This has been extensively utilised across diverse domains owing to its adaptability and efficiency. Ghitany *et al.* (2013) proposed power Lindley distribution by employing power transformation to the Lindley distribution. Sarhan *et al.* (2014) developed a novel distribution and employed the same methodology to present the power-transformed version of this distribution. Al-Babtain *et al.* (2021) developed a novel flexible model named the Weibull Marshall-Olkin power Lindley (WMOPL) distribution to enhance the adaptability of the power Lindley distribution. The suggested WMOPL exhibits flexibility in modelling engineering data. Afify *et al.* (2022a), Meriem *et al.* (2022), Nagy *et al.* (2023), Yıldırım *et al.* (2023), Alsadat *et al.* (2023), Gemeay *et al.* (2023b), and others, have effectively utilised the technique.

The transformation enhances the distribution's adaptability, allowing it to effectively capture complex data patterns that traditional models might overlook. The introduced parameter by transformation is essential in defining the distribution's properties, enabling it to adjust to diverse data patterns more efficiently.

The structure of the article is as follows. The new distribution is formulated in Section 2, along with the potential forms of probability density function (PDF) and its statistical characteristics such as moments, incomplete moments, stochastic orders, *etc.* are discussed in Section 3. Additionally, Section 3 provides the potential forms of the hazard rate function (HRF). We review the strategies for estimating the parameters of our distribution in Section 4. The numerical simulation is examined in Section 5, and the adaptability of the distribution for COVID-19 data is analyzed in Section 6 along with one more real data. The topic is concluded in Section 7.

2. Formulation of power Gemeay distribution

Gemeay *et al.* (2023a) presented a general two-parameter distribution, and they derived a new statistical model called Gemeay distribution (GD), its cumulative density function (CDF) defined as follows

$$G(x) = \frac{\theta^k \left(1 - e^{-\theta x}\right) + \theta^2 \Gamma(k+1, x\theta)}{\theta^2 k! + \theta^k}, \;\; x, \theta, k > 0, \tag{1}$$

where $\Gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt$. The GD's PDF is

$$g(x) = \frac{\theta^{k+1} e^{-\theta x} \left(\frac{1}{\theta^2} + x^k\right)}{k! + \theta^{k-2}}. \tag{2}$$

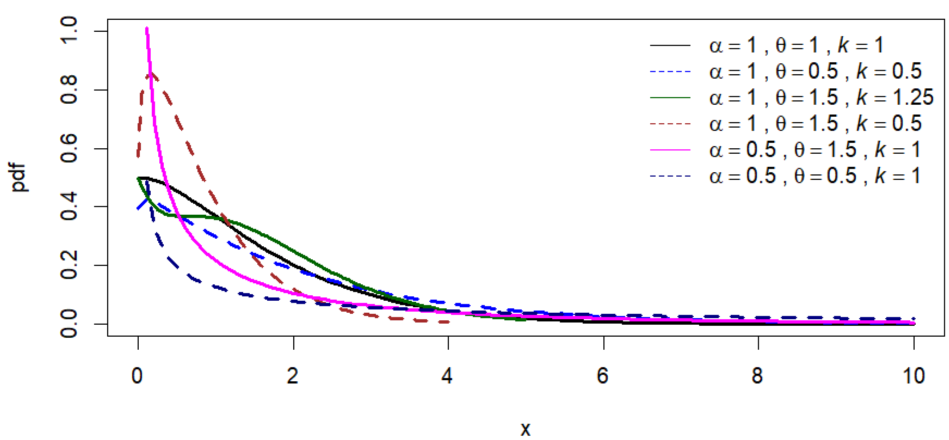
By using the following transformation $X = T^{\frac{1}{\alpha}}$ to (1), T follows GD, we obtain the CDF of the power Gemeay distribution (PGD) (for $x > 0$) as follows

$$F(x) = \frac{\theta^k \left(1 - e^{-\theta x^\alpha}\right) + \theta^2 \Gamma\left(k+1, x^\alpha \theta\right)}{\theta^2 k! + \theta^k}, \;\; x, \theta, k, \alpha > 0, \tag{3}$$

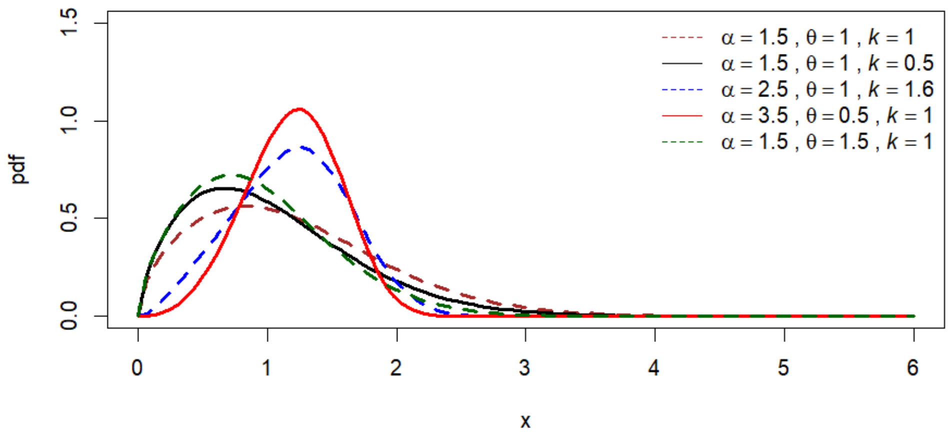
and its PDF is

$$f(x) = \frac{\alpha \theta^{k+1} x^{\alpha - 1} e^{-\theta x^\alpha} \left(\theta^2 x^{\alpha k} + 1\right)}{\theta^2 k! + \theta^k}. \tag{4}$$

The PDFs of PGD for different parameter values are represented graphically in Figure 1.



(a)

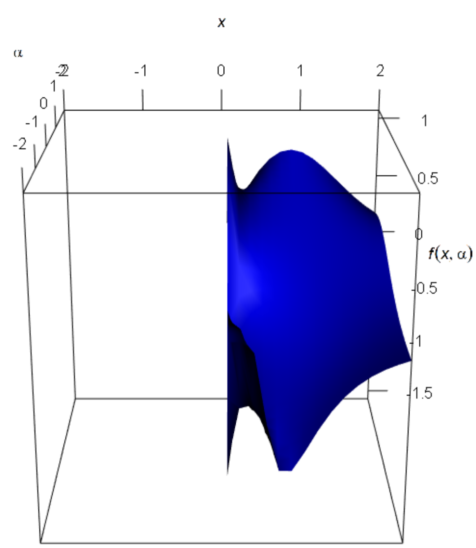


(b)

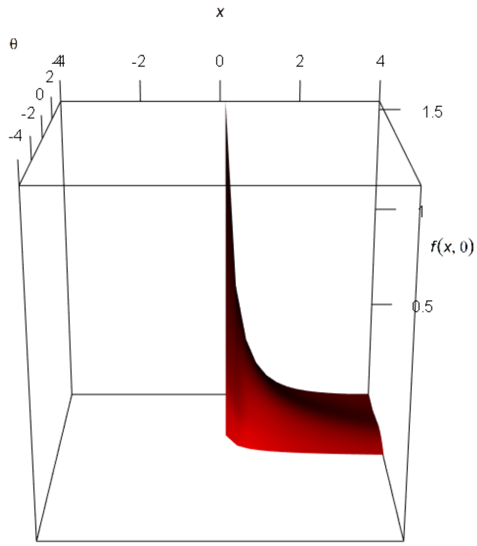
Figure 1: Plots of PDF with different parameter values

Figure 1a shows that when $\alpha \leq 1$, the PDF strictly declines for some choices of θ and k values while showing an increase initially and a decrease afterward for other choices of parameters. Figure 1b shows that the distribution is unimodal when $\alpha > 1$, regardless of the values of θ and k.

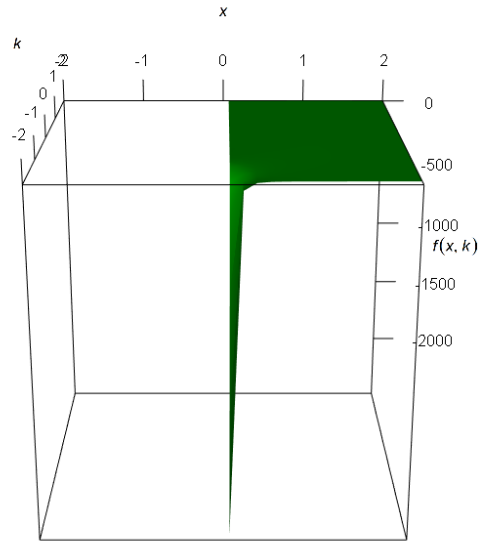
The three-dimensional (3-D) plot of the PDF of PGD with different choices of parameters are illustrated in Figure 2.



(a) PDF as a function of the variable x as well as the parameter α under the conditions of θ =0.75 and k=0.25



(b) PDF as a function of the variable x as well as the parameter θ under the conditions of α =0.5 and k=1.5



(c) PDF as a function of the variable x as well as the parameter k under the conditions of α =2.75 and θ=1.25

Figure 2: 3-D Plots of PDF with different choices of parameters

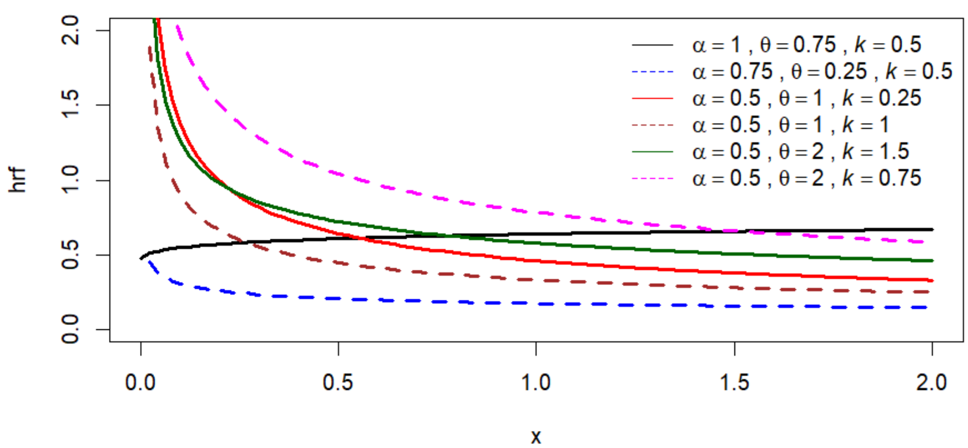
3. Statistical properties

3.1. Hazard rate function

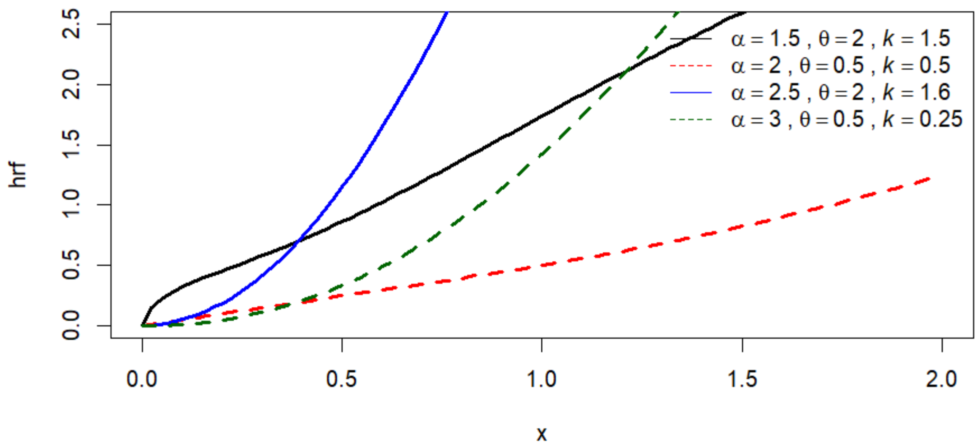
The HRF of the PGD is obtained as,

$$h(x) = \frac{\alpha\theta x^{\alpha-1}(\theta^2 x^{\alpha k} + 1)}{\theta^{2-k} e^{\theta x^{\alpha}} \gamma(k+1, x^{\alpha}\theta) + 1} \tag{5}$$

where $\gamma(a, z) = \int_z^{\infty} t^{a-1} e^{-t} dt$. The HRF of PGD is shown in Figure 3.



(a)



(b)

Figure 3: Plots of HRF with different parameter values

Figure 3a makes it abundantly evident that the HRF is decreasing in nature when $\alpha < 1$, regardless of the values of the other two parameters, θ and k. A constant line may be drawn on the graph when $\alpha = 1$. Figure 3b clearly shows that, regardless of θ and k values, the HRF increases in nature when $\alpha > 1$.

3.2. Moments and related measures

If $X \sim$ PGD,then the r^{th} moment of X can be obtained as follows;

$$E[X^r] = \int_0^\infty x^r f(x)dx = \frac{1}{\theta^{r/\alpha}}\left[\frac{(k+r/\alpha)!}{k!+\theta^{k-2}} + \frac{(r/\alpha)!}{\theta^{2-k}k!+1}\right] \tag{6}$$

The first four moments can be obtained by putting $r = 1, 2, 3, 4$ respectively. Using this, we obtain the mean and variance of the distribution as follows.

$$Mean = \frac{1}{\theta^{1/\alpha}}\left[\frac{(k+1/\alpha)!}{k!+\theta^{k-2}} + \frac{(1/\alpha)!}{\theta^{2-k}k!+1}\right]$$

$$Var(X) = \frac{1}{\theta^{2/\alpha}}\left[\frac{(k+2/\alpha)!}{k+\theta^{k-2}} + \frac{(2/\alpha)!}{\theta^{2-k}k!+1} - \left(\frac{(k+1/\alpha)!}{k!+\theta^{k-2}} + \frac{(1/\alpha)!}{\theta^{2-k}k!+1}\right)^2\right]$$

The skewness and kurtosis of PGD are derived as follows.

$$Skewness = \frac{E[X^3]}{(Var(X))^{3/2}} = \frac{\frac{1}{\theta^{3/\alpha}}\left[\frac{(k+3/\alpha)!}{k!+\theta^{k-2}} + \frac{(3/\alpha)!}{\theta^{2-k}k!+1}\right]}{\left(\frac{1}{\theta^{2/\alpha}}\left[\frac{(k+2/\alpha)!}{k+\theta^{k-2}} + \frac{(2/\alpha)!}{\theta^{2-k}k!+1} - \left(\frac{(k+1/\alpha)!}{k!+\theta^{k-2}} + \frac{(1/\alpha)!}{\theta^{2-k}k!+1}\right)^2\right]\right)^{3/2}}$$

$$Kurtosis = \frac{E[X^4]}{(Var(X))^{2}} = \frac{\frac{1}{\theta^{4/\alpha}}\left[\frac{(k+4/\alpha)!}{k!+\theta^{k-2}} + \frac{(4/\alpha)!}{\theta^{2-k}k!+1}\right]}{\left(\frac{1}{\theta^{2/\alpha}}\left[\frac{(k+2/\alpha)!}{k+\theta^{k-2}} + \frac{(2/\alpha)!}{\theta^{2-k}k!+1} - \left(\frac{(k+1/\alpha)!}{k!+\theta^{k-2}} + \frac{(1/\alpha)!}{\theta^{2-k}k!+1}\right)^2\right]\right)^{2}}$$

The coefficient of variation is given as,

$$\begin{aligned} C.V &= \frac{\sqrt{Var(x)}}{E(X)} \\ &= \frac{\left(\frac{1}{\theta^{2/\alpha}}\left[\frac{(k+2/\alpha)!}{k+\theta^{k-2}}+\frac{(2/\alpha)!}{\theta^{2-k}k!+1}-\left(\frac{(k+1/\alpha)!}{k!+\theta^{k-2}}+\frac{(1/\alpha)!}{\theta^{2-k}k!+1}\right)^2\right]\right)^{1/2}}{\frac{1}{\theta^{1/\alpha}}\left[\frac{(k+1/\alpha)!}{k!+\theta^{k-2}}+\frac{(1/\alpha)!}{\theta^{2-k}k!+1}\right]} \end{aligned}$$

The moment-generating function can be obtained as,

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx = \frac{\alpha\theta^{k+1}}{\theta^2 k! + \theta^k}\left(\theta^2\Gamma(\alpha(k+1))(1-t)^{-\alpha(k+1)} + \Gamma\alpha(1-t)^{-\alpha}\right) \quad (7)$$

for $\alpha > 0$ and $Re(\alpha(k+1)) > 0$.
The characteristic function can be formulated by replacing t with it in Equation 7.

3.3. Entropy

The entropy can be defined as a measure of uncertainty or randomness. The entropy defined by Rényi (1961) is as follows,

$$\begin{aligned} H_z(X) &= \frac{1}{1-z} log\left(\int_0^\infty f^z(x)dx\right) \\ &= \frac{1}{1-z} log\left[\left(\frac{\alpha\theta^{k+1}}{\theta^2 k! + \theta^k}\right)^z \sum_{t=0}^{z}\binom{z}{t}\frac{\theta^{2t}}{\alpha}\frac{\Gamma(kt+z/\alpha+1)}{\theta^{kt+z/\alpha+1}}\right] \end{aligned}$$

$H_z(x)$ tends to Shannon entropy as $z \to 1$.

3.4. Order statistics

Let $X_1, X_2, ...X_n$ be n random sample taken from a population. If the variables are arranged in ascending order of magnitude such that $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(n)}$, then $x_{(i)}$ is said to be the i^{th} order statistic, for $i = 1, 2, ..., n$ and specifically, $x_{(1)}$ is the first order statistic, and $x_{(n)}$ is the n^{th} order statistic.

The PDF of r^{th} order statistic $x_{(r)}$ can be obtained in general as

$$f_r(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x) \quad (8)$$

For PGD, this can be derived as,

$$\begin{aligned} f_r(x) = \frac{n!}{(r-1)!(n-r)!}&\left[\frac{\theta^k(1-e^{-\theta x^\alpha}) + \theta^2\Gamma(k+1, x^\alpha\theta)}{\theta^2 k! + \theta^k}\right]^{r-1} \times \\ &\left[\frac{\theta^2 k! + \theta^k - \theta^k(1-e^{-\theta x^\alpha}) - \theta^2\Gamma(k+1, x^\alpha\theta)}{\theta^2 k! + \theta^k}\right]^{n-r} \times \\ &\left[\frac{\alpha\theta^{k+1}x^{\alpha-1}e^{-\theta x^\alpha}(\theta^2 x^{\alpha k}+1)}{\theta^2 k! + \theta^k}\right] \end{aligned}$$

Putting $r = 1$, the PDF of first-order statistic is given as,

$$f_r(x) = n\left[\frac{\theta^2 k! + \theta^k - \theta^k(1-e^{-\theta x^\alpha}) - \theta^2\Gamma(k+1, x^\alpha\theta)}{\theta^2 k! + \theta^k}\right]^{n-1} \times \left[\frac{\alpha\theta^{k+1}x^{\alpha-1}e^{-\theta x^\alpha}(\theta^2 x^{\alpha k}+1)}{\theta^2 k! + \theta^k}\right]$$

The PDF of n^{th} order statistic is obtained as follows, by putting $r = n$.

$$f_r(x) = n\left[\frac{\theta^k(1-e^{-\theta x^\alpha}) + \theta^2\Gamma(k+1, x^\alpha\theta)}{\theta^2 k! + \theta^k}\right]^{n-1} \times \left[\frac{\alpha\theta^{k+1}x^{\alpha-1}e^{-\theta x^\alpha}(\theta^2 x^{\alpha k}+1)}{\theta^2 k! + \theta^k}\right]$$

3.5. Incomplete moments

According to Butler and McDonald (1989), the h^{th} incomplete moment for a density function can be defined as,

$$I(t;h) = \int_0^t x^h.f(x)dx \ ; \quad for \ \ x < t$$

For PGD, the h^{th} incomplete moment can be obtained as,

$$I(t;h) = \frac{\alpha\theta^{k-h/\alpha+1}}{\theta^2 k! + \theta^k}\left[\frac{(t^\alpha)^{2k}}{\alpha}\left(\Gamma(h/\alpha+k+1) - \Gamma(h/\alpha+k+1, t^\alpha\theta)\right) + \Gamma(h/\alpha) - \Gamma(h/\alpha, t^\alpha\theta)\right]$$

We have the first incomplete moment $I(t,1)$ by putting $h = 1$ in the last equation. Moreover, the normalised incomplete moments are acquired as follows:

$$\phi(t;h) = \frac{I(t;h)}{E(y^h)} \tag{9}$$

where $E(y^h)$ is the h^{th} moment of Y where $y < t$. Putting $h = 0$ and $h = 1$ in Equation 9, we get the first two normalised incomplete moments, which are very useful in economics.

4. Estimation methods

In this section, we discussed the conventional estimation methods for determining the parameters of the PGD. These estimations are derived by optimising an objective function to maximise or minimise its value. For more information about these estimation methods, see Anderson and Darling (1952); Mukhtar *et al.* (2020); Choi and Bulgren (1968); Swain *et al.* (1988); Kao (1958); Torabi (2008).

In the context of estimating PGD parameters, the calculation of the PGD estimator involves utilising maximum likelihood estimation (MLE). This estimation technique revolves around the maximisation of the following equation.

$$\log L = n\log\left(\frac{\alpha\theta^{k+1}}{\theta^2 k! + \theta^k}\right) + \sum_{i=1}^n \log\left(\theta^2 x_i^{\alpha k}+1\right) - \theta\sum_{i=1}^n x_i^\alpha + (\alpha-1)\sum_{i=1}^n \log(x_i)$$

In the process of computing the PGD estimator, Anderson-Darling estimation (ADE) is employed, and this estimation technique revolves around the minimisation of the provided equation.

$$A(x_i) = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\Bigg[\log \frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k} + \log\left(1 - \frac{\theta^k\left(1-e^{-\theta x_{(n-i-1)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(n-i-1)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k}\right)\Bigg]$$

In the process of obtaining the PGD estimator, we utilise right-tail Anderson-Darling estimation (RADE). This estimation procedure involves the minimisation of the equation provided, serving as a fundamental tool in precisely determining the PGD parameters.

$$R(x_i) = \frac{n}{2} - 2\sum_{i=1}^{n}\frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\left(\frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k}\right)$$

The PGD estimator is computed through the application of left-tailed Anderson-Darling estimation (LTADE). Its essence lies in the minimisation of the given equation.

$$L(x_i) = -\frac{3}{2}n + 2\sum_{i=1}^{n}\frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k} - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k}$$

The determination of PGD parameters employs Cramér-von Mises estimation (CVME), a method that centers around the minimisation of the given equation.

$$C(x_i) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k} - \frac{2i-1}{2n}\right]^2$$

In the process of computing the PGD estimator, we rely on the least-squares estimation (LSE) method, which involves minimizing the provided equation.

$$V(x_i) = \sum_{i=1}^{n}\left[\frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k} - \frac{i}{n+1}\right]^2$$

In the computation of the PGD estimator, we employ the weighted least-squares estimation (WLSE) method, which centers on the minimisation of the equation provided.

$$W(x_i) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[\frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k! + \theta^k} - \frac{i}{n+1}\right]^2$$

To derive the PGD estimator, we employ the maximum product of spacing estimation (MPSE) method, which entails maximising the following equation.

$$\delta(x_i) = \frac{1}{n+1}\sum_{i=1}^{n+1}\log \nu_i(x_i),$$
$$\nu_i(x_i) = \frac{\theta^k\left(1-e^{-\theta x_{(i)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i)}^{\alpha}\theta\right)}{\theta^2 k!+\theta^k} - \frac{\theta^k\left(1-e^{-\theta x_{(i-1:n)}^{\alpha}}\right)+\theta^2\Gamma\left(k+1, x_{(i-1:n)}^{\alpha}\theta\right)}{\theta^2 k!+\theta^k}$$

In determining the PGD estimator, the minimum spacing absolute distance estimation (MSADE) method is applied, involving minimising the provided equation.

$$\zeta(x_i) = \sum_{i=1}^{n+1}|\nu_i - \frac{1}{n+1}|$$

The calculation of the PGD estimator relies on the minimum spacing absolute-log distance estimation method (MSALDE). This method entails minimisation of the provided equation and is instrumental in accurately estimating PGD parameters.

$$\Upsilon_2(x_i) = \sum_{i=1}^{n+1}|\log \nu_i - \log\frac{1}{n+1}|$$

5. Numerical simulation

In this section, we delve into evaluating the effectiveness of various parameter estimation techniques for the PGD using an extensive dataset obtained through simulation. Our objective is to assess the performance of these estimation methods under different scenarios. Our simulations generated datasets with varying sample sizes, specifically, $n = 30$, 70, 100, 150, and 250. These datasets were generated by randomly sampling data points from the PGD's quantile function with different sets of parameter values. Our focus lies in understanding the behavior of the PGD estimators and evaluating the performance of these estimation techniques within this context. Additionally, we will employ a variety of metrics to evaluate the efficacy of different estimation methods, including an average of bias ($|Bias(\hat{\boldsymbol{\iota}})| = \frac{1}{n}\sum_{i=1}^{n}|\hat{\boldsymbol{\iota}} - \boldsymbol{\iota}|$), mean squared errors ($MSE = \frac{1}{n}\sum_{i=1}^{n}(\hat{\boldsymbol{\iota}} - \boldsymbol{\iota})^2$), and mean relative errors ($MRE = \frac{1}{n}\sum_{i=1}^{n}|\hat{\boldsymbol{\iota}} - \boldsymbol{\iota}|/\boldsymbol{\iota}$), $\boldsymbol{\iota} = (\theta,\ k,\ \alpha)$.

Tables from 5 to 12 (see Annexure) present the outcomes of simulating PGD parameters using ten distinct estimation techniques by providing the bias, MSE, and MRE for each sample size, for all the estimation techniques. Also ranks from 1 to 10 are assigned to the values starting from the lowest. It is noteworthy that all parameter estimates for the suggested distribution exhibit a high degree of reliability and proximity to their true values. The provided estimation techniques demonstrate precision, and in every considered scenario, all computed metrics show a decline as n increases. The performance of each estimation method is exceptional in identifying the proposed model parameters. Table 13 (see Annexure) provides an overview of the overall rankings for all estimation strategies and each rank is determined from the power presented in Tables 5-12. In our investigation, the MSADE method emerged as the most effective for assessing the parameter values in question, achieving a total score of 80.0, as depicted in Table 13.

6. Real data analysis

To assess the adaptability of the distribution, we apply PGD to real data sets. The data sets have been subjected to descriptive analysis, and several plots, including box plots and time to target (TTT) plots, have also been produced.

We contrast the model's adaptability with the GD and other well-known distributions like the Frechet distribution (FD), Frechet- Weibull distribution(FrWD), and Lomax distribution(LoD). To find the optimum model, metrics like the Akaike information criterion (AIC), Consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), the Kolmogrov-Smirnov (KS) statistic, and Anderson-Darling(A) statistic are utilized.

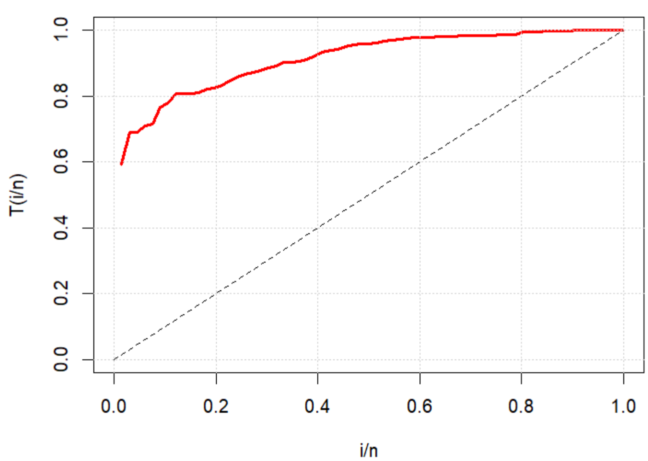
Dataset 1:

The data has been taken from Afify *et al.* (2022b) which indicates the recovery rate of COVID-19 infections in Spain from 3 March to 7 May 2020. The descriptive statistic of the data is given in Table 1.

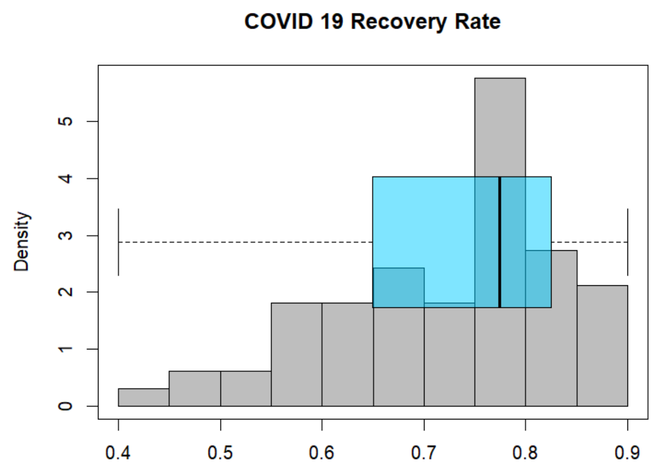
Table 1: Descriptive Statistics: COVID-19 recovery data

Minimum	1^{st} Quartile	Mean	Median	3^{rd} Quartile	Maximum	Skewness	Kurtosis
0.4286	0.6474	0.7240	0.7533	0.7975	0.8628	-0.688966	-0.4761233

Based on the data from Table 1 our data is left skewed and platykurtic in nature. It is evident from the histogram and box plot depicted in Figure 4b. Figure 4a shows the TTT plot, which is increasing.



(a) TTT plot of the COVID-19 data



(b) Histogram and box plot of COVID-19 data

Figure 4: TTT plot, Histogram, and box plot of the COVID-19 recovery data.

The goodness-of-fit statistics for the data set are shown in Table 2 together with the MLEs and SEs of the parameters of PGD and its competitors.

It is clear that PGD outperforms other distributions like GD, FrWD, FD, and LoD. The COVID data set analysis reveals that the model PGD meets all of the model selection criteria, making it the best model overall. It has a higher p-value of 0.9179 and a minimum value for statistics including AIC, CAIC, BIC and A as shown in Table 2. It makes the model adaptive for different lifetime data.

Table 2: Measures analysing goodness of fit of COVID-19 recovery data

Model	MLE (SE)	-L	AIC	CAIC	BIC	KS	A	P value (K-S test)
PGD	$\hat{\alpha}$ = 14.92(4.97) $\hat{\theta}$=14.02 (6.67) $\hat{k}$=-0.56 (0.21)	-59.76	-113.52	-113.13	-106.95	0.0682	0.6958	0.9179
GD	$\hat{\theta}$ = 3.709(0.448) $\hat{k}$=1.911 (0.333)	25.14	54.39	54.58	58.77	0.46	1.78	< 0.00001
LoD	$\hat{\alpha}$ = 17.912(9.68) $\hat{\lambda}$=12.795 (7.04)	46.45	96.913	97.104	101.29	0.481	1.942	< 0.000001
FD	$\hat{a}$=-1.47 (1.021) $\hat{m}$=2.13 (1.024) $\hat{s}$=17.31 (8.21)	-41.029	-76.059	-75.672	-69.49	0.1936	2.937	0.014
FrWD	$\hat{\alpha}$=4.47 (19.12) $\hat{\beta}$=0.976 (2.42) $\hat{\lambda}$=0.668 (1.35) $\hat{k}$=1.17 (5.04)	-35.19	-63.19	-62.53	-54.43	0.208	3.65	0.007

Figure 5 depicts the histogram of the data set along with the fitted PDF. The fitted PDF almost perfectly captures the shape of the histogram, as seen by an analysis of the diagram.

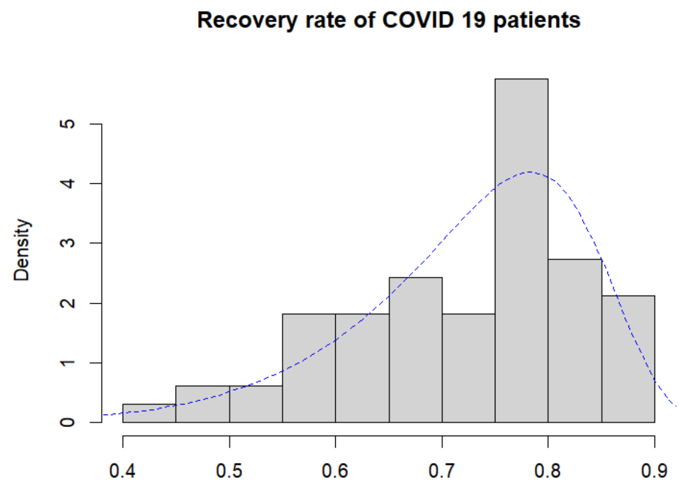


Figure 5: Histogram of COVID 19 recovery data with fitted pdf

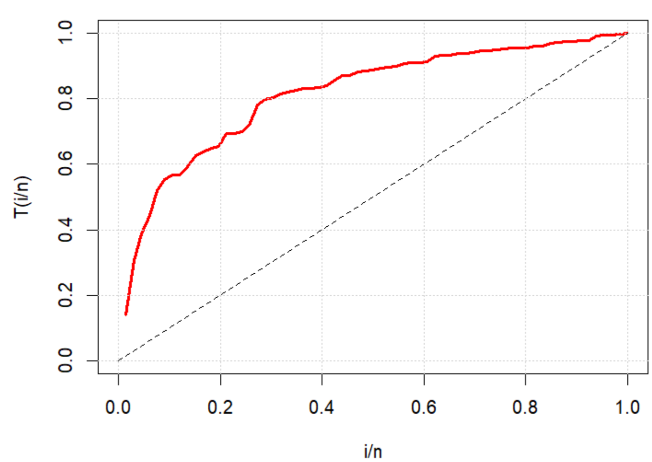
Data set 2:

The second data set, which provides the strengths of 1.5 cm glass fibers, was obtained from Nichols and Padgett (2006). Table 3 displays the descriptive statistics.

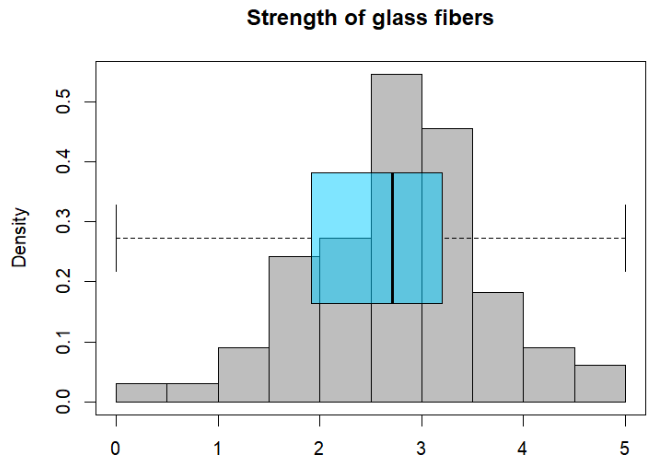
Table 3: Descriptive Statistics of strength of glass fibers data set

Minimum	1^{st} Quartile	Mean	Median	3^{rd} Quartile	Maximum	Skewness	Kurtosis
0.39	2.178	2.760	2.835	3.277	4.90	-0.1284	0.1267

It shows that our data is slightly negatively skewed and it is leptokurtic in nature and Figure 6b represents the histogram and box plot of the data set which justify the data in Table 3. Figure 6a shows the TTT plot which is increasing in nature.



(a) TTT plot of the strength of glass fibers



(b) Histogram and box plot of glass fibers data

Figure 6: TTT plot, Histogram, and box plot of the of glass fibers data.

The MLE, SE, and goodness of fit metrics for the glass fiber data for PGD and its competing models are presented in Table 4.

It is obvious that PGD performs better than other models. The model PGD is the best model overall, according to the analysis of the glass fibers data set, as it satisfies all model selection criteria. Table 4 shows the lowest values for the statistics AIC, CAIC, BIC, and A, as well as the higher p-value of 0.8427 for this study. The model becomes appropriate for various lifetime data as a result.

Table 4: Numerical values analyzing the goodness of fit for strength of glass fibers data

Model	MLE (SE)	-L	AIC	CAIC	BIC	KS	A	P value (K-S test)
PGD	$\hat{\alpha} = 2.163(0.376)$ $\hat{\theta}$=0.312 (0.196) $\hat{k}$=2.542 (0.511)	85.04	176.19	176.58	182.76	0.0758	0.3144	0.8427
GD	$\hat{\theta} = 3.49(0.564)$ $\hat{k}$=8.811 (1.48)	87.86	179.72	179.92	184.11	0.1005	0.684	0.517
LoD	$\hat{\alpha} = 14.38(7.31)$ $\hat{\lambda}$=38.51 (19.89)	137.43	274.06	274.25	278.44	0.361	1.41	$<$0.00001
FD	$\hat{a}$=-9.11 (4.1919) $\hat{m}$=11.38 (4.209) $\hat{s}$=12.105 (4.68)	95.48	195.88	196.275	202.457	0.1560	1.868	0.1212
FrWD	$\hat{\alpha}$=0.3375 (0.43) $\hat{\beta}$=2.0469 (9.83) $\hat{\lambda}$=1.7565 (1.82) $\hat{k}$=4.8753 (6.30)	121.19	250.39	251.045	259.148	0.23	5.38	0.0018

Figure 7 depicts the histogram of the data set along with the fitted PDF. This demonstrates that the distribution accurately depicts the form of the histogram.

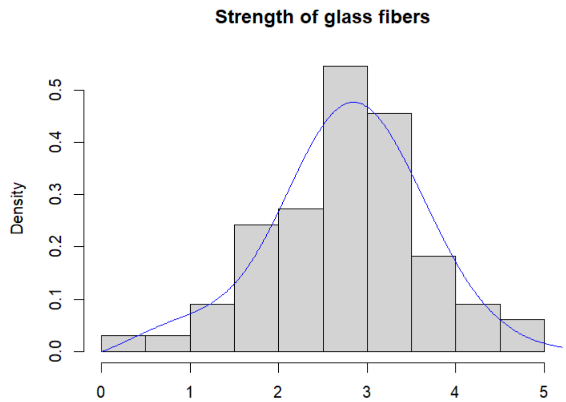


Figure 7: Histogram of glass fibers data with fitted pdf

7. Conclusion

In this paper, we proposed a new power Gemeay distribution with parameters α, θ and k. The PDF, CDF, and HRF are formulated, and the behavior of the PDF and HRF is assessed by evaluating the plots. Furthermore, some statistical properties are derived. The parameters are estimated using several methods like MLE, ADE, RADE, LTADE, CVME, LSE, WLSE, MPSE, MSADE, and MSALDE. A simulation study assesses the effectiveness of all ten estimation techniques. It should be emphasized that all of the distributional estimations show a high degree of accuracy and reliability concerning their true values. The best-fitted model among some existing distributions and the suggested distribution is determined by analysis of two real data sets, and it exceeds all other mentioned distributions in terms of the goodness of fit criterion.

Acknowledgements

We are sincerely grateful for the Editor's help and advice. We are extremely thankful of the reviewer's for insightful comments and suggestions.

Conflict of interest

The authors do not have any financial or non-financial conflict of interest to declare for the research work included in this article.

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ANNEXURE

Table 5: Numerical values of the PGD simulation for $\theta = 0.25$, $k = 0.5$, and $\alpha = 0.75$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.25865^{\{5\}}$	$0.32403^{\{6\}}$	$0.32779^{\{7\}}$	$0.36521^{\{8\}}$	$0.43861^{\{10\}}$	$0.22234^{\{3\}}$	$0.39659^{\{9\}}$	$0.25818^{\{4\}}$	$0.1256^{\{1\}}$	$0.16549^{\{2\}}$
		$\hat{k}$	$0.79281^{\{5\}}$	$0.87253^{\{7\}}$	$0.83048^{\{6\}}$	$0.65183^{\{3\}}$	$0.97738^{\{9\}}$	$0.6618^{\{4\}}$	$0.98085^{\{10\}}$	$0.95445^{\{8\}}$	$0.14546^{\{1\}}$	$0.30555^{\{2\}}$
		$\hat{\alpha}$	$0.10985^{\{2\}}$	$0.12432^{\{5\}}$	$0.13937^{\{7\}}$	$0.1317^{\{6\}}$	$0.15504^{\{10\}}$	$0.11415^{\{4\}}$	$0.14265^{\{9\}}$	$0.14217^{\{8\}}$	$0.11054^{\{3\}}$	$0.10656^{\{1\}}$
	MSE	$\hat{\theta}$	$0.24582^{\{4\}}$	$0.39659^{\{6\}}$	$0.40477^{\{7\}}$	$0.54059^{\{8\}}$	$0.64788^{\{10\}}$	$0.20296^{\{3\}}$	$0.55552^{\{9\}}$	$0.28566^{\{5\}}$	$0.06825^{\{1\}}$	$0.15815^{\{2\}}$
		$\hat{k}$	$1.31754^{\{4\}}$	$1.77562^{\{7\}}$	$1.71171^{\{6\}}$	$1.87424^{\{8\}}$	$2.38251^{\{10\}}$	$0.99702^{\{3\}}$	$2.21934^{\{9\}}$	$1.7032^{\{5\}}$	$0.22187^{\{1\}}$	$0.58968^{\{2\}}$
		$\hat{\alpha}$	$0.02029^{\{3\}}$	$0.02469^{\{5\}}$	$0.03135^{\{7\}}$	$0.02804^{\{6\}}$	$0.03651^{\{10\}}$	$0.02135^{\{4\}}$	$0.03148^{\{8\}}$	$0.03256^{\{9\}}$	$0.0199^{\{2\}}$	$0.01853^{\{1\}}$
	MRE	$\hat{\theta}$	$1.03459^{\{5\}}$	$1.29613^{\{6\}}$	$1.31115^{\{7\}}$	$1.46085^{\{8\}}$	$1.75443^{\{10\}}$	$0.88934^{\{3\}}$	$1.58635^{\{9\}}$	$1.03274^{\{4\}}$	$0.5024^{\{1\}}$	$0.66195^{\{2\}}$
		$\hat{k}$	$1.58562^{\{5\}}$	$1.74507^{\{7\}}$	$1.66097^{\{6\}}$	$1.30366^{\{3\}}$	$1.95477^{\{9\}}$	$1.3236^{\{4\}}$	$1.9617^{\{10\}}$	$1.90891^{\{8\}}$	$0.29092^{\{1\}}$	$0.61111^{\{2\}}$
		$\hat{\alpha}$	$0.14647^{\{2\}}$	$0.16576^{\{5\}}$	$0.18583^{\{7\}}$	$0.1756^{\{6\}}$	$0.20673^{\{10\}}$	$0.1522^{\{4\}}$	$0.1902^{\{9\}}$	$0.18956^{\{8\}}$	$0.14739^{\{3\}}$	$0.14208^{\{1\}}$
	$\sum Ranks$		$35^{\{4\}}$	$54^{\{5\}}$	$60^{\{8\}}$	$56^{\{6\}}$	$88^{\{10\}}$	$32^{\{3\}}$	$82^{\{9\}}$	$59^{\{7\}}$	$14^{\{1\}}$	$15^{\{2\}}$
70	BIAS	$\hat{\theta}$	$0.15963^{\{4\}}$	$0.1918^{\{7\}}$	$0.21943^{\{8\}}$	$0.13679^{\{3\}}$	$0.24127^{\{10\}}$	$0.18063^{\{6\}}$	$0.22178^{\{9\}}$	$0.17231^{\{5\}}$	$0.07683^{\{1\}}$	$0.11258^{\{2\}}$
		$\hat{k}$	$0.63955^{\{4\}}$	$0.67896^{\{6\}}$	$0.71336^{\{7\}}$	$0.22915^{\{2\}}$	$0.73658^{\{8\}}$	$0.64138^{\{5\}}$	$0.74519^{\{9\}}$	$0.80685^{\{10\}}$	$0.09957^{\{1\}}$	$0.27747^{\{3\}}$
		$\hat{\alpha}$	$0.07085^{\{1\}}$	$0.08157^{\{5\}}$	$0.09285^{\{8\}}$	$0.07443^{\{3\}}$	$0.10021^{\{10\}}$	$0.08361^{\{6\}}$	$0.08964^{\{7\}}$	$0.09785^{\{9\}}$	$0.0746^{\{4\}}$	$0.07378^{\{2\}}$
	MSE	$\hat{\theta}$	$0.08664^{\{3\}}$	$0.12066^{\{7\}}$	$0.15022^{\{9\}}$	$0.10584^{\{5\}}$	$0.17945^{\{10\}}$	$0.1035^{\{4\}}$	$0.14769^{\{8\}}$	$0.10676^{\{6\}}$	$0.02032^{\{1\}}$	$0.05612^{\{2\}}$
		$\hat{k}$	$0.73117^{\{4\}}$	$0.83644^{\{6\}}$	$0.95092^{\{7\}}$	$0.43668^{\{3\}}$	$1.03546^{\{10\}}$	$0.73737^{\{5\}}$	$0.95386^{\{8\}}$	$0.99808^{\{9\}}$	$0.1092^{\{1\}}$	$0.33317^{\{2\}}$
		$\hat{\alpha}$	$0.00847^{\{1\}}$	$0.01108^{\{5\}}$	$0.01382^{\{8\}}$	$0.0101^{\{4\}}$	$0.01578^{\{10\}}$	$0.01138^{\{6\}}$	$0.01306^{\{7\}}$	$0.01498^{\{9\}}$	$0.00919^{\{3\}}$	$0.00911^{\{2\}}$
	MRE	$\hat{\theta}$	$0.63853^{\{4\}}$	$0.7672^{\{7\}}$	$0.87771^{\{8\}}$	$0.54717^{\{3\}}$	$0.96508^{\{10\}}$	$0.72254^{\{6\}}$	$0.88713^{\{9\}}$	$0.68924^{\{5\}}$	$0.30734^{\{1\}}$	$0.45034^{\{2\}}$
		$\hat{k}$	$1.2791^{\{4\}}$	$1.35791^{\{6\}}$	$1.42672^{\{7\}}$	$0.4583^{\{2\}}$	$1.47315^{\{8\}}$	$1.28276^{\{5\}}$	$1.49038^{\{9\}}$	$1.6137^{\{10\}}$	$0.19914^{\{1\}}$	$0.55494^{\{3\}}$
		$\hat{\alpha}$	$0.09446^{\{1\}}$	$0.10876^{\{5\}}$	$0.1238^{\{8\}}$	$0.09923^{\{3\}}$	$0.13362^{\{10\}}$	$0.11148^{\{6\}}$	$0.11952^{\{7\}}$	$0.13047^{\{9\}}$	$0.09946^{\{4\}}$	$0.09837^{\{2\}}$
	$\sum Ranks$		$26^{\{3\}}$	$54^{\{6\}}$	$70^{\{7\}}$	$28^{\{4\}}$	$86^{\{10\}}$	$49^{\{5\}}$	$73^{\{9\}}$	$72^{\{8\}}$	$17^{\{1\}}$	$20^{\{2\}}$
100	BIAS	$\hat{\theta}$	$0.13278^{\{4\}}$	$0.15621^{\{6\}}$	$0.18086^{\{8\}}$	$0.08393^{\{2\}}$	$0.20355^{\{10\}}$	$0.15778^{\{7\}}$	$0.18245^{\{9\}}$	$0.14482^{\{5\}}$	$0.06887^{\{1\}}$	$0.08869^{\{3\}}$
		$\hat{k}$	$0.59081^{\{4\}}$	$0.63705^{\{6\}}$	$0.6751^{\{7\}}$	$0.11845^{\{2\}}$	$0.68911^{\{8\}}$	$0.6111^{\{5\}}$	$0.69388^{\{9\}}$	$0.75662^{\{10\}}$	$0.09683^{\{1\}}$	$0.24613^{\{3\}}$
		$\hat{\alpha}$	$0.05959^{\{2\}}$	$0.06873^{\{5\}}$	$0.07822^{\{8\}}$	$0.05754^{\{1\}}$	$0.08547^{\{10\}}$	$0.07155^{\{6\}}$	$0.07664^{\{7\}}$	$0.0813^{\{9\}}$	$0.06404^{\{4\}}$	$0.06114^{\{3\}}$
	MSE	$\hat{\theta}$	$0.05403^{\{4\}}$	$0.07765^{\{7\}}$	$0.10351^{\{9\}}$	$0.05024^{\{3\}}$	$0.12863^{\{10\}}$	$0.0726^{\{6\}}$	$0.09407^{\{8\}}$	$0.06797^{\{5\}}$	$0.01697^{\{1\}}$	$0.03665^{\{2\}}$
		$\hat{k}$	$0.583^{\{4\}}$	$0.6769^{\{6\}}$	$0.79391^{\{8\}}$	$0.21273^{\{2\}}$	$0.85299^{\{10\}}$	$0.62988^{\{5\}}$	$0.76826^{\{7\}}$	$0.82993^{\{9\}}$	$0.1004^{\{1\}}$	$0.25214^{\{3\}}$
		$\hat{\alpha}$	$0.00608^{\{1\}}$	$0.008^{\{5\}}$	$0.00998^{\{8\}}$	$0.00617^{\{2\}}$	$0.01171^{\{10\}}$	$0.00829^{\{6\}}$	$0.00952^{\{7\}}$	$0.01036^{\{9\}}$	$0.00676^{\{4\}}$	$0.00647^{\{3\}}$
	MRE	$\hat{\theta}$	$0.53112^{\{4\}}$	$0.62486^{\{6\}}$	$0.72344^{\{8\}}$	$0.33572^{\{2\}}$	$0.81421^{\{10\}}$	$0.63112^{\{7\}}$	$0.72979^{\{9\}}$	$0.5793^{\{5\}}$	$0.27547^{\{1\}}$	$0.35477^{\{3\}}$
		$\hat{k}$	$1.18162^{\{4\}}$	$1.27409^{\{6\}}$	$1.3502^{\{7\}}$	$0.23689^{\{2\}}$	$1.37823^{\{8\}}$	$1.2222^{\{5\}}$	$1.38777^{\{9\}}$	$1.51324^{\{10\}}$	$0.19365^{\{1\}}$	$0.49226^{\{3\}}$
		$\hat{\alpha}$	$0.07945^{\{2\}}$	$0.09164^{\{5\}}$	$0.10429^{\{8\}}$	$0.07672^{\{1\}}$	$0.11396^{\{10\}}$	$0.0954^{\{6\}}$	$0.10219^{\{7\}}$	$0.1084^{\{9\}}$	$0.08539^{\{4\}}$	$0.08152^{\{3\}}$
	$\sum Ranks$		$29^{\{4\}}$	$52^{\{5\}}$	$71^{\{7.5\}}$	$17^{\{1\}}$	$86^{\{10\}}$	$53^{\{6\}}$	$72^{\{9\}}$	$71^{\{7.5\}}$	$18^{\{2\}}$	$26^{\{3\}}$
150	BIAS	$\hat{\theta}$	$0.10611^{\{4\}}$	$0.12412^{\{6\}}$	$0.15046^{\{9\}}$	$0.0439^{\{1\}}$	$0.16988^{\{10\}}$	$0.13703^{\{8\}}$	$0.13379^{\{7\}}$	$0.12264^{\{5\}}$	$0.05424^{\{2\}}$	$0.07316^{\{3\}}$
		$\hat{k}$	$0.53044^{\{4\}}$	$0.5773^{\{5\}}$	$0.63444^{\{8\}}$	$0.03948^{\{1\}}$	$0.66083^{\{9\}}$	$0.58086^{\{6\}}$	$0.62523^{\{7\}}$	$0.72204^{\{10\}}$	$0.07876^{\{2\}}$	$0.23075^{\{3\}}$
		$\hat{\alpha}$	$0.04914^{\{2\}}$	$0.05551^{\{5\}}$	$0.06505^{\{8\}}$	$0.0419^{\{1\}}$	$0.07134^{\{10\}}$	$0.06173^{\{7\}}$	$0.0587^{\{6\}}$	$0.06866^{\{9\}}$	$0.05276^{\{4\}}$	$0.05052^{\{3\}}$
	MSE	$\hat{\theta}$	$0.03585^{\{4\}}$	$0.04817^{\{6\}}$	$0.06652^{\{9\}}$	$0.01236^{\{2\}}$	$0.08195^{\{10\}}$	$0.05865^{\{8\}}$	$0.05101^{\{7\}}$	$0.04462^{\{5\}}$	$0.01011^{\{1\}}$	$0.02413^{\{3\}}$
		$\hat{k}$	$0.474^{\{4\}}$	$0.53568^{\{5\}}$	$0.64815^{\{8\}}$	$0.05504^{\{1\}}$	$0.70867^{\{9\}}$	$0.55634^{\{6\}}$	$0.58432^{\{7\}}$	$0.72035^{\{10\}}$	$0.07393^{\{2\}}$	$0.20945^{\{3\}}$
		$\hat{\alpha}$	$0.00428^{\{2\}}$	$0.0054^{\{5\}}$	$0.00693^{\{8\}}$	$0.00312^{\{1\}}$	$0.00831^{\{10\}}$	$0.00649^{\{7\}}$	$0.00576^{\{6\}}$	$0.00723^{\{9\}}$	$0.00454^{\{4\}}$	$0.00451^{\{3\}}$
	MRE	$\hat{\theta}$	$0.42442^{\{4\}}$	$0.49649^{\{6\}}$	$0.60186^{\{9\}}$	$0.1756^{\{1\}}$	$0.6795^{\{10\}}$	$0.54811^{\{8\}}$	$0.53515^{\{7\}}$	$0.49056^{\{5\}}$	$0.21698^{\{2\}}$	$0.29266^{\{3\}}$
		$\hat{k}$	$1.06088^{\{4\}}$	$1.15459^{\{5\}}$	$1.26888^{\{8\}}$	$0.07896^{\{1\}}$	$1.32165^{\{9\}}$	$1.16173^{\{6\}}$	$1.25046^{\{7\}}$	$1.44409^{\{10\}}$	$0.15752^{\{2\}}$	$0.4615^{\{3\}}$
		$\hat{\alpha}$	$0.06551^{\{2\}}$	$0.07401^{\{5\}}$	$0.08673^{\{8\}}$	$0.05586^{\{1\}}$	$0.09511^{\{10\}}$	$0.08231^{\{7\}}$	$0.07827^{\{6\}}$	$0.09155^{\{9\}}$	$0.07035^{\{4\}}$	$0.06736^{\{3\}}$
	$\sum Ranks$		$30^{\{4\}}$	$48^{\{5\}}$	$75^{\{9\}}$	$10^{\{1\}}$	$87^{\{10\}}$	$63^{\{7\}}$	$60^{\{6\}}$	$72^{\{8\}}$	$23^{\{2\}}$	$27^{\{3\}}$
250	BIAS	$\hat{\theta}$	$0.07797^{\{4\}}$	$0.0935^{\{5\}}$	$0.11638^{\{9\}}$	$0.02534^{\{1\}}$	$0.12017^{\{10\}}$	$0.10898^{\{8\}}$	$0.10273^{\{7\}}$	$0.09432^{\{6\}}$	$0.04487^{\{2\}}$	$0.05973^{\{3\}}$
		$\hat{k}$	$0.45976^{\{4\}}$	$0.51897^{\{5\}}$	$0.58196^{\{8\}}$	$0.00876^{\{1\}}$	$0.59179^{\{9\}}$	$0.54575^{\{6\}}$	$0.56963^{\{7\}}$	$0.65288^{\{10\}}$	$0.07331^{\{2\}}$	$0.22622^{\{3\}}$
		$\hat{\alpha}$	$0.03701^{\{2\}}$	$0.04027^{\{4\}}$	$0.05051^{\{8\}}$	$0.03021^{\{1\}}$	$0.05111^{\{9\}}$	$0.04923^{\{7\}}$	$0.04455^{\{6\}}$	$0.05227^{\{10\}}$	$0.04185^{\{5\}}$	$0.03874^{\{3\}}$
	MSE	$\hat{\theta}$	$0.01867^{\{4\}}$	$0.02681^{\{6\}}$	$0.03953^{\{9\}}$	$0.00205^{\{1\}}$	$0.0423^{\{10\}}$	$0.03404^{\{8\}}$	$0.02856^{\{7\}}$	$0.02506^{\{5\}}$	$0.00842^{\{2\}}$	$0.01472^{\{3\}}$
		$\hat{k}$	$0.35282^{\{4\}}$	$0.41819^{\{5\}}$	$0.50993^{\{8\}}$	$0.00764^{\{1\}}$	$0.52482^{\{9\}}$	$0.44969^{\{6\}}$	$0.45689^{\{7\}}$	$0.57658^{\{10\}}$	$0.06869^{\{2\}}$	$0.17934^{\{3\}}$
		$\hat{\alpha}$	$0.00246^{\{2\}}$	$0.00304^{\{5\}}$	$0.00436^{\{9\}}$	$0.00153^{\{1\}}$	$0.00443^{\{10\}}$	$0.00412^{\{7\}}$	$0.00335^{\{6\}}$	$0.00423^{\{8\}}$	$0.00292^{\{4\}}$	$0.00276^{\{3\}}$
	MRE	$\hat{\theta}$	$0.31187^{\{4\}}$	$0.374^{\{5\}}$	$0.46553^{\{9\}}$	$0.10135^{\{1\}}$	$0.48068^{\{10\}}$	$0.43594^{\{8\}}$	$0.41091^{\{7\}}$	$0.37728^{\{6\}}$	$0.17948^{\{2\}}$	$0.23892^{\{3\}}$
		$\hat{k}$	$0.91952^{\{4\}}$	$1.03794^{\{5\}}$	$1.16391^{\{8\}}$	$0.01752^{\{1\}}$	$1.18357^{\{9\}}$	$1.0915^{\{6\}}$	$1.13926^{\{7\}}$	$1.30576^{\{10\}}$	$0.14662^{\{2\}}$	$0.45244^{\{3\}}$
		$\hat{\alpha}$	$0.04935^{\{2\}}$	$0.05369^{\{4\}}$	$0.06734^{\{8\}}$	$0.04028^{\{1\}}$	$0.06815^{\{9\}}$	$0.06564^{\{7\}}$	$0.0594^{\{6\}}$	$0.0697^{\{10\}}$	$0.0558^{\{5\}}$	$0.05166^{\{3\}}$
	$\sum Ranks$		$30^{\{4\}}$	$44^{\{5\}}$	$76^{\{9\}}$	$9^{\{1\}}$	$85^{\{10\}}$	$63^{\{7\}}$	$60^{\{6\}}$	$75^{\{8\}}$	$26^{\{2\}}$	$27^{\{3\}}$

Table 6: Numerical values of the PGD simulation for $\theta = 0.2$, $k = 1.2$, and $\alpha = 0.9$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.19033^{\{3\}}$	$0.25544^{\{6\}}$	$0.21025^{\{5\}}$	$0.34281^{\{10\}}$	$0.28372^{\{7\}}$	$0.14505^{\{2\}}$	$0.28952^{\{9\}}$	$0.28864^{\{8\}}$	$0.13824^{\{1\}}$	$0.20542^{\{4\}}$
		$\hat{k}$	$0.71908^{\{3\}}$	$0.8983^{\{6\}}$	$0.83681^{\{5\}}$	$0.9447^{\{10\}}$	$0.92892^{\{7\}}$	$0.74395^{\{4\}}$	$0.94054^{\{9\}}$	$0.92939^{\{8\}}$	$0.2586^{\{1\}}$	$0.48444^{\{2\}}$
		$\hat{\alpha}$	$0.13456^{\{3\}}$	$0.15006^{\{5\}}$	$0.16005^{\{6\}}$	$0.16583^{\{7\}}$	$0.17142^{\{9\}}$	$0.13074^{\{1\}}$	$0.16703^{\{8\}}$	$0.18016^{\{10\}}$	$0.13346^{\{2\}}$	$0.1391^{\{4\}}$
	MSE	$\hat{\theta}$	$0.16931^{\{3\}}$	$0.29393^{\{6\}}$	$0.18286^{\{4\}}$	$0.50761^{\{10\}}$	$0.34001^{\{8\}}$	$0.08595^{\{2\}}$	$0.34515^{\{9\}}$	$0.32224^{\{7\}}$	$0.07425^{\{1\}}$	$0.22873^{\{5\}}$
		$\hat{k}$	$0.94467^{\{4\}}$	$1.38303^{\{6\}}$	$1.05927^{\{5\}}$	$1.80301^{\{10\}}$	$1.39224^{\{7\}}$	$0.76905^{\{2\}}$	$1.46422^{\{9\}}$	$1.42652^{\{8\}}$	$0.24219^{\{1\}}$	$0.77736^{\{3\}}$
		$\hat{\alpha}$	$0.03039^{\{3\}}$	$0.03616^{\{5\}}$	$0.04114^{\{6\}}$	$0.04501^{\{8\}}$	$0.04515^{\{9\}}$	$0.02722^{\{1\}}$	$0.04388^{\{7\}}$	$0.05183^{\{10\}}$	$0.02992^{\{2\}}$	$0.03242^{\{4\}}$
	MRE	$\hat{\theta}$	$0.95164^{\{3\}}$	$1.27721^{\{6\}}$	$1.05127^{\{5\}}$	$1.71406^{\{10\}}$	$1.41862^{\{7\}}$	$0.72526^{\{2\}}$	$1.4476^{\{9\}}$	$1.44321^{\{8\}}$	$0.69122^{\{1\}}$	$1.02711^{\{4\}}$
		$\hat{k}$	$0.59923^{\{3\}}$	$0.74858^{\{6\}}$	$0.69735^{\{5\}}$	$0.78725^{\{10\}}$	$0.7741^{\{7\}}$	$0.61996^{\{4\}}$	$0.78378^{\{9\}}$	$0.77449^{\{8\}}$	$0.2155^{\{1\}}$	$0.4037^{\{2\}}$
		$\hat{\alpha}$	$0.14951^{\{3\}}$	$0.16673^{\{5\}}$	$0.17783^{\{6\}}$	$0.18426^{\{7\}}$	$0.19047^{\{9\}}$	$0.14527^{\{1\}}$	$0.18559^{\{8\}}$	$0.20018^{\{10\}}$	$0.14829^{\{2\}}$	$0.15455^{\{4\}}$
	$\sum Ranks$		$28^{\{3\}}$	$51^{\{6\}}$	$47^{\{5\}}$	$82^{\{10\}}$	$70^{\{7\}}$	$19^{\{2\}}$	$77^{\{8.5\}}$	$77^{\{8.5\}}$	$12^{\{1\}}$	$32^{\{4\}}$
70	BIAS	$\hat{\theta}$	$0.12816^{\{4\}}$	$0.1615^{\{6\}}$	$0.15803^{\{5\}}$	$0.17545^{\{8\}}$	$0.17946^{\{10\}}$	$0.12302^{\{2\}}$	$0.16427^{\{7\}}$	$0.1793^{\{9\}}$	$0.09715^{\{1\}}$	$0.12563^{\{3\}}$
		$\hat{k}$	$0.61492^{\{3\}}$	$0.74746^{\{7\}}$	$0.75036^{\{8\}}$	$0.67901^{\{4\}}$	$0.77227^{\{9\}}$	$0.68958^{\{5\}}$	$0.74733^{\{6\}}$	$0.79249^{\{10\}}$	$0.23065^{\{1\}}$	$0.40529^{\{2\}}$
		$\hat{\alpha}$	$0.0912^{\{1\}}$	$0.1049^{\{5\}}$	$0.11395^{\{8\}}$	$0.10766^{\{6\}}$	$0.12054^{\{10\}}$	$0.09762^{\{2\}}$	$0.11025^{\{7\}}$	$0.12027^{\{9\}}$	$0.0979^{\{4\}}$	$0.09769^{\{3\}}$
	MSE	$\hat{\theta}$	$0.05921^{\{3\}}$	$0.08628^{\{7\}}$	$0.07312^{\{5\}}$	$0.12714^{\{10\}}$	$0.09587^{\{8\}}$	$0.04202^{\{2\}}$	$0.08365^{\{6\}}$	$0.10255^{\{9\}}$	$0.0329^{\{1\}}$	$0.07145^{\{4\}}$
		$\hat{k}$	$0.58159^{\{3\}}$	$0.75488^{\{8\}}$	$0.72456^{\{5\}}$	$0.74983^{\{7\}}$	$0.76942^{\{9\}}$	$0.60355^{\{4\}}$	$0.73955^{\{6\}}$	$0.83978^{\{10\}}$	$0.15619^{\{1\}}$	$0.37018^{\{2\}}$
		$\hat{\alpha}$	$0.0142^{\{1\}}$	$0.01848^{\{5\}}$	$0.02042^{\{7\}}$	$0.02064^{\{8\}}$	$0.02314^{\{9\}}$	$0.01516^{\{2\}}$	$0.01967^{\{6\}}$	$0.02328^{\{10\}}$	$0.01614^{\{3\}}$	$0.01653^{\{4\}}$
	MRE	$\hat{\theta}$	$0.64078^{\{4\}}$	$0.80751^{\{6\}}$	$0.79015^{\{5\}}$	$0.87724^{\{8\}}$	$0.89729^{\{10\}}$	$0.61509^{\{2\}}$	$0.82135^{\{7\}}$	$0.89652^{\{9\}}$	$0.48577^{\{1\}}$	$0.62817^{\{3\}}$
		$\hat{k}$	$0.51244^{\{3\}}$	$0.62289^{\{7\}}$	$0.6253^{\{8\}}$	$0.56584^{\{4\}}$	$0.64356^{\{9\}}$	$0.57465^{\{5\}}$	$0.62277^{\{6\}}$	$0.66041^{\{10\}}$	$0.19221^{\{1\}}$	$0.33774^{\{2\}}$
		$\hat{\alpha}$	$0.10133^{\{1\}}$	$0.11655^{\{5\}}$	$0.12662^{\{8\}}$	$0.11962^{\{6\}}$	$0.13394^{\{10\}}$	$0.10846^{\{2\}}$	$0.1225^{\{7\}}$	$0.13363^{\{9\}}$	$0.10877^{\{4\}}$	$0.10855^{\{3\}}$
	$\sum Ranks$		$23^{\{2\}}$	$56^{\{5\}}$	$59^{\{7\}}$	$61^{\{8\}}$	$84^{\{9\}}$	$26^{\{3.5\}}$	$58^{\{6\}}$	$85^{\{10\}}$	$17^{\{1\}}$	$26^{\{3.5\}}$
100	BIAS	$\hat{\theta}$	$0.11203^{\{2\}}$	$0.13274^{\{5\}}$	$0.13679^{\{7\}}$	$0.14035^{\{8\}}$	$0.16025^{\{10\}}$	$0.11517^{\{4\}}$	$0.13633^{\{6\}}$	$0.14478^{\{9\}}$	$0.08428^{\{1\}}$	$0.11414^{\{3\}}$
		$\hat{k}$	$0.57429^{\{3\}}$	$0.69538^{\{7\}}$	$0.7083^{\{8\}}$	$0.60957^{\{4\}}$	$0.73704^{\{10\}}$	$0.66222^{\{5\}}$	$0.6953^{\{6\}}$	$0.72461^{\{9\}}$	$0.22189^{\{1\}}$	$0.39656^{\{2\}}$
		$\hat{\alpha}$	$0.07711^{\{1\}}$	$0.08896^{\{5\}}$	$0.09713^{\{8\}}$	$0.09071^{\{6\}}$	$0.10616^{\{10\}}$	$0.08739^{\{4\}}$	$0.09268^{\{7\}}$	$0.09933^{\{9\}}$	$0.08334^{\{2\}}$	$0.08731^{\{3\}}$
	MSE	$\hat{\theta}$	$0.04261^{\{3\}}$	$0.0568^{\{7\}}$	$0.0521^{\{4\}}$	$0.07587^{\{9\}}$	$0.0771^{\{10\}}$	$0.03463^{\{2\}}$	$0.0543^{\{6\}}$	$0.05956^{\{8\}}$	$0.0241^{\{1\}}$	$0.0539^{\{5\}}$
		$\hat{k}$	$0.50995^{\{3\}}$	$0.65349^{\{8\}}$	$0.65198^{\{7\}}$	$0.5945^{\{5\}}$	$0.71059^{\{10\}}$	$0.56559^{\{4\}}$	$0.63728^{\{6\}}$	$0.69042^{\{9\}}$	$0.13693^{\{1\}}$	$0.33286^{\{2\}}$
		$\hat{\alpha}$	$0.01051^{\{1\}}$	$0.0135^{\{5\}}$	$0.01524^{\{8\}}$	$0.01508^{\{7\}}$	$0.01848^{\{10\}}$	$0.01223^{\{3\}}$	$0.0143^{\{6\}}$	$0.01576^{\{9\}}$	$0.01159^{\{2\}}$	$0.01339^{\{4\}}$
	MRE	$\hat{\theta}$	$0.56017^{\{2\}}$	$0.66372^{\{5\}}$	$0.68393^{\{7\}}$	$0.70174^{\{8\}}$	$0.80123^{\{10\}}$	$0.57584^{\{4\}}$	$0.68167^{\{6\}}$	$0.72391^{\{9\}}$	$0.42141^{\{1\}}$	$0.57068^{\{3\}}$
		$\hat{k}$	$0.47857^{\{3\}}$	$0.57949^{\{7\}}$	$0.59025^{\{8\}}$	$0.50797^{\{4\}}$	$0.6142^{\{10\}}$	$0.55185^{\{5\}}$	$0.57941^{\{6\}}$	$0.60384^{\{9\}}$	$0.18491^{\{1\}}$	$0.33047^{\{2\}}$
		$\hat{\alpha}$	$0.08568^{\{1\}}$	$0.09885^{\{5\}}$	$0.10793^{\{8\}}$	$0.10079^{\{6\}}$	$0.11796^{\{10\}}$	$0.0971^{\{4\}}$	$0.10297^{\{7\}}$	$0.11036^{\{9\}}$	$0.09259^{\{2\}}$	$0.09701^{\{3\}}$
	$\sum Ranks$		$19^{\{2\}}$	$54^{\{5\}}$	$65^{\{8\}}$	$57^{\{7\}}$	$90^{\{10\}}$	$35^{\{4\}}$	$56^{\{6\}}$	$80^{\{9\}}$	$12^{\{1\}}$	$27^{\{3\}}$
150	BIAS	$\hat{\theta}$	$0.095^{\{3\}}$	$0.10904^{\{6\}}$	$0.11971^{\{9\}}$	$0.10059^{\{4\}}$	$0.12692^{\{10\}}$	$0.10269^{\{5\}}$	$0.11112^{\{7\}}$	$0.11813^{\{8\}}$	$0.07157^{\{1\}}$	$0.09127^{\{2\}}$
		$\hat{k}$	$0.53059^{\{4\}}$	$0.63298^{\{6\}}$	$0.66666^{\{8\}}$	$0.51678^{\{3\}}$	$0.67516^{\{9\}}$	$0.62817^{\{5\}}$	$0.63955^{\{7\}}$	$0.68851^{\{10\}}$	$0.22038^{\{1\}}$	$0.37888^{\{2\}}$
		$\hat{\alpha}$	$0.06452^{\{1\}}$	$0.07315^{\{5\}}$	$0.08072^{\{9\}}$	$0.07^{\{2\}}$	$0.08489^{\{10\}}$	$0.07443^{\{6\}}$	$0.07645^{\{7\}}$	$0.07879^{\{8\}}$	$0.07122^{\{3\}}$	$0.07275^{\{4\}}$
	MSE	$\hat{\theta}$	$0.02955^{\{3\}}$	$0.03613^{\{7\}}$	$0.04152^{\{9\}}$	$0.03605^{\{6\}}$	$0.04838^{\{10\}}$	$0.02686^{\{2\}}$	$0.0336^{\{5\}}$	$0.0379^{\{8\}}$	$0.01615^{\{1\}}$	$0.03224^{\{4\}}$
		$\hat{k}$	$0.44308^{\{4\}}$	$0.56131^{\{7\}}$	$0.59891^{\{8\}}$	$0.44168^{\{3\}}$	$0.61265^{\{9\}}$	$0.5256^{\{5\}}$	$0.55746^{\{6\}}$	$0.62665^{\{10\}}$	$0.11414^{\{1\}}$	$0.27554^{\{2\}}$
		$\hat{\alpha}$	$0.00759^{\{1\}}$	$0.00945^{\{5\}}$	$0.01104^{\{9\}}$	$0.00923^{\{3\}}$	$0.01227^{\{10\}}$	$0.00925^{\{4\}}$	$0.00996^{\{7\}}$	$0.01037^{\{8\}}$	$0.00847^{\{2\}}$	$0.00955^{\{6\}}$
	MRE	$\hat{\theta}$	$0.47501^{\{3\}}$	$0.54522^{\{6\}}$	$0.59857^{\{9\}}$	$0.50295^{\{4\}}$	$0.63458^{\{10\}}$	$0.51345^{\{5\}}$	$0.55559^{\{7\}}$	$0.59064^{\{8\}}$	$0.35783^{\{1\}}$	$0.45635^{\{2\}}$
		$\hat{k}$	$0.44216^{\{4\}}$	$0.52748^{\{6\}}$	$0.55555^{\{8\}}$	$0.43065^{\{3\}}$	$0.56263^{\{9\}}$	$0.52348^{\{5\}}$	$0.53296^{\{7\}}$	$0.57376^{\{10\}}$	$0.18365^{\{1\}}$	$0.31573^{\{2\}}$
		$\hat{\alpha}$	$0.07169^{\{1\}}$	$0.08128^{\{5\}}$	$0.08969^{\{9\}}$	$0.07778^{\{2\}}$	$0.09432^{\{10\}}$	$0.0827^{\{6\}}$	$0.08494^{\{7\}}$	$0.08755^{\{8\}}$	$0.07913^{\{3\}}$	$0.08084^{\{4\}}$
	$\sum Ranks$		$24^{\{2\}}$	$53^{\{6\}}$	$78^{\{8.5\}}$	$30^{\{4\}}$	$87^{\{10\}}$	$43^{\{5\}}$	$60^{\{7\}}$	$78^{\{8.5\}}$	$14^{\{1\}}$	$28^{\{3\}}$
250	BIAS	$\hat{\theta}$	$0.07031^{\{3\}}$	$0.07813^{\{5\}}$	$0.0941^{\{9\}}$	$0.07232^{\{4\}}$	$0.09648^{\{10\}}$	$0.08536^{\{7\}}$	$0.08431^{\{6\}}$	$0.08763^{\{8\}}$	$0.06325^{\{1\}}$	$0.0701^{\{2\}}$
		$\hat{k}$	$0.46386^{\{4\}}$	$0.57061^{\{5\}}$	$0.61433^{\{8\}}$	$0.43574^{\{3\}}$	$0.61758^{\{9\}}$	$0.58132^{\{7\}}$	$0.57518^{\{6\}}$	$0.62223^{\{10\}}$	$0.20038^{\{1\}}$	$0.35785^{\{2\}}$
		$\hat{\alpha}$	$0.0492^{\{1\}}$	$0.0536^{\{3\}}$	$0.06337^{\{9\}}$	$0.05235^{\{2\}}$	$0.06473^{\{10\}}$	$0.05957^{\{8\}}$	$0.05628^{\{5\}}$	$0.05912^{\{7\}}$	$0.05782^{\{6\}}$	$0.05477^{\{4\}}$
	MSE	$\hat{\theta}$	$0.013^{\{2\}}$	$0.01725^{\{4\}}$	$0.0238^{\{9\}}$	$0.01749^{\{6.5\}}$	$0.02594^{\{10\}}$	$0.01887^{\{8\}}$	$0.01749^{\{6.5\}}$	$0.01737^{\{5\}}$	$0.01263^{\{1\}}$	$0.01724^{\{3\}}$
		$\hat{k}$	$0.35609^{\{4\}}$	$0.48251^{\{7\}}$	$0.52675^{\{8\}}$	$0.35005^{\{3\}}$	$0.53038^{\{9\}}$	$0.4748^{\{5\}}$	$0.4779^{\{6\}}$	$0.53493^{\{10\}}$	$0.12826^{\{1\}}$	$0.23922^{\{2\}}$
		$\hat{\alpha}$	$0.00439^{\{1\}}$	$0.00526^{\{2\}}$	$0.00694^{\{9\}}$	$0.00536^{\{3\}}$	$0.00738^{\{10\}}$	$0.00624^{\{8\}}$	$0.00555^{\{4\}}$	$0.00581^{\{7\}}$	$0.00572^{\{5.5\}}$	$0.00572^{\{5.5\}}$
	MRE	$\hat{\theta}$	$0.35157^{\{3\}}$	$0.39063^{\{5\}}$	$0.47052^{\{9\}}$	$0.3616^{\{4\}}$	$0.48242^{\{10\}}$	$0.42682^{\{7\}}$	$0.42153^{\{6\}}$	$0.43817^{\{8\}}$	$0.31623^{\{1\}}$	$0.35052^{\{2\}}$
		$\hat{k}$	$0.38655^{\{4\}}$	$0.47551^{\{5\}}$	$0.51194^{\{8\}}$	$0.36312^{\{3\}}$	$0.51465^{\{9\}}$	$0.48443^{\{7\}}$	$0.47932^{\{6\}}$	$0.51852^{\{10\}}$	$0.17443^{\{1\}}$	$0.29821^{\{2\}}$
		$\hat{\alpha}$	$0.05467^{\{1\}}$	$0.05955^{\{3\}}$	$0.07041^{\{9\}}$	$0.05817^{\{2\}}$	$0.07192^{\{10\}}$	$0.06618^{\{8\}}$	$0.06253^{\{5\}}$	$0.06569^{\{7\}}$	$0.06425^{\{6\}}$	$0.06086^{\{4\}}$
	$\sum Ranks$		$23^{\{1\}}$	$39^{\{5\}}$	$78^{\{9\}}$	$30.5^{\{4\}}$	$87^{\{10\}}$	$65^{\{7\}}$	$50.5^{\{6\}}$	$72^{\{8\}}$	$23.5^{\{2\}}$	$26.5^{\{3\}}$

Table 7: Numerical values of the PGD simulation for $\theta = 1.5$, $k = 0.25$, and $\alpha = 1.5$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.39865^{\{2\}}$	$0.51479^{\{6\}}$	$0.54992^{\{7\}}$	$0.41497^{\{4\}}$	$0.56405^{\{8\}}$	$0.49898^{\{5\}}$	$0.56877^{\{9\}}$	$0.65377^{\{10\}}$	$0.26397^{\{1\}}$	$0.40666^{\{3\}}$
		$\hat{k}$	$0.37927^{\{2\}}$	$0.54978^{\{7\}}$	$0.54937^{\{6\}}$	$0.4719^{\{4\}}$	$0.62634^{\{10\}}$	$0.52726^{\{5\}}$	$0.62017^{\{9\}}$	$0.59021^{\{8\}}$	$0.26025^{\{1\}}$	$0.44806^{\{3\}}$
		$\hat{\alpha}$	$0.20564^{\{3\}}$	$0.19023^{\{1\}}$	$0.21628^{\{8\}}$	$0.20789^{\{4\}}$	$0.2081^{\{6\}}$	$0.20794^{\{5\}}$	$0.19973^{\{2\}}$	$0.57164^{\{10\}}$	$0.21384^{\{7\}}$	$0.21672^{\{9\}}$
	MSE	$\hat{\theta}$	$0.35523^{\{2\}}$	$0.57749^{\{6\}}$	$0.7166^{\{8\}}$	$0.39132^{\{3\}}$	$0.75187^{\{9\}}$	$0.55496^{\{5\}}$	$0.71403^{\{7\}}$	$0.77454^{\{10\}}$	$0.21591^{\{1\}}$	$0.45168^{\{4\}}$
		$\hat{k}$	$0.53666^{\{2\}}$	$0.95779^{\{7\}}$	$1.16749^{\{9\}}$	$0.66765^{\{3\}}$	$1.30282^{\{10\}}$	$0.88596^{\{6\}}$	$1.13786^{\{8\}}$	$0.7946^{\{5\}}$	$0.31973^{\{1\}}$	$0.76925^{\{4\}}$
		$\hat{\alpha}$	$0.07242^{\{7\}}$	$0.05817^{\{1\}}$	$0.08619^{\{9\}}$	$0.06432^{\{2\}}$	$0.06995^{\{4\}}$	$0.07128^{\{5\}}$	$0.06567^{\{3\}}$	$1.38559^{\{10\}}$	$0.07431^{\{8\}}$	$0.07153^{\{6\}}$
	MRE	$\hat{\theta}$	$0.26577^{\{2\}}$	$0.34319^{\{6\}}$	$0.36661^{\{7\}}$	$0.27665^{\{4\}}$	$0.37603^{\{8\}}$	$0.33266^{\{5\}}$	$0.37918^{\{9\}}$	$0.43584^{\{10\}}$	$0.17598^{\{1\}}$	$0.2711^{\{3\}}$
		$\hat{k}$	$1.51706^{\{2\}}$	$2.1991^{\{7\}}$	$2.19747^{\{6\}}$	$1.8876^{\{4\}}$	$2.50537^{\{10\}}$	$2.10906^{\{5\}}$	$2.48069^{\{9\}}$	$2.36086^{\{8\}}$	$1.04101^{\{1\}}$	$1.79225^{\{3\}}$
		$\hat{\alpha}$	$0.13709^{\{3\}}$	$0.12682^{\{1\}}$	$0.14419^{\{8\}}$	$0.13859^{\{4\}}$	$0.13873^{\{6\}}$	$0.13863^{\{5\}}$	$0.13315^{\{2\}}$	$0.38109^{\{10\}}$	$0.14256^{\{7\}}$	$0.14448^{\{9\}}$
	$\sum Ranks$		$25^{\{1\}}$	$42^{\{4\}}$	$68^{\{8\}}$	$32^{\{3\}}$	$71^{\{9\}}$	$46^{\{6\}}$	$58^{\{7\}}$	$81^{\{10\}}$	$28^{\{2\}}$	$44^{\{5\}}$
70	BIAS	$\hat{\theta}$	$0.31477^{\{3\}}$	$0.39061^{\{7\}}$	$0.38683^{\{5\}}$	$0.31416^{\{2\}}$	$0.42466^{\{9\}}$	$0.3906^{\{6\}}$	$0.40746^{\{8\}}$	$0.5016^{\{10\}}$	$0.21014^{\{1\}}$	$0.35523^{\{4\}}$
		$\hat{k}$	$0.32839^{\{2\}}$	$0.41404^{\{6\}}$	$0.39748^{\{5\}}$	$0.35935^{\{3\}}$	$0.44388^{\{9\}}$	$0.41979^{\{7\}}$	$0.42892^{\{8\}}$	$0.48296^{\{10\}}$	$0.23047^{\{1\}}$	$0.39056^{\{4\}}$
		$\hat{\alpha}$	$0.14773^{\{7\}}$	$0.13446^{\{1\}}$	$0.14285^{\{4\}}$	$0.14488^{\{5\}}$	$0.14102^{\{3\}}$	$0.14848^{\{8\}}$	$0.1362^{\{2\}}$	$0.31633^{\{10\}}$	$0.14657^{\{6\}}$	$0.15951^{\{9\}}$
	MSE	$\hat{\theta}$	$0.22597^{\{2\}}$	$0.32212^{\{6\}}$	$0.35489^{\{8\}}$	$0.24593^{\{3\}}$	$0.39725^{\{9\}}$	$0.31618^{\{4\}}$	$0.34876^{\{7\}}$	$0.48076^{\{10\}}$	$0.14092^{\{1\}}$	$0.32165^{\{5\}}$
		$\hat{k}$	$0.30556^{\{2\}}$	$0.4459^{\{6\}}$	$0.49991^{\{8\}}$	$0.36082^{\{3\}}$	$0.55611^{\{10\}}$	$0.44143^{\{4\}}$	$0.47326^{\{7\}}$	$0.53094^{\{9\}}$	$0.22578^{\{1\}}$	$0.44354^{\{5\}}$
		$\hat{\alpha}$	$0.03465^{\{8\}}$	$0.02791^{\{1\}}$	$0.03311^{\{5\}}$	$0.03162^{\{4\}}$	$0.03133^{\{3\}}$	$0.03416^{\{7\}}$	$0.02883^{\{2\}}$	$0.25562^{\{10\}}$	$0.03386^{\{6\}}$	$0.03914^{\{9\}}$
	MRE	$\hat{\theta}$	$0.20984^{\{3\}}$	$0.26041^{\{7\}}$	$0.25789^{\{5\}}$	$0.20944^{\{2\}}$	$0.2831^{\{9\}}$	$0.2604^{\{6\}}$	$0.27164^{\{8\}}$	$0.3344^{\{10\}}$	$0.14009^{\{1\}}$	$0.23682^{\{4\}}$
		$\hat{k}$	$1.31354^{\{2\}}$	$1.65617^{\{6\}}$	$1.58991^{\{5\}}$	$1.43739^{\{3\}}$	$1.77553^{\{9\}}$	$1.67916^{\{7\}}$	$1.71566^{\{8\}}$	$1.93183^{\{10\}}$	$0.92188^{\{1\}}$	$1.56223^{\{4\}}$
		$\hat{\alpha}$	$0.09848^{\{7\}}$	$0.08964^{\{1\}}$	$0.09524^{\{4\}}$	$0.09658^{\{5\}}$	$0.09402^{\{3\}}$	$0.09898^{\{8\}}$	$0.0908^{\{2\}}$	$0.21089^{\{10\}}$	$0.09771^{\{6\}}$	$0.10634^{\{9\}}$
	$\sum Ranks$		$36^{\{3\}}$	$41^{\{4\}}$	$49^{\{5\}}$	$30^{\{2\}}$	$64^{\{9\}}$	$57^{\{8\}}$	$52^{\{6\}}$	$89^{\{10\}}$	$24^{\{1\}}$	$53^{\{7\}}$
100	BIAS	$\hat{\theta}$	$0.29897^{\{3\}}$	$0.35942^{\{9\}}$	$0.34142^{\{4\}}$	$0.26356^{\{2\}}$	$0.35079^{\{6\}}$	$0.3423^{\{5\}}$	$0.35896^{\{8\}}$	$0.46651^{\{10\}}$	$0.19741^{\{1\}}$	$0.35588^{\{7\}}$
		$\hat{k}$	$0.30727^{\{3\}}$	$0.38326^{\{8\}}$	$0.34874^{\{4\}}$	$0.29363^{\{2\}}$	$0.37443^{\{6\}}$	$0.35272^{\{5\}}$	$0.38069^{\{7\}}$	$0.45798^{\{10\}}$	$0.21528^{\{1\}}$	$0.39944^{\{9\}}$
		$\hat{\alpha}$	$0.12845^{\{7\}}$	$0.12111^{\{2\}}$	$0.12328^{\{3\}}$	$0.12384^{\{4\}}$	$0.12541^{\{5\}}$	$0.13845^{\{8\}}$	$0.11839^{\{1\}}$	$0.27236^{\{10\}}$	$0.12715^{\{6\}}$	$0.14087^{\{9\}}$
	MSE	$\hat{\theta}$	$0.18713^{\{3\}}$	$0.26535^{\{5\}}$	$0.26993^{\{7\}}$	$0.18384^{\{2\}}$	$0.27462^{\{8\}}$	$0.21979^{\{4\}}$	$0.26952^{\{6\}}$	$0.41321^{\{10\}}$	$0.12227^{\{1\}}$	$0.29982^{\{9\}}$
		$\hat{k}$	$0.23127^{\{2\}}$	$0.35979^{\{7\}}$	$0.35312^{\{5\}}$	$0.25614^{\{3\}}$	$0.37912^{\{8\}}$	$0.26341^{\{4\}}$	$0.35372^{\{6\}}$	$0.47608^{\{10\}}$	$0.18215^{\{1\}}$	$0.41237^{\{9\}}$
		$\hat{\alpha}$	$0.02565^{\{7\}}$	$0.02234^{\{2\}}$	$0.02451^{\{5\}}$	$0.02367^{\{3\}}$	$0.02447^{\{4\}}$	$0.02901^{\{8\}}$	$0.02171^{\{1\}}$	$0.17345^{\{10\}}$	$0.0255^{\{6\}}$	$0.03019^{\{9\}}$
	MRE	$\hat{\theta}$	$0.19931^{\{3\}}$	$0.23961^{\{9\}}$	$0.22761^{\{4\}}$	$0.1757^{\{2\}}$	$0.23386^{\{6\}}$	$0.2282^{\{5\}}$	$0.23931^{\{8\}}$	$0.31101^{\{10\}}$	$0.13161^{\{1\}}$	$0.23725^{\{7\}}$
		$\hat{k}$	$1.2291^{\{3\}}$	$1.53306^{\{8\}}$	$1.39496^{\{4\}}$	$1.17451^{\{2\}}$	$1.49773^{\{6\}}$	$1.41089^{\{5\}}$	$1.52276^{\{7\}}$	$1.83194^{\{10\}}$	$0.8611^{\{1\}}$	$1.59776^{\{9\}}$
		$\hat{\alpha}$	$0.08563^{\{7\}}$	$0.08074^{\{2\}}$	$0.08219^{\{3\}}$	$0.08256^{\{4\}}$	$0.08361^{\{5\}}$	$0.0923^{\{8\}}$	$0.07893^{\{1\}}$	$0.18157^{\{10\}}$	$0.08476^{\{6\}}$	$0.09391^{\{9\}}$
	$\sum Ranks$		$38^{\{3\}}$	$52^{\{6.5\}}$	$39^{\{4\}}$	$24^{\{1.5\}}$	$54^{\{8\}}$	$52^{\{6.5\}}$	$45^{\{5\}}$	$90^{\{10\}}$	$24^{\{1.5\}}$	$77^{\{9\}}$
150	BIAS	$\hat{\theta}$	$0.27788^{\{3\}}$	$0.31689^{\{6\}}$	$0.31072^{\{5\}}$	$0.21965^{\{2\}}$	$0.32485^{\{8\}}$	$0.31893^{\{7\}}$	$0.30926^{\{4\}}$	$0.39575^{\{10\}}$	$0.18625^{\{1\}}$	$0.33863^{\{9\}}$
		$\hat{k}$	$0.29491^{\{3\}}$	$0.33484^{\{7\}}$	$0.32051^{\{4\}}$	$0.23451^{\{2\}}$	$0.34198^{\{8\}}$	$0.33041^{\{6\}}$	$0.32606^{\{5\}}$	$0.38985^{\{10\}}$	$0.19857^{\{1\}}$	$0.36986^{\{9\}}$
		$\hat{\alpha}$	$0.11864^{\{7\}}$	$0.10693^{\{1\}}$	$0.11344^{\{6\}}$	$0.1075^{\{2\}}$	$0.10927^{\{4\}}$	$0.12551^{\{8\}}$	$0.10768^{\{3\}}$	$0.22849^{\{10\}}$	$0.11285^{\{5\}}$	$0.13053^{\{9\}}$
	MSE	$\hat{\theta}$	$0.15744^{\{3\}}$	$0.20246^{\{7\}}$	$0.2013^{\{6\}}$	$0.14045^{\{2\}}$	$0.2203^{\{8\}}$	$0.18165^{\{4\}}$	$0.19057^{\{5\}}$	$0.28278^{\{10\}}$	$0.10805^{\{1\}}$	$0.25843^{\{9\}}$
		$\hat{k}$	$0.19323^{\{3\}}$	$0.25693^{\{7\}}$	$0.25117^{\{6\}}$	$0.17837^{\{2\}}$	$0.28179^{\{8\}}$	$0.20787^{\{4\}}$	$0.23715^{\{5\}}$	$0.30872^{\{9\}}$	$0.14395^{\{1\}}$	$0.32224^{\{10\}}$
		$\hat{\alpha}$	$0.0216^{\{7\}}$	$0.01734^{\{1\}}$	$0.02048^{\{5\}}$	$0.0179^{\{3\}}$	$0.01857^{\{4\}}$	$0.02385^{\{8\}}$	$0.01759^{\{2\}}$	$0.12105^{\{10\}}$	$0.02062^{\{6\}}$	$0.02566^{\{9\}}$
	MRE	$\hat{\theta}$	$0.18525^{\{3\}}$	$0.21126^{\{6\}}$	$0.20715^{\{5\}}$	$0.14643^{\{2\}}$	$0.21657^{\{8\}}$	$0.21262^{\{7\}}$	$0.20617^{\{4\}}$	$0.26383^{\{10\}}$	$0.12417^{\{1\}}$	$0.22576^{\{9\}}$
		$\hat{k}$	$1.17963^{\{3\}}$	$1.33937^{\{7\}}$	$1.28206^{\{4\}}$	$0.93802^{\{2\}}$	$1.36793^{\{8\}}$	$1.32164^{\{6\}}$	$1.30425^{\{5\}}$	$1.55941^{\{10\}}$	$0.79427^{\{1\}}$	$1.47943^{\{9\}}$
		$\hat{\alpha}$	$0.07909^{\{7\}}$	$0.07128^{\{1\}}$	$0.07563^{\{6\}}$	$0.07166^{\{2\}}$	$0.07285^{\{4\}}$	$0.08367^{\{8\}}$	$0.07179^{\{3\}}$	$0.15232^{\{10\}}$	$0.07523^{\{5\}}$	$0.08702^{\{9\}}$
	$\sum Ranks$		$39^{\{4\}}$	$43^{\{5\}}$	$47^{\{6\}}$	$19^{\{1\}}$	$60^{\{8\}}$	$58^{\{7\}}$	$36^{\{3\}}$	$89^{\{10\}}$	$22^{\{2\}}$	$82^{\{9\}}$
250	BIAS	$\hat{\theta}$	$0.25092^{\{3\}}$	$0.27934^{\{8\}}$	$0.26736^{\{5\}}$	$0.14555^{\{1\}}$	$0.27747^{\{7\}}$	$0.26651^{\{4\}}$	$0.27347^{\{6\}}$	$0.34017^{\{10\}}$	$0.17047^{\{2\}}$	$0.301^{\{9\}}$
		$\hat{k}$	$0.26835^{\{3\}}$	$0.29507^{\{8\}}$	$0.27311^{\{4\}}$	$0.15134^{\{1\}}$	$0.28767^{\{6\}}$	$0.28357^{\{5\}}$	$0.29215^{\{7\}}$	$0.3383^{\{10\}}$	$0.18266^{\{2\}}$	$0.32906^{\{9\}}$
		$\hat{\alpha}$	$0.10298^{\{7\}}$	$0.09768^{\{6\}}$	$0.09718^{\{4\}}$	$0.07824^{\{1\}}$	$0.09767^{\{5\}}$	$0.1114^{\{8\}}$	$0.09633^{\{3\}}$	$0.18221^{\{10\}}$	$0.09207^{\{2\}}$	$0.11484^{\{9\}}$
	MSE	$\hat{\theta}$	$0.11918^{\{4\}}$	$0.14629^{\{7\}}$	$0.1423^{\{6\}}$	$0.07558^{\{1\}}$	$0.14871^{\{8\}}$	$0.11612^{\{3\}}$	$0.14148^{\{5\}}$	$0.207^{\{10\}}$	$0.09031^{\{2\}}$	$0.19491^{\{9\}}$
		$\hat{k}$	$0.1447^{\{4\}}$	$0.17809^{\{8\}}$	$0.16967^{\{5\}}$	$0.09165^{\{1\}}$	$0.17672^{\{6\}}$	$0.13103^{\{3\}}$	$0.17736^{\{7\}}$	$0.22592^{\{9\}}$	$0.11428^{\{2\}}$	$0.23803^{\{10\}}$
		$\hat{\alpha}$	$0.01604^{\{7\}}$	$0.01432^{\{4\}}$	$0.01465^{\{6\}}$	$0.01019^{\{1\}}$	$0.01438^{\{5\}}$	$0.01811^{\{8\}}$	$0.014^{\{3\}}$	$0.07173^{\{10\}}$	$0.01364^{\{2\}}$	$0.01984^{\{9\}}$
	MRE	$\hat{\theta}$	$0.16728^{\{3\}}$	$0.18623^{\{8\}}$	$0.17824^{\{5\}}$	$0.09703^{\{1\}}$	$0.18498^{\{7\}}$	$0.17767^{\{4\}}$	$0.18231^{\{6\}}$	$0.22678^{\{10\}}$	$0.11365^{\{2\}}$	$0.20067^{\{9\}}$
		$\hat{k}$	$1.07339^{\{3\}}$	$1.1803^{\{8\}}$	$1.09245^{\{4\}}$	$0.60537^{\{1\}}$	$1.15067^{\{6\}}$	$1.13428^{\{5\}}$	$1.16859^{\{7\}}$	$1.35319^{\{10\}}$	$0.73065^{\{2\}}$	$1.31624^{\{9\}}$
		$\hat{\alpha}$	$0.06865^{\{7\}}$	$0.06512^{\{6\}}$	$0.06479^{\{4\}}$	$0.05216^{\{1\}}$	$0.06511^{\{5\}}$	$0.07426^{\{8\}}$	$0.06422^{\{3\}}$	$0.12147^{\{10\}}$	$0.06138^{\{2\}}$	$0.07656^{\{9\}}$
	$\sum Ranks$		$41^{\{3\}}$	$63^{\{8\}}$	$43^{\{4\}}$	$9^{\{1\}}$	$55^{\{7\}}$	$48^{\{6\}}$	$47^{\{5\}}$	$89^{\{10\}}$	$18^{\{2\}}$	$82^{\{9\}}$

Table 8: Numerical values of the PGD simulation for $\theta = 2.0$, $k = 2.5$, and $\alpha = 0.5$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.68641^{\{5\}}$	$0.67819^{\{4\}}$	$0.89874^{\{8\}}$	$0.64778^{\{3\}}$	$0.9267^{\{9\}}$	$1.0066^{\{10\}}$	$0.87473^{\{6\}}$	$0.88315^{\{7\}}$	$0.50833^{\{1\}}$	$0.53425^{\{2\}}$
		$\hat{k}$	$1.05377^{\{5\}}$	$1.05034^{\{4\}}$	$1.40682^{\{7\}}$	$0.99692^{\{3\}}$	$1.43881^{\{9\}}$	$1.56529^{\{10\}}$	$1.33514^{\{6\}}$	$1.40914^{\{8\}}$	$0.73837^{\{1\}}$	$0.82327^{\{2\}}$
		$\hat{\alpha}$	$0.08368^{\{6\}}$	$0.07658^{\{4\}}$	$0.10353^{\{9\}}$	$0.05802^{\{2\}}$	$0.08802^{\{7\}}$	$0.09485^{\{8\}}$	$0.0832^{\{5\}}$	$0.10587^{\{10\}}$	$0.07211^{\{3\}}$	$0.05735^{\{1\}}$
	MSE	$\hat{\theta}$	$0.83033^{\{5\}}$	$0.82821^{\{4\}}$	$1.1658^{\{8\}}$	$0.75697^{\{3\}}$	$1.24545^{\{9\}}$	$1.44951^{\{10\}}$	$1.14533^{\{7\}}$	$1.11712^{\{6\}}$	$0.56523^{\{1\}}$	$0.6413^{\{2\}}$
		$\hat{k}$	$2.07884^{\{4\}}$	$2.11181^{\{5\}}$	$3.02022^{\{8\}}$	$1.90557^{\{3\}}$	$3.17024^{\{9\}}$	$3.82206^{\{10\}}$	$2.87884^{\{6\}}$	$3.00749^{\{7\}}$	$1.46234^{\{1\}}$	$1.70225^{\{2\}}$
		$\hat{\alpha}$	$0.01644^{\{7\}}$	$0.01391^{\{5\}}$	$0.0209^{\{9\}}$	$0.00724^{\{1\}}$	$0.01482^{\{6\}}$	$0.01672^{\{8\}}$	$0.01291^{\{4\}}$	$0.02346^{\{10\}}$	$0.00972^{\{3\}}$	$0.00778^{\{2\}}$
	MRE	$\hat{\theta}$	$0.34321^{\{5\}}$	$0.33909^{\{4\}}$	$0.44937^{\{8\}}$	$0.32389^{\{3\}}$	$0.46335^{\{9\}}$	$0.5033^{\{10\}}$	$0.43737^{\{6\}}$	$0.44158^{\{7\}}$	$0.25417^{\{1\}}$	$0.26712^{\{2\}}$
		$\hat{k}$	$0.42151^{\{5\}}$	$0.42014^{\{4\}}$	$0.56273^{\{7\}}$	$0.39877^{\{3\}}$	$0.57552^{\{9\}}$	$0.62612^{\{10\}}$	$0.53406^{\{6\}}$	$0.56366^{\{8\}}$	$0.29535^{\{1\}}$	$0.32931^{\{2\}}$
		$\hat{\alpha}$	$0.16736^{\{6\}}$	$0.15316^{\{4\}}$	$0.20707^{\{9\}}$	$0.11603^{\{2\}}$	$0.17604^{\{7\}}$	$0.18969^{\{8\}}$	$0.1664^{\{5\}}$	$0.21173^{\{10\}}$	$0.14422^{\{3\}}$	$0.11469^{\{1\}}$
	$\sum Ranks$		$48^{\{5\}}$	$38^{\{4\}}$	$73^{\{7.5\}}$	$23^{\{3\}}$	$74^{\{9\}}$	$84^{\{10\}}$	$51^{\{6\}}$	$73^{\{7.5\}}$	$15^{\{1\}}$	$16^{\{2\}}$
70	BIAS	$\hat{\theta}$	$0.38937^{\{5\}}$	$0.35492^{\{3\}}$	$0.6773^{\{8\}}$	$0.34136^{\{2\}}$	$0.67818^{\{9\}}$	$0.73042^{\{10\}}$	$0.62083^{\{6\}}$	$0.62949^{\{7\}}$	$0.36695^{\{4\}}$	$0.3331^{\{1\}}$
		$\hat{k}$	$0.57748^{\{5\}}$	$0.52603^{\{3\}}$	$1.00068^{\{9\}}$	$0.51511^{\{2\}}$	$0.9953^{\{8\}}$	$1.08324^{\{10\}}$	$0.92772^{\{6\}}$	$0.96283^{\{7\}}$	$0.55581^{\{4\}}$	$0.50521^{\{1\}}$
		$\hat{\alpha}$	$0.04467^{\{5\}}$	$0.03954^{\{3\}}$	$0.07057^{\{9\}}$	$0.03115^{\{1\}}$	$0.06894^{\{7\}}$	$0.07284^{\{10\}}$	$0.06075^{\{6\}}$	$0.06942^{\{8\}}$	$0.0417^{\{4\}}$	$0.03196^{\{2\}}$
	MSE	$\hat{\theta}$	$0.37645^{\{5\}}$	$0.31563^{\{3\}}$	$0.67497^{\{8\}}$	$0.3039^{\{2\}}$	$0.67904^{\{9\}}$	$0.793^{\{10\}}$	$0.57574^{\{6\}}$	$0.59681^{\{7\}}$	$0.31583^{\{4\}}$	$0.30082^{\{1\}}$
		$\hat{k}$	$0.89576^{\{5\}}$	$0.71499^{\{1\}}$	$1.56196^{\{9\}}$	$0.73637^{\{3\}}$	$1.53132^{\{8\}}$	$1.86001^{\{10\}}$	$1.33279^{\{6\}}$	$1.50102^{\{7\}}$	$0.82738^{\{4\}}$	$0.72904^{\{2\}}$
		$\hat{\alpha}$	$0.00611^{\{5\}}$	$0.00508^{\{4\}}$	$0.00954^{\{9\}}$	$0.00304^{\{1\}}$	$0.00872^{\{7\}}$	$0.00998^{\{10\}}$	$0.00706^{\{6\}}$	$0.00922^{\{8\}}$	$0.00365^{\{3\}}$	$0.00348^{\{2\}}$
	MRE	$\hat{\theta}$	$0.19469^{\{5\}}$	$0.17746^{\{3\}}$	$0.33865^{\{8\}}$	$0.17068^{\{2\}}$	$0.33909^{\{9\}}$	$0.36521^{\{10\}}$	$0.31042^{\{6\}}$	$0.31475^{\{7\}}$	$0.18347^{\{4\}}$	$0.16655^{\{1\}}$
		$\hat{k}$	$0.23099^{\{5\}}$	$0.21041^{\{3\}}$	$0.40027^{\{9\}}$	$0.20604^{\{2\}}$	$0.39812^{\{8\}}$	$0.4333^{\{10\}}$	$0.37109^{\{6\}}$	$0.38513^{\{7\}}$	$0.22232^{\{4\}}$	$0.20208^{\{1\}}$
		$\hat{\alpha}$	$0.08934^{\{5\}}$	$0.07908^{\{3\}}$	$0.14114^{\{9\}}$	$0.06231^{\{1\}}$	$0.13788^{\{7\}}$	$0.14567^{\{10\}}$	$0.1215^{\{6\}}$	$0.13883^{\{8\}}$	$0.0834^{\{4\}}$	$0.06392^{\{2\}}$
	$\sum Ranks$		$45^{\{5\}}$	$26^{\{3\}}$	$78^{\{9\}}$	$16^{\{2\}}$	$72^{\{8\}}$	$90^{\{10\}}$	$54^{\{6\}}$	$66^{\{7\}}$	$35^{\{4\}}$	$13^{\{1\}}$
100	BIAS	$\hat{\theta}$	$0.26088^{\{4\}}$	$0.24428^{\{1\}}$	$0.5659^{\{8\}}$	$0.24738^{\{2\}}$	$0.57232^{\{9\}}$	$0.64628^{\{10\}}$	$0.50496^{\{6\}}$	$0.56189^{\{7\}}$	$0.34338^{\{5\}}$	$0.25452^{\{3\}}$
		$\hat{k}$	$0.37157^{\{3\}}$	$0.35078^{\{1\}}$	$0.81104^{\{7\}}$	$0.36542^{\{2\}}$	$0.82814^{\{9\}}$	$0.94636^{\{10\}}$	$0.7277^{\{6\}}$	$0.81868^{\{8\}}$	$0.50452^{\{5\}}$	$0.38331^{\{4\}}$
		$\hat{\alpha}$	$0.02861^{\{4\}}$	$0.02574^{\{3\}}$	$0.05982^{\{9\}}$	$0.02239^{\{1\}}$	$0.05542^{\{7\}}$	$0.06508^{\{10\}}$	$0.04883^{\{6\}}$	$0.05879^{\{8\}}$	$0.03928^{\{5\}}$	$0.02367^{\{2\}}$
	MSE	$\hat{\theta}$	$0.21059^{\{4\}}$	$0.19318^{\{1\}}$	$0.47902^{\{8\}}$	$0.19815^{\{2\}}$	$0.50328^{\{9\}}$	$0.63382^{\{10\}}$	$0.40902^{\{6\}}$	$0.47549^{\{7\}}$	$0.28267^{\{5\}}$	$0.21017^{\{3\}}$
		$\hat{k}$	$0.4637^{\{3\}}$	$0.4279^{\{1\}}$	$1.03391^{\{7\}}$	$0.45659^{\{2\}}$	$1.09981^{\{9\}}$	$1.43409^{\{10\}}$	$0.90927^{\{6\}}$	$1.08117^{\{8\}}$	$0.66863^{\{5\}}$	$0.50608^{\{4\}}$
		$\hat{\alpha}$	$0.00318^{\{4\}}$	$0.00258^{\{3\}}$	$0.00672^{\{9\}}$	$0.002^{\{1\}}$	$0.00567^{\{7\}}$	$0.00833^{\{10\}}$	$0.00448^{\{6\}}$	$0.00655^{\{8\}}$	$0.00416^{\{5\}}$	$0.00216^{\{2\}}$
	MRE	$\hat{\theta}$	$0.13044^{\{4\}}$	$0.12214^{\{1\}}$	$0.28295^{\{8\}}$	$0.12369^{\{2\}}$	$0.28616^{\{9\}}$	$0.32314^{\{10\}}$	$0.25248^{\{6\}}$	$0.28095^{\{7\}}$	$0.17169^{\{5\}}$	$0.12726^{\{3\}}$
		$\hat{k}$	$0.14863^{\{3\}}$	$0.14031^{\{1\}}$	$0.32442^{\{7\}}$	$0.14617^{\{2\}}$	$0.33126^{\{9\}}$	$0.37854^{\{10\}}$	$0.29108^{\{6\}}$	$0.32747^{\{8\}}$	$0.20181^{\{5\}}$	$0.15332^{\{4\}}$
		$\hat{\alpha}$	$0.05721^{\{4\}}$	$0.05148^{\{3\}}$	$0.11963^{\{9\}}$	$0.04478^{\{1\}}$	$0.11084^{\{7\}}$	$0.13015^{\{10\}}$	$0.09767^{\{6\}}$	$0.11758^{\{8\}}$	$0.07857^{\{5\}}$	$0.04733^{\{2\}}$
	$\sum Ranks$		$33^{\{4\}}$	$15^{\{1.5\}}$	$72^{\{8\}}$	$15^{\{1.5\}}$	$75^{\{9\}}$	$90^{\{10\}}$	$54^{\{6\}}$	$69^{\{7\}}$	$45^{\{5\}}$	$27^{\{3\}}$
150	BIAS	$\hat{\theta}$	$0.15549^{\{3\}}$	$0.14704^{\{2\}}$	$0.47431^{\{9\}}$	$0.14019^{\{1\}}$	$0.46006^{\{8\}}$	$0.51142^{\{10\}}$	$0.42005^{\{6\}}$	$0.45043^{\{7\}}$	$0.27066^{\{5\}}$	$0.16384^{\{4\}}$
		$\hat{k}$	$0.22275^{\{3\}}$	$0.20304^{\{2\}}$	$0.67238^{\{9\}}$	$0.19823^{\{1\}}$	$0.64712^{\{8\}}$	$0.72561^{\{10\}}$	$0.59108^{\{6\}}$	$0.64169^{\{7\}}$	$0.37254^{\{5\}}$	$0.23117^{\{4\}}$
		$\hat{\alpha}$	$0.01712^{\{4\}}$	$0.01683^{\{2\}}$	$0.04609^{\{7\}}$	$0.01241^{\{1\}}$	$0.04691^{\{8\}}$	$0.05031^{\{10\}}$	$0.03901^{\{6\}}$	$0.04728^{\{9\}}$	$0.03307^{\{5\}}$	$0.0169^{\{3\}}$
	MSE	$\hat{\theta}$	$0.10863^{\{3\}}$	$0.10409^{\{2\}}$	$0.35274^{\{9\}}$	$0.09317^{\{1\}}$	$0.3331^{\{8\}}$	$0.41139^{\{10\}}$	$0.27391^{\{6\}}$	$0.31463^{\{7\}}$	$0.19272^{\{5\}}$	$0.12297^{\{4\}}$
		$\hat{k}$	$0.23048^{\{3\}}$	$0.20546^{\{2\}}$	$0.73107^{\{9\}}$	$0.19361^{\{1\}}$	$0.68367^{\{8\}}$	$0.86376^{\{10\}}$	$0.55859^{\{6\}}$	$0.67892^{\{7\}}$	$0.41086^{\{5\}}$	$0.25772^{\{4\}}$
		$\hat{\alpha}$	$0.00153^{\{3\}}$	$0.00166^{\{4\}}$	$0.00416^{\{9\}}$	$0.00091^{\{1\}}$	$0.00413^{\{8\}}$	$0.00512^{\{10\}}$	$0.00296^{\{5\}}$	$0.00402^{\{7\}}$	$0.00377^{\{6\}}$	$0.00123^{\{2\}}$
	MRE	$\hat{\theta}$	$0.07775^{\{3\}}$	$0.07352^{\{2\}}$	$0.23715^{\{9\}}$	$0.07009^{\{1\}}$	$0.23003^{\{8\}}$	$0.25571^{\{10\}}$	$0.21003^{\{6\}}$	$0.22521^{\{7\}}$	$0.13533^{\{5\}}$	$0.08192^{\{4\}}$
		$\hat{k}$	$0.0891^{\{3\}}$	$0.08121^{\{2\}}$	$0.26895^{\{9\}}$	$0.07929^{\{1\}}$	$0.25885^{\{8\}}$	$0.29024^{\{10\}}$	$0.23643^{\{6\}}$	$0.25668^{\{7\}}$	$0.14901^{\{5\}}$	$0.09247^{\{4\}}$
		$\hat{\alpha}$	$0.03424^{\{4\}}$	$0.03365^{\{2\}}$	$0.09218^{\{7\}}$	$0.02481^{\{1\}}$	$0.09383^{\{8\}}$	$0.10063^{\{10\}}$	$0.07803^{\{6\}}$	$0.09457^{\{9\}}$	$0.06614^{\{5\}}$	$0.03379^{\{3\}}$
	$\sum Ranks$		$29^{\{3\}}$	$20^{\{2\}}$	$77^{\{9\}}$	$9^{\{1\}}$	$72^{\{8\}}$	$90^{\{10\}}$	$53^{\{6\}}$	$67^{\{7\}}$	$46^{\{5\}}$	$32^{\{4\}}$
250	BIAS	$\hat{\theta}$	$0.06276^{\{4\}}$	$0.06025^{\{1\}}$	$0.36003^{\{9\}}$	$0.06054^{\{2\}}$	$0.34474^{\{7\}}$	$0.40634^{\{10\}}$	$0.32146^{\{6\}}$	$0.35544^{\{8\}}$	$0.17883^{\{5\}}$	$0.06055^{\{3\}}$
		$\hat{k}$	$0.08466^{\{4\}}$	$0.08238^{\{2\}}$	$0.50324^{\{9\}}$	$0.0834^{\{3\}}$	$0.47774^{\{7\}}$	$0.55528^{\{10\}}$	$0.44517^{\{6\}}$	$0.49596^{\{8\}}$	$0.23422^{\{5\}}$	$0.08164^{\{1\}}$
		$\hat{\alpha}$	$0.00721^{\{4\}}$	$0.0064^{\{2\}}$	$0.03396^{\{8\}}$	$0.0056^{\{1\}}$	$0.03335^{\{7\}}$	$0.03756^{\{9\}}$	$0.03014^{\{6\}}$	$0.03923^{\{10\}}$	$0.02116^{\{5\}}$	$0.00651^{\{3\}}$
	MSE	$\hat{\theta}$	$0.03576^{\{2\}}$	$0.03544^{\{1\}}$	$0.2063^{\{9\}}$	$0.03784^{\{4\}}$	$0.19356^{\{7\}}$	$0.26847^{\{10\}}$	$0.15874^{\{6\}}$	$0.20613^{\{8\}}$	$0.1256^{\{5\}}$	$0.03737^{\{3\}}$
		$\hat{k}$	$0.0656^{\{2\}}$	$0.06491^{\{1\}}$	$0.41363^{\{8\}}$	$0.08037^{\{4\}}$	$0.38115^{\{7\}}$	$0.52546^{\{10\}}$	$0.31186^{\{6\}}$	$0.41671^{\{9\}}$	$0.23066^{\{5\}}$	$0.07075^{\{3\}}$
		$\hat{\alpha}$	$0.00052^{\{4\}}$	$0.00042^{\{2\}}$	$0.00218^{\{8\}}$	$0.00034^{\{1\}}$	$0.00202^{\{6.5\}}$	$0.00281^{\{10\}}$	$0.00158^{\{5\}}$	$0.00257^{\{9\}}$	$0.00202^{\{6.5\}}$	$0.00043^{\{3\}}$
	MRE	$\hat{\theta}$	$0.03138^{\{4\}}$	$0.03012^{\{1\}}$	$0.18001^{\{9\}}$	$0.03027^{\{2.5\}}$	$0.17237^{\{7\}}$	$0.20317^{\{10\}}$	$0.16073^{\{6\}}$	$0.17772^{\{8\}}$	$0.08942^{\{5\}}$	$0.03027^{\{2.5\}}$
		$\hat{k}$	$0.03387^{\{4\}}$	$0.03295^{\{2\}}$	$0.20129^{\{9\}}$	$0.03336^{\{3\}}$	$0.1911^{\{7\}}$	$0.22211^{\{10\}}$	$0.17807^{\{6\}}$	$0.19838^{\{8\}}$	$0.09369^{\{5\}}$	$0.03266^{\{1\}}$
		$\hat{\alpha}$	$0.01441^{\{4\}}$	$0.0128^{\{2\}}$	$0.06792^{\{8\}}$	$0.01121^{\{1\}}$	$0.0667^{\{7\}}$	$0.07513^{\{9\}}$	$0.06027^{\{6\}}$	$0.07846^{\{10\}}$	$0.04232^{\{5\}}$	$0.01302^{\{3\}}$
	$\sum Ranks$		$32^{\{4\}}$	$14^{\{1\}}$	$77^{\{8\}}$	$21.5^{\{2\}}$	$62.5^{\{7\}}$	$88^{\{10\}}$	$53^{\{6\}}$	$78^{\{9\}}$	$46.5^{\{5\}}$	$22.5^{\{3\}}$

Table 9: Numerical values of the PGD simulation for $\theta = 0.5$, $k = 1.5$, and $\alpha = 2.5$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.38242^{\{2\}}$	$0.54166^{\{7\}}$	$0.52674^{\{5\}}$	$0.6097^{\{8\}}$	$0.67283^{\{10\}}$	$0.52859^{\{6\}}$	$0.63371^{\{9\}}$	$0.48975^{\{3\}}$	$0.33305^{\{1\}}$	$0.4898^{\{4\}}$
		$\hat{k}$	$0.85665^{\{2\}}$	$1.16018^{\{7\}}$	$1.15991^{\{6\}}$	$1.19179^{\{8\}}$	$1.35032^{\{10\}}$	$1.15534^{\{5\}}$	$1.27672^{\{9\}}$	$1.12234^{\{4\}}$	$0.60399^{\{1\}}$	$0.90601^{\{3\}}$
		$\hat{\alpha}$	$0.42151^{\{2\}}$	$0.48617^{\{6\}}$	$0.46463^{\{5\}}$	$0.4923^{\{7\}}$	$0.53337^{\{10\}}$	$0.43717^{\{3\}}$	$0.52343^{\{9\}}$	$0.5112^{\{8\}}$	$0.37548^{\{1\}}$	$0.45308^{\{4\}}$
	MSE	$\hat{\theta}$	$0.4092^{\{2\}}$	$0.74692^{\{5\}}$	$0.78359^{\{6\}}$	$0.88812^{\{8\}}$	$1.23521^{\{10\}}$	$0.81818^{\{7\}}$	$0.9586^{\{9\}}$	$0.57516^{\{3\}}$	$0.38798^{\{1\}}$	$0.66622^{\{4\}}$
		$\hat{k}$	$1.51856^{\{2\}}$	$2.43628^{\{5\}}$	$2.59493^{\{7\}}$	$2.50059^{\{6\}}$	$3.76395^{\{10\}}$	$2.607^{\{8\}}$	$2.92356^{\{9\}}$	$2.06313^{\{4\}}$	$1.05837^{\{1\}}$	$1.78033^{\{3\}}$
		$\hat{\alpha}$	$0.28367^{\{2\}}$	$0.34345^{\{6\}}$	$0.32068^{\{4\}}$	$0.36124^{\{7\}}$	$0.40574^{\{9\}}$	$0.30142^{\{3\}}$	$0.38686^{\{8\}}$	$0.41734^{\{10\}}$	$0.23155^{\{1\}}$	$0.32278^{\{5\}}$
	MRE	$\hat{\theta}$	$0.76485^{\{2\}}$	$1.08332^{\{7\}}$	$1.05347^{\{5\}}$	$1.21939^{\{8\}}$	$1.34566^{\{10\}}$	$1.05717^{\{6\}}$	$1.26742^{\{9\}}$	$0.9795^{\{3\}}$	$0.6661^{\{1\}}$	$0.9796^{\{4\}}$
		$\hat{k}$	$0.5711^{\{2\}}$	$0.77345^{\{7\}}$	$0.77327^{\{6\}}$	$0.79452^{\{8\}}$	$0.90021^{\{10\}}$	$0.77023^{\{5\}}$	$0.85114^{\{9\}}$	$0.74823^{\{4\}}$	$0.40266^{\{1\}}$	$0.60401^{\{3\}}$
		$\hat{\alpha}$	$0.16861^{\{2\}}$	$0.19447^{\{6\}}$	$0.18585^{\{5\}}$	$0.19692^{\{7\}}$	$0.21335^{\{10\}}$	$0.17487^{\{3\}}$	$0.20937^{\{9\}}$	$0.20448^{\{8\}}$	$0.15019^{\{1\}}$	$0.18123^{\{4\}}$
	$\sum Ranks$		$18^{\{2\}}$	$56^{\{7\}}$	$49^{\{6\}}$	$67^{\{8\}}$	$89^{\{10\}}$	$46^{\{4\}}$	$80^{\{9\}}$	$47^{\{5\}}$	$9^{\{1\}}$	$34^{\{3\}}$
70	BIAS	$\hat{\theta}$	$0.27755^{\{2\}}$	$0.32299^{\{5\}}$	$0.32695^{\{6\}}$	$0.39841^{\{10\}}$	$0.39527^{\{9\}}$	$0.35632^{\{8\}}$	$0.34141^{\{7\}}$	$0.3066^{\{3\}}$	$0.23548^{\{1\}}$	$0.31744^{\{4\}}$
		$\hat{k}$	$0.67782^{\{3\}}$	$0.74039^{\{4\}}$	$0.7946^{\{6\}}$	$0.82619^{\{9\}}$	$0.8986^{\{10\}}$	$0.80748^{\{7\}}$	$0.75003^{\{5\}}$	$0.81299^{\{8\}}$	$0.46464^{\{1\}}$	$0.64873^{\{2\}}$
		$\hat{\alpha}$	$0.30513^{\{2\}}$	$0.33961^{\{5\}}$	$0.35183^{\{8\}}$	$0.37375^{\{9\}}$	$0.3927^{\{10\}}$	$0.34417^{\{6\}}$	$0.35131^{\{7\}}$	$0.3394^{\{4\}}$	$0.28615^{\{1\}}$	$0.31663^{\{3\}}$
	MSE	$\hat{\theta}$	$0.16235^{\{2\}}$	$0.21019^{\{4\}}$	$0.224^{\{5\}}$	$0.32223^{\{8\}}$	$0.33496^{\{10\}}$	$0.32581^{\{9\}}$	$0.23997^{\{6\}}$	$0.18402^{\{3\}}$	$0.16014^{\{1\}}$	$0.25086^{\{7\}}$
		$\hat{k}$	$0.79002^{\{3\}}$	$0.88505^{\{4\}}$	$1.01734^{\{7\}}$	$1.07377^{\{8\}}$	$1.27994^{\{10\}}$	$1.1909^{\{9\}}$	$0.92141^{\{5\}}$	$1.01137^{\{6\}}$	$0.4847^{\{1\}}$	$0.78597^{\{2\}}$
		$\hat{\alpha}$	$0.14383^{\{2\}}$	$0.1742^{\{4\}}$	$0.18673^{\{8\}}$	$0.20885^{\{9\}}$	$0.22701^{\{10\}}$	$0.18458^{\{7\}}$	$0.18028^{\{6\}}$	$0.17947^{\{5\}}$	$0.13965^{\{1\}}$	$0.17025^{\{3\}}$
	MRE	$\hat{\theta}$	$0.55511^{\{2\}}$	$0.64598^{\{5\}}$	$0.6539^{\{6\}}$	$0.79683^{\{10\}}$	$0.79054^{\{9\}}$	$0.71263^{\{8\}}$	$0.68281^{\{7\}}$	$0.6132^{\{3\}}$	$0.47096^{\{1\}}$	$0.63487^{\{4\}}$
		$\hat{k}$	$0.45188^{\{3\}}$	$0.49359^{\{4\}}$	$0.52973^{\{6\}}$	$0.55079^{\{9\}}$	$0.59906^{\{10\}}$	$0.53832^{\{7\}}$	$0.50002^{\{5\}}$	$0.542^{\{8\}}$	$0.30976^{\{1\}}$	$0.43249^{\{2\}}$
		$\hat{\alpha}$	$0.12205^{\{2\}}$	$0.13584^{\{5\}}$	$0.14073^{\{8\}}$	$0.1495^{\{9\}}$	$0.15708^{\{10\}}$	$0.13767^{\{6\}}$	$0.14053^{\{7\}}$	$0.13576^{\{4\}}$	$0.11446^{\{1\}}$	$0.12665^{\{3\}}$
	$\sum Ranks$		$21^{\{2\}}$	$40^{\{4\}}$	$60^{\{7\}}$	$81^{\{9\}}$	$88^{\{10\}}$	$67^{\{8\}}$	$55^{\{6\}}$	$44^{\{5\}}$	$9^{\{1\}}$	$30^{\{3\}}$
100	BIAS	$\hat{\theta}$	$0.2466^{\{2\}}$	$0.27262^{\{4\}}$	$0.30183^{\{7\}}$	$0.32147^{\{8\}}$	$0.34189^{\{10\}}$	$0.32708^{\{9\}}$	$0.29051^{\{5\}}$	$0.26738^{\{3\}}$	$0.20685^{\{1\}}$	$0.3^{\{6\}}$
		$\hat{k}$	$0.59251^{\{3\}}$	$0.66561^{\{4\}}$	$0.74558^{\{8\}}$	$0.67171^{\{5\}}$	$0.75917^{\{9\}}$	$0.76199^{\{10\}}$	$0.68518^{\{6\}}$	$0.69513^{\{7\}}$	$0.40982^{\{1\}}$	$0.59189^{\{2\}}$
		$\hat{\alpha}$	$0.27614^{\{2\}}$	$0.29416^{\{3\}}$	$0.31982^{\{9\}}$	$0.31149^{\{8\}}$	$0.34329^{\{10\}}$	$0.30203^{\{6\}}$	$0.30971^{\{7\}}$	$0.30028^{\{5\}}$	$0.25062^{\{1\}}$	$0.29877^{\{4\}}$
	MSE	$\hat{\theta}$	$0.11954^{\{2\}}$	$0.14291^{\{4\}}$	$0.19068^{\{6\}}$	$0.21175^{\{8\}}$	$0.24253^{\{9\}}$	$0.26779^{\{10\}}$	$0.14932^{\{5\}}$	$0.12441^{\{3\}}$	$0.1186^{\{1\}}$	$0.21167^{\{7\}}$
		$\hat{k}$	$0.61763^{\{2\}}$	$0.71973^{\{4\}}$	$0.91444^{\{8\}}$	$0.73815^{\{6\}}$	$0.9476^{\{9\}}$	$1.02668^{\{10\}}$	$0.72937^{\{5\}}$	$0.75853^{\{7\}}$	$0.37038^{\{1\}}$	$0.65586^{\{3\}}$
		$\hat{\alpha}$	$0.11823^{\{2\}}$	$0.13292^{\{3\}}$	$0.15427^{\{9\}}$	$0.15149^{\{8\}}$	$0.17559^{\{10\}}$	$0.14361^{\{6\}}$	$0.14185^{\{5\}}$	$0.13663^{\{4\}}$	$0.10705^{\{1\}}$	$0.14828^{\{7\}}$
	MRE	$\hat{\theta}$	$0.49321^{\{2\}}$	$0.54525^{\{4\}}$	$0.60365^{\{7\}}$	$0.64293^{\{8\}}$	$0.68378^{\{10\}}$	$0.65416^{\{9\}}$	$0.58103^{\{5\}}$	$0.53477^{\{3\}}$	$0.4137^{\{1\}}$	$0.6^{\{6\}}$
		$\hat{k}$	$0.39501^{\{3\}}$	$0.44374^{\{4\}}$	$0.49705^{\{8\}}$	$0.44781^{\{5\}}$	$0.50611^{\{9\}}$	$0.50799^{\{10\}}$	$0.45678^{\{6\}}$	$0.46342^{\{7\}}$	$0.27321^{\{1\}}$	$0.3946^{\{2\}}$
		$\hat{\alpha}$	$0.11046^{\{2\}}$	$0.11766^{\{3\}}$	$0.12793^{\{9\}}$	$0.1246^{\{8\}}$	$0.13732^{\{10\}}$	$0.12081^{\{6\}}$	$0.12388^{\{7\}}$	$0.12011^{\{5\}}$	$0.10025^{\{1\}}$	$0.11951^{\{4\}}$
	$\sum Ranks$		$20^{\{2\}}$	$33^{\{3\}}$	$71^{\{8\}}$	$64^{\{7\}}$	$86^{\{10\}}$	$76^{\{9\}}$	$51^{\{6\}}$	$44^{\{5\}}$	$9^{\{1\}}$	$41^{\{4\}}$
150	BIAS	$\hat{\theta}$	$0.20376^{\{2\}}$	$0.22971^{\{5\}}$	$0.25039^{\{8\}}$	$0.24397^{\{6\}}$	$0.26617^{\{9\}}$	$0.26618^{\{10\}}$	$0.22919^{\{4\}}$	$0.22237^{\{3\}}$	$0.19287^{\{1\}}$	$0.25013^{\{7\}}$
		$\hat{k}$	$0.4983^{\{2\}}$	$0.55241^{\{5\}}$	$0.62486^{\{8\}}$	$0.53261^{\{4\}}$	$0.63468^{\{9\}}$	$0.66456^{\{10\}}$	$0.55307^{\{6\}}$	$0.59418^{\{7\}}$	$0.39133^{\{1\}}$	$0.52523^{\{3\}}$
		$\hat{\alpha}$	$0.22743^{\{1\}}$	$0.26072^{\{7\}}$	$0.27816^{\{8\}}$	$0.25014^{\{4\}}$	$0.28029^{\{10\}}$	$0.27819^{\{9\}}$	$0.24895^{\{3\}}$	$0.25575^{\{5\}}$	$0.22797^{\{2\}}$	$0.2596^{\{6\}}$
	MSE	$\hat{\theta}$	$0.0781^{\{1\}}$	$0.09274^{\{3\}}$	$0.11074^{\{6\}}$	$0.11969^{\{7\}}$	$0.13144^{\{8\}}$	$0.14541^{\{10\}}$	$0.09861^{\{5\}}$	$0.08688^{\{2\}}$	$0.09468^{\{4\}}$	$0.13941^{\{9\}}$
		$\hat{k}$	$0.45246^{\{2\}}$	$0.52667^{\{5\}}$	$0.6476^{\{8\}}$	$0.49321^{\{3\}}$	$0.66161^{\{9\}}$	$0.73722^{\{10\}}$	$0.53711^{\{6\}}$	$0.59999^{\{7\}}$	$0.31946^{\{1\}}$	$0.50141^{\{4\}}$
		$\hat{\alpha}$	$0.08149^{\{1\}}$	$0.10338^{\{6\}}$	$0.11785^{\{10\}}$	$0.09994^{\{4\}}$	$0.11734^{\{9\}}$	$0.11619^{\{8\}}$	$0.09366^{\{3\}}$	$0.10209^{\{5\}}$	$0.08792^{\{2\}}$	$0.11185^{\{7\}}$
	MRE	$\hat{\theta}$	$0.40752^{\{2\}}$	$0.45941^{\{5\}}$	$0.50078^{\{8\}}$	$0.48793^{\{6\}}$	$0.53234^{\{9\}}$	$0.53236^{\{10\}}$	$0.45839^{\{4\}}$	$0.44474^{\{3\}}$	$0.38575^{\{1\}}$	$0.50025^{\{7\}}$
		$\hat{k}$	$0.3322^{\{2\}}$	$0.36827^{\{5\}}$	$0.41657^{\{8\}}$	$0.35507^{\{4\}}$	$0.42312^{\{9\}}$	$0.44304^{\{10\}}$	$0.36871^{\{6\}}$	$0.39612^{\{7\}}$	$0.26089^{\{1\}}$	$0.35015^{\{3\}}$
		$\hat{\alpha}$	$0.09097^{\{1\}}$	$0.10429^{\{7\}}$	$0.11126^{\{8\}}$	$0.10006^{\{4\}}$	$0.11212^{\{10\}}$	$0.11128^{\{9\}}$	$0.09958^{\{3\}}$	$0.1023^{\{5\}}$	$0.09119^{\{2\}}$	$0.10384^{\{6\}}$
	$\sum Ranks$		$14^{\{1\}}$	$48^{\{6\}}$	$72^{\{8\}}$	$42^{\{4\}}$	$82^{\{9\}}$	$86^{\{10\}}$	$40^{\{3\}}$	$44^{\{5\}}$	$15^{\{2\}}$	$52^{\{7\}}$
250	BIAS	$\hat{\theta}$	$0.16202^{\{1\}}$	$0.16951^{\{3\}}$	$0.20108^{\{8\}}$	$0.18616^{\{6\}}$	$0.20902^{\{9\}}$	$0.21365^{\{10\}}$	$0.17568^{\{4\}}$	$0.18335^{\{5\}}$	$0.16524^{\{2\}}$	$0.19602^{\{7\}}$
		$\hat{k}$	$0.37514^{\{2\}}$	$0.41545^{\{5\}}$	$0.47727^{\{8\}}$	$0.38633^{\{3\}}$	$0.51795^{\{9\}}$	$0.52296^{\{10\}}$	$0.40251^{\{4\}}$	$0.4743^{\{7\}}$	$0.33952^{\{1\}}$	$0.43819^{\{6\}}$
		$\hat{\alpha}$	$0.18974^{\{1\}}$	$0.19207^{\{2\}}$	$0.22647^{\{8\}}$	$0.21213^{\{7\}}$	$0.23243^{\{10\}}$	$0.22867^{\{9\}}$	$0.19897^{\{4\}}$	$0.2029^{\{5\}}$	$0.19569^{\{3\}}$	$0.20536^{\{6\}}$
	MSE	$\hat{\theta}$	$0.04488^{\{1\}}$	$0.04866^{\{2\}}$	$0.06909^{\{7\}}$	$0.06883^{\{6\}}$	$0.07635^{\{8\}}$	$0.0883^{\{10\}}$	$0.05117^{\{3\}}$	$0.05498^{\{4\}}$	$0.06487^{\{5\}}$	$0.07782^{\{9\}}$
		$\hat{k}$	$0.27424^{\{2\}}$	$0.33679^{\{5\}}$	$0.41223^{\{7\}}$	$0.28768^{\{3\}}$	$0.47177^{\{9\}}$	$0.48462^{\{10\}}$	$0.30569^{\{4\}}$	$0.42377^{\{8\}}$	$0.24349^{\{1\}}$	$0.34919^{\{6\}}$
		$\hat{\alpha}$	$0.05514^{\{1\}}$	$0.05621^{\{2\}}$	$0.077^{\{8\}}$	$0.0707^{\{7\}}$	$0.08137^{\{10\}}$	$0.08113^{\{9\}}$	$0.05947^{\{3\}}$	$0.06318^{\{4\}}$	$0.06669^{\{5\}}$	$0.07046^{\{6\}}$
	MRE	$\hat{\theta}$	$0.32405^{\{1\}}$	$0.33902^{\{3\}}$	$0.40216^{\{8\}}$	$0.37232^{\{6\}}$	$0.41804^{\{9\}}$	$0.4273^{\{10\}}$	$0.35136^{\{4\}}$	$0.3667^{\{5\}}$	$0.33048^{\{2\}}$	$0.39204^{\{7\}}$
		$\hat{k}$	$0.25009^{\{2\}}$	$0.27697^{\{5\}}$	$0.31818^{\{8\}}$	$0.25755^{\{3\}}$	$0.3453^{\{9\}}$	$0.34864^{\{10\}}$	$0.26834^{\{4\}}$	$0.3162^{\{7\}}$	$0.22635^{\{1\}}$	$0.29213^{\{6\}}$
		$\hat{\alpha}$	$0.0759^{\{1\}}$	$0.07683^{\{2\}}$	$0.09059^{\{8\}}$	$0.08485^{\{7\}}$	$0.09297^{\{10\}}$	$0.09147^{\{9\}}$	$0.07959^{\{4\}}$	$0.08116^{\{5\}}$	$0.07828^{\{3\}}$	$0.08214^{\{6\}}$
	$\sum Ranks$		$12^{\{1\}}$	$29^{\{3\}}$	$70^{\{8\}}$	$48^{\{5\}}$	$83^{\{9\}}$	$87^{\{10\}}$	$34^{\{4\}}$	$50^{\{6\}}$	$23^{\{2\}}$	$59^{\{7\}}$

Table 10: Numerical values of the PGD simulation for $\theta = 2.5$, $k = 1.5$, and $\alpha = 2.5$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.93923^{\{8\}}$	$0.85634^{\{4\}}$	$0.97288^{\{10\}}$	$0.78383^{\{3\}}$	$0.88991^{\{6\}}$	$0.95194^{\{9\}}$	$0.88865^{\{5\}}$	$0.93358^{\{7\}}$	$0.67018^{\{1\}}$	$0.77213^{\{2\}}$
		$\hat{k}$	$1.15822^{\{7\}}$	$1.03411^{\{4\}}$	$1.1675^{\{8\}}$	$0.91233^{\{3\}}$	$1.07246^{\{5\}}$	$1.18694^{\{10\}}$	$1.08699^{\{6\}}$	$1.1785^{\{9\}}$	$0.7726^{\{1\}}$	$0.8833^{\{2\}}$
		$\hat{\alpha}$	$0.49567^{\{8\}}$	$0.44167^{\{4\}}$	$0.5388^{\{9\}}$	$0.35914^{\{1\}}$	$0.45275^{\{6\}}$	$0.46876^{\{7\}}$	$0.4418^{\{5\}}$	$0.7977^{\{10\}}$	$0.42771^{\{3\}}$	$0.40009^{\{2\}}$
	MSE	$\hat{\theta}$	$1.2309^{\{8\}}$	$1.061^{\{4\}}$	$1.36703^{\{10\}}$	$0.89876^{\{2\}}$	$1.13982^{\{5\}}$	$1.27038^{\{9\}}$	$1.17523^{\{7\}}$	$1.16497^{\{6\}}$	$0.76733^{\{1\}}$	$0.90469^{\{3\}}$
		$\hat{k}$	$2.07807^{\{8\}}$	$1.70581^{\{4\}}$	$2.10838^{\{9\}}$	$1.31775^{\{2\}}$	$1.78986^{\{5\}}$	$2.31976^{\{10\}}$	$2.01056^{\{7\}}$	$1.85148^{\{6\}}$	$1.16608^{\{1\}}$	$1.41586^{\{3\}}$
		$\hat{\alpha}$	$0.44133^{\{8\}}$	$0.34419^{\{4\}}$	$0.52611^{\{9\}}$	$0.20548^{\{1\}}$	$0.35043^{\{5\}}$	$0.39264^{\{7\}}$	$0.35387^{\{6\}}$	$2.18571^{\{10\}}$	$0.33535^{\{3\}}$	$0.26867^{\{2\}}$
	MRE	$\hat{\theta}$	$0.37569^{\{8\}}$	$0.34254^{\{4\}}$	$0.38915^{\{10\}}$	$0.31353^{\{3\}}$	$0.35597^{\{6\}}$	$0.38078^{\{9\}}$	$0.35546^{\{5\}}$	$0.37343^{\{7\}}$	$0.26807^{\{1\}}$	$0.30885^{\{2\}}$
		$\hat{k}$	$0.77215^{\{7\}}$	$0.68941^{\{4\}}$	$0.77833^{\{8\}}$	$0.60822^{\{3\}}$	$0.71498^{\{5\}}$	$0.79129^{\{10\}}$	$0.72466^{\{6\}}$	$0.78567^{\{9\}}$	$0.51507^{\{1\}}$	$0.58887^{\{2\}}$
		$\hat{\alpha}$	$0.19827^{\{8\}}$	$0.17667^{\{4\}}$	$0.21552^{\{9\}}$	$0.14366^{\{1\}}$	$0.1811^{\{6\}}$	$0.18751^{\{7\}}$	$0.17672^{\{5\}}$	$0.31908^{\{10\}}$	$0.17109^{\{3\}}$	$0.16004^{\{2\}}$
	$\sum Ranks$		$70^{\{7\}}$	$36^{\{4\}}$	$82^{\{10\}}$	$19^{\{2\}}$	$49^{\{5\}}$	$78^{\{9\}}$	$52^{\{6\}}$	$74^{\{8\}}$	$15^{\{1\}}$	$20^{\{3\}}$
70	BIAS	$\hat{\theta}$	$0.71623^{\{7\}}$	$0.6438^{\{4\}}$	$0.71014^{\{6\}}$	$0.53655^{\{2\}}$	$0.73008^{\{9\}}$	$0.72043^{\{8\}}$	$0.66158^{\{5\}}$	$0.82648^{\{10\}}$	$0.53274^{\{1\}}$	$0.59957^{\{3\}}$
		$\hat{k}$	$0.84792^{\{7\}}$	$0.78379^{\{5\}}$	$0.84295^{\{6\}}$	$0.61946^{\{2\}}$	$0.85243^{\{8\}}$	$0.86852^{\{9\}}$	$0.7797^{\{4\}}$	$1.01174^{\{10\}}$	$0.59744^{\{1\}}$	$0.68724^{\{3\}}$
		$\hat{\alpha}$	$0.31654^{\{7\}}$	$0.28998^{\{5\}}$	$0.33472^{\{9\}}$	$0.2454^{\{1\}}$	$0.33258^{\{8\}}$	$0.29659^{\{6\}}$	$0.28322^{\{3\}}$	$0.51437^{\{10\}}$	$0.28481^{\{4\}}$	$0.27018^{\{2\}}$
	MSE	$\hat{\theta}$	$0.72691^{\{6\}}$	$0.62417^{\{4\}}$	$0.72897^{\{7\}}$	$0.47121^{\{1\}}$	$0.75513^{\{9\}}$	$0.73648^{\{8\}}$	$0.63182^{\{5\}}$	$0.95416^{\{10\}}$	$0.49207^{\{2\}}$	$0.56362^{\{3\}}$
		$\hat{k}$	$1.05889^{\{6\}}$	$0.94992^{\{5\}}$	$1.06551^{\{7\}}$	$0.6594^{\{1\}}$	$1.12588^{\{9\}}$	$1.11346^{\{8\}}$	$0.90929^{\{4\}}$	$1.45848^{\{10\}}$	$0.66493^{\{2\}}$	$0.78317^{\{3\}}$
		$\hat{\alpha}$	$0.18339^{\{7\}}$	$0.13928^{\{5\}}$	$0.19222^{\{8\}}$	$0.09511^{\{1\}}$	$0.19457^{\{9\}}$	$0.14752^{\{6\}}$	$0.13449^{\{4\}}$	$0.66164^{\{10\}}$	$0.13121^{\{3\}}$	$0.11817^{\{2\}}$
	MRE	$\hat{\theta}$	$0.28649^{\{7\}}$	$0.25752^{\{4\}}$	$0.28405^{\{6\}}$	$0.21462^{\{2\}}$	$0.29203^{\{9\}}$	$0.28817^{\{8\}}$	$0.26463^{\{5\}}$	$0.33059^{\{10\}}$	$0.2131^{\{1\}}$	$0.23983^{\{3\}}$
		$\hat{k}$	$0.56528^{\{7\}}$	$0.52253^{\{5\}}$	$0.56196^{\{6\}}$	$0.41297^{\{2\}}$	$0.56829^{\{8\}}$	$0.57901^{\{9\}}$	$0.5198^{\{4\}}$	$0.67449^{\{10\}}$	$0.39829^{\{1\}}$	$0.45816^{\{3\}}$
		$\hat{\alpha}$	$0.12662^{\{7\}}$	$0.11599^{\{5\}}$	$0.13389^{\{9\}}$	$0.09816^{\{1\}}$	$0.13303^{\{8\}}$	$0.11864^{\{6\}}$	$0.11329^{\{3\}}$	$0.20575^{\{10\}}$	$0.11392^{\{4\}}$	$0.10807^{\{2\}}$
	$\sum Ranks$		$61^{\{6\}}$	$42^{\{5\}}$	$64^{\{7\}}$	$13^{\{1\}}$	$77^{\{9\}}$	$68^{\{8\}}$	$37^{\{4\}}$	$90^{\{10\}}$	$19^{\{2\}}$	$24^{\{3\}}$
100	BIAS	$\hat{\theta}$	$0.6526^{\{8\}}$	$0.5885^{\{4\}}$	$0.67378^{\{9\}}$	$0.50432^{\{1\}}$	$0.64718^{\{7\}}$	$0.64441^{\{6\}}$	$0.58907^{\{5\}}$	$0.69952^{\{10\}}$	$0.50682^{\{2\}}$	$0.55763^{\{3\}}$
		$\hat{k}$	$0.76335^{\{8\}}$	$0.66573^{\{4\}}$	$0.78315^{\{9\}}$	$0.57051^{\{2\}}$	$0.75839^{\{7\}}$	$0.75794^{\{6\}}$	$0.6891^{\{5\}}$	$0.83653^{\{10\}}$	$0.56934^{\{1\}}$	$0.63196^{\{3\}}$
		$\hat{\alpha}$	$0.26711^{\{8\}}$	$0.25509^{\{6\}}$	$0.30498^{\{9\}}$	$0.21484^{\{1\}}$	$0.26114^{\{7\}}$	$0.2536^{\{5\}}$	$0.23801^{\{3\}}$	$0.37479^{\{10\}}$	$0.24791^{\{4\}}$	$0.22332^{\{2\}}$
	MSE	$\hat{\theta}$	$0.62827^{\{8\}}$	$0.51453^{\{4\}}$	$0.66142^{\{9\}}$	$0.41213^{\{1\}}$	$0.59744^{\{7\}}$	$0.57976^{\{6\}}$	$0.51667^{\{5\}}$	$0.7324^{\{10\}}$	$0.4306^{\{2\}}$	$0.48924^{\{3\}}$
		$\hat{k}$	$0.89846^{\{8\}}$	$0.668^{\{3\}}$	$0.93431^{\{9\}}$	$0.55041^{\{1\}}$	$0.85326^{\{7\}}$	$0.80439^{\{6\}}$	$0.72522^{\{5\}}$	$1.0748^{\{10\}}$	$0.56871^{\{2\}}$	$0.67368^{\{4\}}$
		$\hat{\alpha}$	$0.12961^{\{8\}}$	$0.10632^{\{5\}}$	$0.15336^{\{9\}}$	$0.07725^{\{1\}}$	$0.11376^{\{7\}}$	$0.10672^{\{6\}}$	$0.09913^{\{3\}}$	$0.32293^{\{10\}}$	$0.1006^{\{4\}}$	$0.08261^{\{2\}}$
	MRE	$\hat{\theta}$	$0.26104^{\{8\}}$	$0.2354^{\{4\}}$	$0.26951^{\{9\}}$	$0.20173^{\{1\}}$	$0.25887^{\{7\}}$	$0.25776^{\{6\}}$	$0.23563^{\{5\}}$	$0.27981^{\{10\}}$	$0.20273^{\{2\}}$	$0.22305^{\{3\}}$
		$\hat{k}$	$0.5089^{\{8\}}$	$0.44382^{\{4\}}$	$0.5221^{\{9\}}$	$0.38034^{\{2\}}$	$0.50559^{\{7\}}$	$0.50529^{\{6\}}$	$0.4594^{\{5\}}$	$0.55768^{\{10\}}$	$0.37956^{\{1\}}$	$0.4213^{\{3\}}$
		$\hat{\alpha}$	$0.10684^{\{8\}}$	$0.10203^{\{6\}}$	$0.12199^{\{9\}}$	$0.08594^{\{1\}}$	$0.10445^{\{7\}}$	$0.10144^{\{5\}}$	$0.0952^{\{3\}}$	$0.14991^{\{10\}}$	$0.09916^{\{4\}}$	$0.08933^{\{2\}}$
	$\sum Ranks$		$72^{\{8\}}$	$40^{\{5\}}$	$81^{\{9\}}$	$11^{\{1\}}$	$63^{\{7\}}$	$52^{\{6\}}$	$39^{\{4\}}$	$90^{\{10\}}$	$22^{\{2\}}$	$25^{\{3\}}$
150	BIAS	$\hat{\theta}$	$0.54254^{\{7\}}$	$0.49281^{\{5\}}$	$0.55019^{\{9\}}$	$0.41393^{\{1\}}$	$0.52908^{\{6\}}$	$0.54649^{\{8\}}$	$0.49115^{\{4\}}$	$0.64252^{\{10\}}$	$0.4352^{\{2\}}$	$0.46715^{\{3\}}$
		$\hat{k}$	$0.63208^{\{8\}}$	$0.57886^{\{5\}}$	$0.63153^{\{7\}}$	$0.46058^{\{1\}}$	$0.61174^{\{6\}}$	$0.64514^{\{9\}}$	$0.57182^{\{4\}}$	$0.74752^{\{10\}}$	$0.50133^{\{2\}}$	$0.53809^{\{3\}}$
		$\hat{\alpha}$	$0.22174^{\{7\}}$	$0.19692^{\{2\}}$	$0.22551^{\{9\}}$	$0.17187^{\{1\}}$	$0.22225^{\{8\}}$	$0.21232^{\{6\}}$	$0.20433^{\{4\}}$	$0.31608^{\{10\}}$	$0.20534^{\{5\}}$	$0.20053^{\{3\}}$
	MSE	$\hat{\theta}$	$0.46395^{\{9\}}$	$0.37465^{\{4\}}$	$0.45241^{\{8\}}$	$0.29609^{\{1\}}$	$0.42852^{\{6\}}$	$0.43721^{\{7\}}$	$0.38307^{\{5\}}$	$0.64952^{\{10\}}$	$0.32656^{\{2\}}$	$0.34592^{\{3\}}$
		$\hat{k}$	$0.63457^{\{9\}}$	$0.52214^{\{4\}}$	$0.6016^{\{7\}}$	$0.38396^{\{1\}}$	$0.58275^{\{6\}}$	$0.6071^{\{8\}}$	$0.53386^{\{5\}}$	$0.90545^{\{10\}}$	$0.45573^{\{2\}}$	$0.46945^{\{3\}}$
		$\hat{\alpha}$	$0.09438^{\{9\}}$	$0.06791^{\{4\}}$	$0.08849^{\{8\}}$	$0.05103^{\{1\}}$	$0.08323^{\{7\}}$	$0.07571^{\{6\}}$	$0.07337^{\{5\}}$	$0.24015^{\{10\}}$	$0.0678^{\{3\}}$	$0.06552^{\{2\}}$
	MRE	$\hat{\theta}$	$0.21701^{\{7\}}$	$0.19713^{\{5\}}$	$0.22007^{\{9\}}$	$0.16557^{\{1\}}$	$0.21163^{\{6\}}$	$0.21859^{\{8\}}$	$0.19646^{\{4\}}$	$0.25701^{\{10\}}$	$0.17408^{\{2\}}$	$0.18686^{\{3\}}$
		$\hat{k}$	$0.42139^{\{8\}}$	$0.38591^{\{5\}}$	$0.42102^{\{7\}}$	$0.30705^{\{1\}}$	$0.40783^{\{6\}}$	$0.43009^{\{9\}}$	$0.38121^{\{4\}}$	$0.49835^{\{10\}}$	$0.33422^{\{2\}}$	$0.35873^{\{3\}}$
		$\hat{\alpha}$	$0.08869^{\{7\}}$	$0.07877^{\{2\}}$	$0.09021^{\{9\}}$	$0.06875^{\{1\}}$	$0.0889^{\{8\}}$	$0.08493^{\{6\}}$	$0.08173^{\{4\}}$	$0.12643^{\{10\}}$	$0.08214^{\{5\}}$	$0.08021^{\{3\}}$
	$\sum Ranks$		$71^{\{8\}}$	$36^{\{4\}}$	$73^{\{9\}}$	$9^{\{1\}}$	$59^{\{6\}}$	$67^{\{7\}}$	$39^{\{5\}}$	$90^{\{10\}}$	$25^{\{2\}}$	$26^{\{3\}}$
250	BIAS	$\hat{\theta}$	$0.41302^{\{6\}}$	$0.40123^{\{4\}}$	$0.45982^{\{9\}}$	$0.32337^{\{1\}}$	$0.45279^{\{7\}}$	$0.4582^{\{8\}}$	$0.40341^{\{5\}}$	$0.51113^{\{10\}}$	$0.37113^{\{2\}}$	$0.39843^{\{3\}}$
		$\hat{k}$	$0.46939^{\{6\}}$	$0.45903^{\{5\}}$	$0.5231^{\{8\}}$	$0.35651^{\{1\}}$	$0.51411^{\{7\}}$	$0.53984^{\{9\}}$	$0.45518^{\{4\}}$	$0.58743^{\{10\}}$	$0.42006^{\{2\}}$	$0.45481^{\{3\}}$
		$\hat{\alpha}$	$0.15601^{\{4\}}$	$0.15098^{\{3\}}$	$0.17886^{\{9\}}$	$0.13145^{\{1\}}$	$0.17145^{\{7\}}$	$0.17213^{\{8\}}$	$0.16157^{\{6\}}$	$0.22492^{\{10\}}$	$0.15644^{\{5\}}$	$0.15063^{\{2\}}$
	MSE	$\hat{\theta}$	$0.28525^{\{6\}}$	$0.25951^{\{4\}}$	$0.33734^{\{9\}}$	$0.19344^{\{1\}}$	$0.31345^{\{7\}}$	$0.32222^{\{8\}}$	$0.26552^{\{5\}}$	$0.45129^{\{10\}}$	$0.2515^{\{2\}}$	$0.25326^{\{3\}}$
		$\hat{k}$	$0.37273^{\{6\}}$	$0.34124^{\{3\}}$	$0.44265^{\{9\}}$	$0.24228^{\{1\}}$	$0.40186^{\{7\}}$	$0.43607^{\{8\}}$	$0.34278^{\{5\}}$	$0.61226^{\{10\}}$	$0.34227^{\{4\}}$	$0.33478^{\{2\}}$
		$\hat{\alpha}$	$0.05025^{\{6\}}$	$0.04182^{\{3\}}$	$0.055^{\{9\}}$	$0.0292^{\{1\}}$	$0.05094^{\{7\}}$	$0.0512^{\{8\}}$	$0.04703^{\{5\}}$	$0.12227^{\{10\}}$	$0.04257^{\{4\}}$	$0.03826^{\{2\}}$
	MRE	$\hat{\theta}$	$0.16521^{\{6\}}$	$0.16049^{\{4\}}$	$0.18393^{\{9\}}$	$0.12935^{\{1\}}$	$0.18112^{\{7\}}$	$0.18328^{\{8\}}$	$0.16137^{\{5\}}$	$0.20445^{\{10\}}$	$0.14845^{\{2\}}$	$0.15937^{\{3\}}$
		$\hat{k}$	$0.31292^{\{6\}}$	$0.30602^{\{5\}}$	$0.34873^{\{8\}}$	$0.23768^{\{1\}}$	$0.34274^{\{7\}}$	$0.35989^{\{9\}}$	$0.30345^{\{4\}}$	$0.39162^{\{10\}}$	$0.28004^{\{2\}}$	$0.30321^{\{3\}}$
		$\hat{\alpha}$	$0.0624^{\{4\}}$	$0.06039^{\{3\}}$	$0.07154^{\{9\}}$	$0.05258^{\{1\}}$	$0.06858^{\{7\}}$	$0.06885^{\{8\}}$	$0.06463^{\{6\}}$	$0.08997^{\{10\}}$	$0.06258^{\{5\}}$	$0.06025^{\{2\}}$
	$\sum Ranks$		$50^{\{6\}}$	$34^{\{4\}}$	$79^{\{9\}}$	$9^{\{1\}}$	$63^{\{7\}}$	$74^{\{8\}}$	$45^{\{5\}}$	$90^{\{10\}}$	$28^{\{3\}}$	$23^{\{2\}}$

Table 11: Numerical values of the PGD simulation for $\theta = 0.9$, $k = 0.75$, and $\alpha = 1.5$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.58439^{\{3\}}$	$0.5961^{\{4\}}$	$0.66069^{\{9\}}$	$0.63149^{\{7\}}$	$0.71647^{\{10\}}$	$0.62649^{\{6\}}$	$0.65894^{\{8\}}$	$0.61168^{\{5\}}$	$0.42206^{\{1\}}$	$0.53852^{\{2\}}$
		$\hat{k}$	$1.01641^{\{6\}}$	$0.96645^{\{4\}}$	$1.08116^{\{9\}}$	$0.96451^{\{3\}}$	$1.11592^{\{10\}}$	$1.01073^{\{5\}}$	$1.0639^{\{8\}}$	$1.05118^{\{7\}}$	$0.57634^{\{1\}}$	$0.76996^{\{2\}}$
		$\hat{\alpha}$	$0.22655^{\{2\}}$	$0.22225^{\{1\}}$	$0.25631^{\{9\}}$	$0.24311^{\{6\}}$	$0.25261^{\{8\}}$	$0.2276^{\{3\}}$	$0.24665^{\{7\}}$	$0.32263^{\{10\}}$	$0.23531^{\{5\}}$	$0.23306^{\{4\}}$
	MSE	$\hat{\theta}$	$0.71287^{\{2\}}$	$0.7287^{\{4\}}$	$0.91571^{\{8\}}$	$0.83496^{\{6\}}$	$1.01676^{\{10\}}$	$0.91105^{\{7\}}$	$0.92174^{\{9\}}$	$0.77534^{\{5\}}$	$0.53045^{\{1\}}$	$0.72236^{\{3\}}$
		$\hat{k}$	$1.71554^{\{4\}}$	$1.62462^{\{3\}}$	$2.16201^{\{9\}}$	$1.81706^{\{6\}}$	$2.22168^{\{10\}}$	$2.05349^{\{7\}}$	$2.14458^{\{8\}}$	$1.79919^{\{5\}}$	$1.16974^{\{1\}}$	$1.5625^{\{2\}}$
		$\hat{\alpha}$	$0.08408^{\{4\}}$	$0.07781^{\{1\}}$	$0.10529^{\{9\}}$	$0.08435^{\{5\}}$	$0.09623^{\{8\}}$	$0.08233^{\{3\}}$	$0.09258^{\{7\}}$	$0.18482^{\{10\}}$	$0.0876^{\{6\}}$	$0.082^{\{2\}}$
	MRE	$\hat{\theta}$	$0.64932^{\{3\}}$	$0.66233^{\{4\}}$	$0.7341^{\{9\}}$	$0.70165^{\{7\}}$	$0.79607^{\{10\}}$	$0.6961^{\{6\}}$	$0.73216^{\{8\}}$	$0.67964^{\{5\}}$	$0.46896^{\{1\}}$	$0.59835^{\{2\}}$
		$\hat{k}$	$1.35521^{\{6\}}$	$1.28861^{\{4\}}$	$1.44155^{\{9\}}$	$1.28601^{\{3\}}$	$1.48789^{\{10\}}$	$1.34764^{\{5\}}$	$1.41853^{\{8\}}$	$1.40158^{\{7\}}$	$0.76845^{\{1\}}$	$1.02662^{\{2\}}$
		$\hat{\alpha}$	$0.15103^{\{2\}}$	$0.14817^{\{1\}}$	$0.17087^{\{9\}}$	$0.16207^{\{6\}}$	$0.16841^{\{8\}}$	$0.15173^{\{3\}}$	$0.16443^{\{7\}}$	$0.21509^{\{10\}}$	$0.15687^{\{5\}}$	$0.15538^{\{4\}}$
	$\sum Ranks$		$32^{\{4\}}$	$26^{\{3\}}$	$80^{\{9\}}$	$49^{\{6\}}$	$84^{\{10\}}$	$45^{\{5\}}$	$70^{\{8\}}$	$64^{\{7\}}$	$22^{\{1\}}$	$23^{\{2\}}$
70	BIAS	$\hat{\theta}$	$0.40185^{\{3\}}$	$0.40082^{\{2\}}$	$0.45616^{\{7\}}$	$0.42308^{\{5\}}$	$0.49656^{\{10\}}$	$0.49304^{\{9\}}$	$0.42716^{\{6\}}$	$0.4811^{\{8\}}$	$0.304^{\{1\}}$	$0.40193^{\{4\}}$
		$\hat{k}$	$0.73063^{\{5\}}$	$0.69854^{\{4\}}$	$0.79882^{\{7\}}$	$0.66646^{\{3\}}$	$0.82215^{\{9\}}$	$0.80953^{\{8\}}$	$0.7346^{\{6\}}$	$0.8538^{\{10\}}$	$0.44242^{\{1\}}$	$0.62368^{\{2\}}$
		$\hat{\alpha}$	$0.16315^{\{1\}}$	$0.16327^{\{2\}}$	$0.17466^{\{7\}}$	$0.16697^{\{4\}}$	$0.18543^{\{9\}}$	$0.17702^{\{8\}}$	$0.17361^{\{6\}}$	$0.22623^{\{10\}}$	$0.16658^{\{3\}}$	$0.17267^{\{5\}}$
	MSE	$\hat{\theta}$	$0.30147^{\{2\}}$	$0.30346^{\{3\}}$	$0.42333^{\{8\}}$	$0.35751^{\{6\}}$	$0.48861^{\{9\}}$	$0.50832^{\{10\}}$	$0.35298^{\{4\}}$	$0.41798^{\{7\}}$	$0.25256^{\{1\}}$	$0.35366^{\{5\}}$
		$\hat{k}$	$0.77348^{\{3\}}$	$0.73315^{\{2\}}$	$1.02302^{\{7\}}$	$0.79956^{\{5\}}$	$1.09177^{\{9\}}$	$1.11616^{\{10\}}$	$0.83668^{\{6\}}$	$1.05331^{\{8\}}$	$0.56882^{\{1\}}$	$0.78444^{\{4\}}$
		$\hat{\alpha}$	$0.03995^{\{1\}}$	$0.04058^{\{2\}}$	$0.04781^{\{8\}}$	$0.04131^{\{3\}}$	$0.05092^{\{9\}}$	$0.04715^{\{7\}}$	$0.04698^{\{6\}}$	$0.09205^{\{10\}}$	$0.04333^{\{4\}}$	$0.04523^{\{5\}}$
	MRE	$\hat{\theta}$	$0.4465^{\{3\}}$	$0.44536^{\{2\}}$	$0.50684^{\{7\}}$	$0.47009^{\{5\}}$	$0.55173^{\{10\}}$	$0.54782^{\{9\}}$	$0.47462^{\{6\}}$	$0.53456^{\{8\}}$	$0.33778^{\{1\}}$	$0.44659^{\{4\}}$
		$\hat{k}$	$0.97418^{\{5\}}$	$0.93139^{\{4\}}$	$1.0651^{\{7\}}$	$0.88862^{\{3\}}$	$1.0962^{\{9\}}$	$1.07938^{\{8\}}$	$0.97946^{\{6\}}$	$1.1384^{\{10\}}$	$0.58989^{\{1\}}$	$0.83157^{\{2\}}$
		$\hat{\alpha}$	$0.10876^{\{1\}}$	$0.10885^{\{2\}}$	$0.11644^{\{7\}}$	$0.11132^{\{4\}}$	$0.12362^{\{9\}}$	$0.11801^{\{8\}}$	$0.11574^{\{6\}}$	$0.15082^{\{10\}}$	$0.11105^{\{3\}}$	$0.11511^{\{5\}}$
	$\sum Ranks$		$24^{\{3\}}$	$23^{\{2\}}$	$65^{\{7\}}$	$38^{\{5\}}$	$83^{\{10\}}$	$77^{\{8\}}$	$52^{\{6\}}$	$81^{\{9\}}$	$16^{\{1\}}$	$36^{\{4\}}$
100	BIAS	$\hat{\theta}$	$0.37478^{\{6\}}$	$0.36944^{\{4\}}$	$0.41492^{\{8\}}$	$0.36739^{\{3\}}$	$0.44452^{\{10\}}$	$0.42126^{\{9\}}$	$0.37176^{\{5\}}$	$0.40226^{\{7\}}$	$0.28807^{\{1\}}$	$0.35353^{\{2\}}$
		$\hat{k}$	$0.67306^{\{6\}}$	$0.66037^{\{4\}}$	$0.72899^{\{8\}}$	$0.56453^{\{2\}}$	$0.75797^{\{10\}}$	$0.71676^{\{7\}}$	$0.66208^{\{5\}}$	$0.73865^{\{9\}}$	$0.43402^{\{1\}}$	$0.57423^{\{3\}}$
		$\hat{\alpha}$	$0.15132^{\{6\}}$	$0.1414^{\{1\}}$	$0.15734^{\{8\}}$	$0.1484^{\{2\}}$	$0.15529^{\{7\}}$	$0.15897^{\{9\}}$	$0.14905^{\{4\}}$	$0.18566^{\{10\}}$	$0.15104^{\{5\}}$	$0.14843^{\{3\}}$
	MSE	$\hat{\theta}$	$0.24512^{\{2\}}$	$0.25066^{\{3\}}$	$0.33403^{\{8\}}$	$0.26963^{\{5\}}$	$0.37899^{\{10\}}$	$0.34707^{\{9\}}$	$0.26089^{\{4\}}$	$0.2814^{\{7\}}$	$0.22479^{\{1\}}$	$0.27208^{\{6\}}$
		$\hat{k}$	$0.63393^{\{5\}}$	$0.63371^{\{4\}}$	$0.80391^{\{8\}}$	$0.59042^{\{2\}}$	$0.89504^{\{10\}}$	$0.81099^{\{9\}}$	$0.65371^{\{6\}}$	$0.75701^{\{7\}}$	$0.5174^{\{1\}}$	$0.63296^{\{3\}}$
		$\hat{\alpha}$	$0.03443^{\{5\}}$	$0.03097^{\{1\}}$	$0.03851^{\{9\}}$	$0.03298^{\{2\}}$	$0.03682^{\{7\}}$	$0.03797^{\{8\}}$	$0.03402^{\{3\}}$	$0.05964^{\{10\}}$	$0.0351^{\{6\}}$	$0.03421^{\{4\}}$
	MRE	$\hat{\theta}$	$0.41642^{\{6\}}$	$0.41048^{\{4\}}$	$0.46102^{\{8\}}$	$0.40821^{\{3\}}$	$0.49391^{\{10\}}$	$0.46806^{\{9\}}$	$0.41307^{\{5\}}$	$0.44695^{\{7\}}$	$0.32008^{\{1\}}$	$0.39281^{\{2\}}$
		$\hat{k}$	$0.89741^{\{6\}}$	$0.88049^{\{4\}}$	$0.97198^{\{8\}}$	$0.7527^{\{2\}}$	$1.01063^{\{10\}}$	$0.95568^{\{7\}}$	$0.88278^{\{5\}}$	$0.98487^{\{9\}}$	$0.5787^{\{1\}}$	$0.76564^{\{3\}}$
		$\hat{\alpha}$	$0.10088^{\{6\}}$	$0.09427^{\{1\}}$	$0.10489^{\{8\}}$	$0.09894^{\{2\}}$	$0.10353^{\{7\}}$	$0.10598^{\{9\}}$	$0.09936^{\{4\}}$	$0.12377^{\{10\}}$	$0.10069^{\{5\}}$	$0.09895^{\{3\}}$
	$\sum Ranks$		$48^{\{6\}}$	$26^{\{3\}}$	$73^{\{7\}}$	$23^{\{2\}}$	$81^{\{10\}}$	$76^{\{8.5\}}$	$41^{\{5\}}$	$76^{\{8.5\}}$	$22^{\{1\}}$	$29^{\{4\}}$
150	BIAS	$\hat{\theta}$	$0.31596^{\{3\}}$	$0.31784^{\{4\}}$	$0.37434^{\{10\}}$	$0.27316^{\{2\}}$	$0.36418^{\{9\}}$	$0.36411^{\{8\}}$	$0.32498^{\{6\}}$	$0.34173^{\{7\}}$	$0.26146^{\{1\}}$	$0.32309^{\{5\}}$
		$\hat{k}$	$0.58953^{\{6\}}$	$0.57809^{\{4\}}$	$0.66676^{\{10\}}$	$0.43899^{\{2\}}$	$0.64391^{\{7\}}$	$0.64798^{\{8\}}$	$0.5881^{\{5\}}$	$0.66571^{\{9\}}$	$0.40163^{\{1\}}$	$0.52433^{\{3\}}$
		$\hat{\alpha}$	$0.12816^{\{3\}}$	$0.1248^{\{2\}}$	$0.13865^{\{9\}}$	$0.11572^{\{1\}}$	$0.13496^{\{8\}}$	$0.12919^{\{5\}}$	$0.12865^{\{4\}}$	$0.1585^{\{10\}}$	$0.13037^{\{6\}}$	$0.13255^{\{7\}}$
	MSE	$\hat{\theta}$	$0.15959^{\{2\}}$	$0.17716^{\{4\}}$	$0.25053^{\{10\}}$	$0.15171^{\{1\}}$	$0.23734^{\{8\}}$	$0.24609^{\{9\}}$	$0.18182^{\{5\}}$	$0.19232^{\{6\}}$	$0.17101^{\{3\}}$	$0.21419^{\{7\}}$
		$\hat{k}$	$0.46852^{\{3\}}$	$0.47401^{\{4\}}$	$0.62542^{\{10\}}$	$0.36723^{\{1\}}$	$0.58829^{\{7\}}$	$0.62334^{\{9\}}$	$0.4854^{\{5\}}$	$0.59746^{\{8\}}$	$0.38535^{\{2\}}$	$0.50011^{\{6\}}$
		$\hat{\alpha}$	$0.02424^{\{3\}}$	$0.02362^{\{2\}}$	$0.0296^{\{9\}}$	$0.02055^{\{1\}}$	$0.02696^{\{8\}}$	$0.02585^{\{5\}}$	$0.02514^{\{4\}}$	$0.0434^{\{10\}}$	$0.02621^{\{6\}}$	$0.02683^{\{7\}}$
	MRE	$\hat{\theta}$	$0.35107^{\{3\}}$	$0.35316^{\{4\}}$	$0.41593^{\{10\}}$	$0.30351^{\{2\}}$	$0.40465^{\{9\}}$	$0.40456^{\{8\}}$	$0.36109^{\{6\}}$	$0.3797^{\{7\}}$	$0.29051^{\{1\}}$	$0.35899^{\{5\}}$
		$\hat{k}$	$0.78603^{\{6\}}$	$0.77079^{\{4\}}$	$0.88901^{\{10\}}$	$0.58532^{\{2\}}$	$0.85854^{\{7\}}$	$0.86398^{\{8\}}$	$0.78413^{\{5\}}$	$0.88761^{\{9\}}$	$0.53551^{\{1\}}$	$0.69911^{\{3\}}$
		$\hat{\alpha}$	$0.08544^{\{3\}}$	$0.0832^{\{2\}}$	$0.09244^{\{9\}}$	$0.07715^{\{1\}}$	$0.08997^{\{8\}}$	$0.08613^{\{5\}}$	$0.08577^{\{4\}}$	$0.10567^{\{10\}}$	$0.08691^{\{6\}}$	$0.08836^{\{7\}}$
	$\sum Ranks$		$32^{\{4\}}$	$30^{\{3\}}$	$87^{\{10\}}$	$13^{\{1\}}$	$71^{\{8\}}$	$65^{\{7\}}$	$44^{\{5\}}$	$76^{\{9\}}$	$27^{\{2\}}$	$50^{\{6\}}$
250	BIAS	$\hat{\theta}$	$0.2643^{\{4\}}$	$0.26031^{\{3\}}$	$0.3078^{\{9\}}$	$0.22152^{\{2\}}$	$0.30862^{\{10\}}$	$0.2939^{\{7\}}$	$0.28233^{\{6\}}$	$0.29842^{\{8\}}$	$0.21444^{\{1\}}$	$0.268^{\{5\}}$
		$\hat{k}$	$0.51495^{\{5\}}$	$0.49119^{\{4\}}$	$0.56836^{\{8\}}$	$0.35625^{\{2\}}$	$0.57362^{\{9\}}$	$0.5442^{\{7\}}$	$0.52328^{\{6\}}$	$0.57636^{\{10\}}$	$0.34268^{\{1\}}$	$0.45263^{\{3\}}$
		$\hat{\alpha}$	$0.10507^{\{4\}}$	$0.10514^{\{5\}}$	$0.11647^{\{8\}}$	$0.09687^{\{1\}}$	$0.11661^{\{9\}}$	$0.1111^{\{7\}}$	$0.11102^{\{6\}}$	$0.1335^{\{10\}}$	$0.0984^{\{2\}}$	$0.10456^{\{3\}}$
	MSE	$\hat{\theta}$	$0.10719^{\{2\}}$	$0.10751^{\{3\}}$	$0.15387^{\{9\}}$	$0.10997^{\{4\}}$	$0.15866^{\{10\}}$	$0.14563^{\{8\}}$	$0.12962^{\{5\}}$	$0.13995^{\{7\}}$	$0.10371^{\{1\}}$	$0.13551^{\{6\}}$
		$\hat{k}$	$0.35108^{\{5\}}$	$0.32697^{\{3\}}$	$0.4329^{\{8\}}$	$0.27488^{\{2\}}$	$0.44497^{\{9\}}$	$0.4058^{\{7\}}$	$0.37426^{\{6\}}$	$0.45161^{\{10\}}$	$0.25564^{\{1\}}$	$0.3379^{\{4\}}$
		$\hat{\alpha}$	$0.01606^{\{3\}}$	$0.01639^{\{4\}}$	$0.02005^{\{9\}}$	$0.01439^{\{1\}}$	$0.01969^{\{8\}}$	$0.01833^{\{6\}}$	$0.01841^{\{7\}}$	$0.0288^{\{10\}}$	$0.01576^{\{2\}}$	$0.01724^{\{5\}}$
	MRE	$\hat{\theta}$	$0.29367^{\{4\}}$	$0.28924^{\{3\}}$	$0.342^{\{9\}}$	$0.24613^{\{2\}}$	$0.34291^{\{10\}}$	$0.32656^{\{7\}}$	$0.3137^{\{6\}}$	$0.33158^{\{8\}}$	$0.23827^{\{1\}}$	$0.29778^{\{5\}}$
		$\hat{k}$	$0.6866^{\{5\}}$	$0.65492^{\{4\}}$	$0.75782^{\{8\}}$	$0.475^{\{2\}}$	$0.76483^{\{9\}}$	$0.7256^{\{7\}}$	$0.69771^{\{6\}}$	$0.76847^{\{10\}}$	$0.45691^{\{1\}}$	$0.6035^{\{3\}}$
		$\hat{\alpha}$	$0.07005^{\{4\}}$	$0.07009^{\{5\}}$	$0.07765^{\{8\}}$	$0.06458^{\{1\}}$	$0.07774^{\{9\}}$	$0.07407^{\{7\}}$	$0.07401^{\{6\}}$	$0.089^{\{10\}}$	$0.0656^{\{2\}}$	$0.0697^{\{3\}}$
	$\sum Ranks$		$36^{\{4\}}$	$34^{\{3\}}$	$76^{\{8\}}$	$17^{\{2\}}$	$83^{\{9.5\}}$	$63^{\{7\}}$	$54^{\{6\}}$	$83^{\{9.5\}}$	$12^{\{1\}}$	$37^{\{5\}}$

Table 12: Numerical values of the PGD simulation for $\theta = 1.5$, $k = 0.15$, and $\alpha = 0.9$.

n	Mea.	$\widehat{Est.}$	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
30	BIAS	$\hat{\theta}$	$0.33765^{\{3\}}$	$0.42926^{\{5\}}$	$0.47714^{\{7\}}$	$0.35674^{\{4\}}$	$0.47474^{\{6\}}$	$0.48237^{\{8\}}$	$0.48326^{\{9\}}$	$0.6893^{\{10\}}$	$0.21973^{\{1\}}$	$0.31049^{\{2\}}$
		$\hat{k}$	$0.263^{\{2\}}$	$0.40435^{\{6\}}$	$0.40122^{\{5\}}$	$0.3725^{\{4\}}$	$0.45401^{\{8\}}$	$0.4425^{\{7\}}$	$0.45957^{\{9\}}$	$0.61184^{\{10\}}$	$0.18532^{\{1\}}$	$0.30064^{\{3\}}$
		$\hat{\alpha}$	$0.12046^{\{2\}}$	$0.11544^{\{1\}}$	$0.13143^{\{8\}}$	$0.12612^{\{5\}}$	$0.13041^{\{7\}}$	$0.13018^{\{6\}}$	$0.12415^{\{3\}}$	$0.38283^{\{10\}}$	$0.13638^{\{9\}}$	$0.12437^{\{4\}}$
	MSE	$\hat{\theta}$	$0.25463^{\{2\}}$	$0.39536^{\{5\}}$	$0.52462^{\{8\}}$	$0.29585^{\{4\}}$	$0.52913^{\{9\}}$	$0.50583^{\{6\}}$	$0.51172^{\{7\}}$	$0.98991^{\{10\}}$	$0.13554^{\{1\}}$	$0.25521^{\{3\}}$
		$\hat{k}$	$0.32529^{\{2\}}$	$0.52018^{\{5\}}$	$0.6221^{\{6\}}$	$0.43563^{\{4\}}$	$0.7326^{\{9\}}$	$0.6308^{\{7\}}$	$0.65601^{\{8\}}$	$1.1982^{\{10\}}$	$0.15812^{\{1\}}$	$0.36002^{\{3\}}$
		$\hat{\alpha}$	$0.02576^{\{5\}}$	$0.02124^{\{1\}}$	$0.03024^{\{9\}}$	$0.0237^{\{2\}}$	$0.02704^{\{6\}}$	$0.0273^{\{7\}}$	$0.02469^{\{4\}}$	$0.73584^{\{10\}}$	$0.03011^{\{8\}}$	$0.02425^{\{3\}}$
	MRE	$\hat{\theta}$	$0.2251^{\{3\}}$	$0.28618^{\{5\}}$	$0.3181^{\{7\}}$	$0.23783^{\{4\}}$	$0.3165^{\{6\}}$	$0.32158^{\{8\}}$	$0.32218^{\{9\}}$	$0.45954^{\{10\}}$	$0.14649^{\{1\}}$	$0.20699^{\{2\}}$
		$\hat{k}$	$1.75332^{\{2\}}$	$2.69567^{\{6\}}$	$2.67481^{\{5\}}$	$2.48336^{\{4\}}$	$3.02676^{\{8\}}$	$2.95^{\{7\}}$	$3.0638^{\{9\}}$	$4.07896^{\{10\}}$	$1.23544^{\{1\}}$	$2.00425^{\{3\}}$
		$\hat{\alpha}$	$0.13385^{\{2\}}$	$0.12827^{\{1\}}$	$0.14604^{\{8\}}$	$0.14014^{\{5\}}$	$0.14489^{\{7\}}$	$0.14464^{\{6\}}$	$0.13795^{\{3\}}$	$0.42537^{\{10\}}$	$0.15153^{\{9\}}$	$0.13819^{\{4\}}$
	$\sum Ranks$		$23^{\{1\}}$	$35^{\{4\}}$	$63^{\{8\}}$	$36^{\{5\}}$	$66^{\{9\}}$	$62^{\{7\}}$	$61^{\{6\}}$	$90^{\{10\}}$	$32^{\{3\}}$	$27^{\{2\}}$
70	BIAS	$\hat{\theta}$	$0.24514^{\{2\}}$	$0.31942^{\{5\}}$	$0.33095^{\{6\}}$	$0.25261^{\{3\}}$	$0.34618^{\{7\}}$	$0.34789^{\{8\}}$	$0.34972^{\{9\}}$	$0.50612^{\{10\}}$	$0.17794^{\{1\}}$	$0.26644^{\{4\}}$
		$\hat{k}$	$0.21105^{\{2\}}$	$0.29149^{\{6\}}$	$0.28596^{\{5\}}$	$0.26655^{\{3\}}$	$0.31514^{\{7\}}$	$0.32202^{\{8\}}$	$0.33411^{\{9\}}$	$0.46841^{\{10\}}$	$0.16688^{\{1\}}$	$0.26914^{\{4\}}$
		$\hat{\alpha}$	$0.08224^{\{2\}}$	$0.07968^{\{1\}}$	$0.08946^{\{5\}}$	$0.08809^{\{4\}}$	$0.09096^{\{6\}}$	$0.09869^{\{9\}}$	$0.08568^{\{3\}}$	$0.209^{\{10\}}$	$0.09324^{\{8\}}$	$0.09113^{\{7\}}$
	MSE	$\hat{\theta}$	$0.12359^{\{2\}}$	$0.20424^{\{5\}}$	$0.22686^{\{6\}}$	$0.15537^{\{3\}}$	$0.24384^{\{8\}}$	$0.23505^{\{7\}}$	$0.24767^{\{9\}}$	$0.48191^{\{10\}}$	$0.09105^{\{1\}}$	$0.17607^{\{4\}}$
		$\hat{k}$	$0.11793^{\{2\}}$	$0.20897^{\{4\}}$	$0.21906^{\{7\}}$	$0.18887^{\{3\}}$	$0.255^{\{8\}}$	$0.21251^{\{6\}}$	$0.28346^{\{9\}}$	$0.48684^{\{10\}}$	$0.10379^{\{1\}}$	$0.21202^{\{5\}}$
		$\hat{\alpha}$	$0.01091^{\{2\}}$	$0.00997^{\{1\}}$	$0.01282^{\{6\}}$	$0.01175^{\{4\}}$	$0.01264^{\{5\}}$	$0.01457^{\{9\}}$	$0.01132^{\{3\}}$	$0.12138^{\{10\}}$	$0.0134^{\{8\}}$	$0.01329^{\{7\}}$
	MRE	$\hat{\theta}$	$0.16343^{\{2\}}$	$0.21295^{\{5\}}$	$0.22064^{\{6\}}$	$0.1684^{\{3\}}$	$0.23078^{\{7\}}$	$0.23193^{\{8\}}$	$0.23315^{\{9\}}$	$0.33742^{\{10\}}$	$0.11863^{\{1\}}$	$0.17763^{\{4\}}$
		$\hat{k}$	$1.40698^{\{2\}}$	$1.9433^{\{6\}}$	$1.90643^{\{5\}}$	$1.777^{\{3\}}$	$2.10091^{\{7\}}$	$2.14678^{\{8\}}$	$2.22739^{\{9\}}$	$3.12271^{\{10\}}$	$1.11254^{\{1\}}$	$1.79429^{\{4\}}$
		$\hat{\alpha}$	$0.09138^{\{2\}}$	$0.08854^{\{1\}}$	$0.0994^{\{5\}}$	$0.09787^{\{4\}}$	$0.10106^{\{6\}}$	$0.10965^{\{9\}}$	$0.0952^{\{3\}}$	$0.23223^{\{10\}}$	$0.1036^{\{8\}}$	$0.10125^{\{7\}}$
	$\sum Ranks$		$18^{\{1\}}$	$34^{\{4\}}$	$51^{\{6\}}$	$30^{\{2.5\}}$	$61^{\{7\}}$	$72^{\{9\}}$	$63^{\{8\}}$	$90^{\{10\}}$	$30^{\{2.5\}}$	$46^{\{5\}}$
100	BIAS	$\hat{\theta}$	$0.22343^{\{3\}}$	$0.27286^{\{5\}}$	$0.28746^{\{6\}}$	$0.20584^{\{2\}}$	$0.30929^{\{8\}}$	$0.33422^{\{9\}}$	$0.30172^{\{7\}}$	$0.43374^{\{10\}}$	$0.15952^{\{1\}}$	$0.26197^{\{4\}}$
		$\hat{k}$	$0.20198^{\{2\}}$	$0.25513^{\{4\}}$	$0.2576^{\{5\}}$	$0.20982^{\{3\}}$	$0.29298^{\{8\}}$	$0.31101^{\{9\}}$	$0.28396^{\{7\}}$	$0.40914^{\{10\}}$	$0.15395^{\{1\}}$	$0.2755^{\{6\}}$
		$\hat{\alpha}$	$0.07036^{\{2\}}$	$0.06885^{\{1\}}$	$0.07773^{\{5\}}$	$0.07225^{\{3\}}$	$0.07948^{\{7\}}$	$0.08792^{\{9\}}$	$0.07482^{\{4\}}$	$0.17341^{\{10\}}$	$0.07923^{\{6\}}$	$0.08294^{\{8\}}$
	MSE	$\hat{\theta}$	$0.10766^{\{3\}}$	$0.14874^{\{4\}}$	$0.16344^{\{5\}}$	$0.10718^{\{2\}}$	$0.19199^{\{8\}}$	$0.20495^{\{9\}}$	$0.17889^{\{7\}}$	$0.34977^{\{10\}}$	$0.07153^{\{1\}}$	$0.17411^{\{6\}}$
		$\hat{k}$	$0.1021^{\{2\}}$	$0.14427^{\{4\}}$	$0.15506^{\{5\}}$	$0.12147^{\{3\}}$	$0.21242^{\{9\}}$	$0.17622^{\{6\}}$	$0.17933^{\{7\}}$	$0.34386^{\{10\}}$	$0.07684^{\{1\}}$	$0.21069^{\{8\}}$
		$\hat{\alpha}$	$0.00796^{\{2\}}$	$0.00756^{\{1\}}$	$0.00925^{\{5\}}$	$0.0083^{\{3\}}$	$0.00963^{\{6\}}$	$0.01135^{\{9\}}$	$0.00858^{\{4\}}$	$0.07348^{\{10\}}$	$0.01008^{\{7\}}$	$0.0109^{\{8\}}$
	MRE	$\hat{\theta}$	$0.14895^{\{3\}}$	$0.18191^{\{5\}}$	$0.19164^{\{6\}}$	$0.13722^{\{2\}}$	$0.20619^{\{8\}}$	$0.22281^{\{9\}}$	$0.20115^{\{7\}}$	$0.28916^{\{10\}}$	$0.10635^{\{1\}}$	$0.17465^{\{4\}}$
		$\hat{k}$	$1.34655^{\{2\}}$	$1.70085^{\{4\}}$	$1.71736^{\{5\}}$	$1.3988^{\{3\}}$	$1.95319^{\{8\}}$	$2.07341^{\{9\}}$	$1.89304^{\{7\}}$	$2.72763^{\{10\}}$	$1.02633^{\{1\}}$	$1.83664^{\{6\}}$
		$\hat{\alpha}$	$0.07818^{\{2\}}$	$0.0765^{\{1\}}$	$0.08637^{\{5\}}$	$0.08028^{\{3\}}$	$0.08831^{\{7\}}$	$0.09769^{\{9\}}$	$0.08313^{\{4\}}$	$0.19268^{\{10\}}$	$0.08804^{\{6\}}$	$0.09215^{\{8\}}$
	$\sum Ranks$		$21^{\{1\}}$	$29^{\{4\}}$	$47^{\{5\}}$	$24^{\{2\}}$	$69^{\{8\}}$	$78^{\{9\}}$	$54^{\{6\}}$	$90^{\{10\}}$	$25^{\{3\}}$	$58^{\{7\}}$
150	BIAS	$\hat{\theta}$	$0.1926^{\{3\}}$	$0.24545^{\{5\}}$	$0.26674^{\{7\}}$	$0.17074^{\{2\}}$	$0.27777^{\{8\}}$	$0.30574^{\{9\}}$	$0.2634^{\{6\}}$	$0.39494^{\{10\}}$	$0.14301^{\{1\}}$	$0.24086^{\{4\}}$
		$\hat{k}$	$0.18053^{\{3\}}$	$0.22984^{\{4\}}$	$0.24629^{\{5\}}$	$0.17349^{\{2\}}$	$0.25923^{\{8\}}$	$0.28904^{\{9\}}$	$0.24982^{\{6\}}$	$0.37815^{\{10\}}$	$0.14414^{\{1\}}$	$0.25187^{\{7\}}$
		$\hat{\alpha}$	$0.06185^{\{3\}}$	$0.06153^{\{1\}}$	$0.06797^{\{5\}}$	$0.06159^{\{2\}}$	$0.07049^{\{7\}}$	$0.08092^{\{9\}}$	$0.06677^{\{4\}}$	$0.14685^{\{10\}}$	$0.06834^{\{6\}}$	$0.07151^{\{8\}}$
	MSE	$\hat{\theta}$	$0.07644^{\{2\}}$	$0.11513^{\{4\}}$	$0.13828^{\{6\}}$	$0.07959^{\{3\}}$	$0.14573^{\{7\}}$	$0.16755^{\{9\}}$	$0.12961^{\{5\}}$	$0.29423^{\{10\}}$	$0.05726^{\{1\}}$	$0.14576^{\{8\}}$
		$\hat{k}$	$0.07042^{\{2\}}$	$0.10249^{\{4\}}$	$0.12177^{\{5\}}$	$0.08406^{\{3\}}$	$0.13916^{\{7\}}$	$0.14351^{\{8\}}$	$0.12394^{\{6\}}$	$0.29916^{\{10\}}$	$0.0619^{\{1\}}$	$0.16941^{\{9\}}$
		$\hat{\alpha}$	$0.00608^{\{1\}}$	$0.0061^{\{2\}}$	$0.00714^{\{5\}}$	$0.00618^{\{3\}}$	$0.00754^{\{7\}}$	$0.00939^{\{9\}}$	$0.00675^{\{4\}}$	$0.04876^{\{10\}}$	$0.00734^{\{6\}}$	$0.00831^{\{8\}}$
	MRE	$\hat{\theta}$	$0.1284^{\{3\}}$	$0.16363^{\{5\}}$	$0.17782^{\{7\}}$	$0.11383^{\{2\}}$	$0.18518^{\{8\}}$	$0.20383^{\{9\}}$	$0.1756^{\{6\}}$	$0.26329^{\{10\}}$	$0.09534^{\{1\}}$	$0.16057^{\{4\}}$
		$\hat{k}$	$1.20351^{\{3\}}$	$1.53228^{\{4\}}$	$1.64191^{\{5\}}$	$1.15658^{\{2\}}$	$1.72822^{\{8\}}$	$1.92692^{\{9\}}$	$1.66548^{\{6\}}$	$2.52098^{\{10\}}$	$0.96092^{\{1\}}$	$1.67912^{\{7\}}$
		$\hat{\alpha}$	$0.06872^{\{3\}}$	$0.06837^{\{1\}}$	$0.07552^{\{5\}}$	$0.06843^{\{2\}}$	$0.07832^{\{7\}}$	$0.08991^{\{9\}}$	$0.07419^{\{4\}}$	$0.16317^{\{10\}}$	$0.07594^{\{6\}}$	$0.07946^{\{8\}}$
	$\sum Ranks$		$23^{\{2\}}$	$30^{\{4\}}$	$50^{\{6\}}$	$21^{\{1\}}$	$67^{\{8\}}$	$80^{\{9\}}$	$47^{\{5\}}$	$90^{\{10\}}$	$24^{\{3\}}$	$63^{\{7\}}$
250	BIAS	$\hat{\theta}$	$0.17447^{\{3\}}$	$0.20452^{\{5\}}$	$0.24567^{\{7\}}$	$0.11444^{\{1\}}$	$0.25323^{\{8\}}$	$0.2924^{\{9\}}$	$0.22953^{\{6\}}$	$0.3336^{\{10\}}$	$0.14195^{\{2\}}$	$0.20204^{\{4\}}$
		$\hat{k}$	$0.16822^{\{3\}}$	$0.19586^{\{4\}}$	$0.22789^{\{7\}}$	$0.11398^{\{1\}}$	$0.23698^{\{8\}}$	$0.2811^{\{9\}}$	$0.22167^{\{6\}}$	$0.32115^{\{10\}}$	$0.14187^{\{2\}}$	$0.21089^{\{5\}}$
		$\hat{\alpha}$	$0.05201^{\{2\}}$	$0.05339^{\{3\}}$	$0.06156^{\{7\}}$	$0.04486^{\{1\}}$	$0.06218^{\{8\}}$	$0.07326^{\{9\}}$	$0.05938^{\{6\}}$	$0.11952^{\{10\}}$	$0.05777^{\{5\}}$	$0.05745^{\{4\}}$
	MSE	$\hat{\theta}$	$0.06205^{\{3\}}$	$0.07772^{\{4\}}$	$0.10725^{\{7\}}$	$0.04157^{\{1\}}$	$0.1144^{\{8\}}$	$0.14545^{\{9\}}$	$0.09106^{\{5\}}$	$0.2028^{\{10\}}$	$0.05789^{\{2\}}$	$0.09878^{\{6\}}$
		$\hat{k}$	$0.05799^{\{2\}}$	$0.06797^{\{4\}}$	$0.09051^{\{6\}}$	$0.04061^{\{1\}}$	$0.10076^{\{7\}}$	$0.12891^{\{9\}}$	$0.08194^{\{5\}}$	$0.20189^{\{10\}}$	$0.05831^{\{3\}}$	$0.10689^{\{8\}}$
		$\hat{\alpha}$	$0.00439^{\{2\}}$	$0.00456^{\{3\}}$	$0.00557^{\{6\}}$	$0.00358^{\{1\}}$	$0.00558^{\{7\}}$	$0.00744^{\{9\}}$	$0.00527^{\{4\}}$	$0.02993^{\{10\}}$	$0.00536^{\{5\}}$	$0.00565^{\{8\}}$
	MRE	$\hat{\theta}$	$0.11632^{\{3\}}$	$0.13635^{\{5\}}$	$0.16378^{\{7\}}$	$0.07629^{\{1\}}$	$0.16882^{\{8\}}$	$0.19493^{\{9\}}$	$0.15302^{\{6\}}$	$0.2224^{\{10\}}$	$0.09464^{\{2\}}$	$0.1347^{\{4\}}$
		$\hat{k}$	$1.12144^{\{3\}}$	$1.30574^{\{4\}}$	$1.51924^{\{7\}}$	$0.75985^{\{1\}}$	$1.57989^{\{8\}}$	$1.87398^{\{9\}}$	$1.47782^{\{6\}}$	$2.14098^{\{10\}}$	$0.94582^{\{2\}}$	$1.40596^{\{5\}}$
		$\hat{\alpha}$	$0.05779^{\{2\}}$	$0.05932^{\{3\}}$	$0.0684^{\{7\}}$	$0.04984^{\{1\}}$	$0.06909^{\{8\}}$	$0.0814^{\{9\}}$	$0.06598^{\{6\}}$	$0.1328^{\{10\}}$	$0.06419^{\{5\}}$	$0.06383^{\{4\}}$
	$\sum Ranks$		$23^{\{2\}}$	$35^{\{4\}}$	$61^{\{7\}}$	$9^{\{1\}}$	$70^{\{8\}}$	$81^{\{9\}}$	$50^{\{6\}}$	$90^{\{10\}}$	$28^{\{3\}}$	$48^{\{5\}}$

Table 13: Partial and overall ranks of the techniques for estimation of the PGD using various values of parameters.

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	RTADE	WLSE	LTADE	MSADE	MSALDE
$\theta = 0.25,\ k = 0.5,\ \alpha = 0.75$	30	4.0	5.0	8.0	6.0	10.0	3.0	9.0	7.0	1.0	2.0
	70	3.0	6.0	7.0	4.0	10.0	5.0	9.0	8.0	1.0	2.0
	100	4.0	5.0	7.5	1.0	10.0	6.0	9.0	7.5	2.0	3.0
	150	4.0	5.0	9.0	1.0	10.0	7.0	6.0	8.0	2.0	3.0
	250	4.0	5.0	9.0	1.0	10.0	7.0	6.0	8.0	2.0	3.0
$\theta = 0.2,\ k = 1.2,\ \alpha = 0.9$	30	3.0	6.0	5.0	10.0	7.0	2.0	8.5	8.5	1.0	4.0
	70	2.0	5.0	7.0	8.0	9.0	3.5	6.0	10.0	1.0	3.5
	100	2.0	5.0	8.0	7.0	10.0	4.0	6.0	9.0	1.0	3.0
	150	2.0	6.0	8.5	4.0	10.0	5.0	7.0	8.5	1.0	3.0
	250	1.0	5.0	9.0	4.0	10.0	7.0	6.0	8.0	2.0	3.0
$\theta = 1.5,\ k = 0.25,\ \alpha = 1.5$	30	1.0	4.0	8.0	3.0	9.0	6.0	7.0	10.0	2.0	5.0
	70	3.0	4.0	5.0	2.0	9.0	8.0	6.0	10.0	1.0	7.0
	100	3.0	6.5	4.0	1.5	8.0	6.5	5.0	10.0	1.5	9.0
	150	4.0	5.0	6.0	1.0	8.0	7.0	3.0	10.0	2.0	9.0
	250	3.0	8.0	4.0	1.0	7.0	6.0	5.0	10.0	2.0	9.0
$\theta = 2.0,\ k = 2.5,\ \alpha = 0.5$	30	5.0	4.0	7.5	3.0	9.0	10.0	6.0	7.5	1.0	2.0
	70	5.0	3.0	9.0	2.0	8.0	10.0	6.0	7.0	4.0	1.0
	100	4.0	1.5	8.0	1.5	9.0	10.0	6.0	7.0	5.0	3.0
	150	3.0	2.0	9.0	1.0	8.0	10.0	6.0	7.0	5.0	4.0
	250	4.0	1.0	8.0	2.0	7.0	10.0	6.0	9.0	5.0	3.0
$\theta = 0.5,\ k = 1.5,\ \alpha = 2.5$	30	2.0	7.0	6.0	8.0	10.0	4.0	9.0	5.0	1.0	3.0
	70	2.0	4.0	7.0	9.0	10.0	8.0	6.0	5.0	1.0	3.0
	100	2.0	3.0	8.0	7.0	10.0	9.0	6.0	5.0	1.0	4.0
	150	1.0	6.0	8.0	4.0	9.0	10.0	3.0	5.0	2.0	7.0
	250	1.0	3.0	8.0	5.0	9.0	10.0	4.0	6.0	2.0	7.0
$\theta = 2.5,\ k = 1.5,\ \alpha = 2.5$	30	7.0	4.0	10.0	2.0	5.0	9.0	6.0	8.0	1.0	3.0
	70	6.0	5.0	7.0	1.0	9.0	8.0	4.0	10.0	2.0	3.0
	100	8.0	5.0	9.0	1.0	7.0	6.0	4.0	10.0	2.0	3.0
	150	8.0	4.0	9.0	1.0	6.0	7.0	5.0	10.0	2.0	3.0
	250	6.0	4.0	9.0	1.0	7.0	8.0	5.0	10.0	3.0	2.0
$\theta = 0.9,\ k = 0.75,\ \alpha = 1.5$	30	4.0	3.0	9.0	6.0	10.0	5.0	8.0	7.0	1.0	2.0
	70	3.0	2.0	7.0	5.0	10.0	8.0	6.0	9.0	1.0	4.0
	100	6.0	3.0	7.0	2.0	10.0	8.5	5.0	8.5	1.0	4.0
	150	4.0	3.0	10.0	1.0	8.0	7.0	5.0	9.0	2.0	6.0
	250	4.0	3.0	8.0	2.0	9.5	7.0	6.0	9.5	1.0	5.0
$\theta = 1.5,\ k = 0.15,\ \alpha = 0.9$	30	1.0	4.0	8.0	5.0	9.0	7.0	6.0	10.0	3.0	2.0
	70	1.0	4.0	6.0	2.5	7.0	9.0	8.0	10.0	2.5	5.0
	100	1.0	4.0	5.0	2.0	8.0	9.0	6.0	10.0	3.0	7.0
	150	2.0	4.0	6.0	1.0	8.0	9.0	5.0	10.0	3.0	7.0
	250	2.0	4.0	7.0	1.0	8.0	9.0	6.0	10.0	3.0	5.0
$\sum$ Ranks		135.0	171.0	300.5	130.5	347.5	290.5	241.5	337.0	80.0	166.5
Overall Rank		3	5	8	2	10	7	6	9	1	4