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On DUS Transformed Weibull Distribution and its Properties

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Abstract

In Statistics literature, there are a number of methods to develop the new distributions. In this paper, a new distribution is developed using DUS transformation. A number of structural properties of this distribution such as moments, moment generating function, mean, median, mode, hazard rate and its shape, survival function and reverse hazard rate among others are derived. Further, the parameters of the newly developed distribution are estimated using method of moments, MLE and through simulation. The newly derived distribution was applied to two real data for the real life applications. The distribution will be a viable model for life-length of components and systems.

Key words: DUS Transformation; Survival Analysis; Hazard Rate; Cumulative distribution function; Maximum likelihood estimation.

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1. Introduction

There are several methods to propose a new distribution using some baseline distribution. For example, Gupta *et al*. (1998) have proposed the cumulative distribution function (cdf) $G_1(x)$ of new distribution corresponding to the cdf, $F(x)$ of baseline distribution as,

$$G_1(x) = \{F(x)\}^a$$

where, $a > 0$ is the shape parameter.

Shaw and Buckley (2009) have developed a stimulating method called the quadratic rank transmutation map (QRTM) to develop the new distribution. It was used in order to form flexible distribution families by adding a new parameter to an existing distribution. Such family is called the transmuted extended distribution that holds the parental distribution as a special case and offers additional suppleness in order to model the numerous types of data sets.

If $G_2(x)$ is the cumulative function of transmuted distribution consistent to the baseline distribution having $F(x)$, then

$$G_2(x) = (1+ \lambda)F(x) - \lambda\{F(x)\}^2$$

where $|\lambda| \leq 1$.

Recently, various generalizations have been introduced based on QRTM such as transmuted extreme value distribution [see, Aryal and Tsokos (2011)], transmuted inverse

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Weibull distribution [see, Khan *et al*. (2014)], transmuted modified Weibull distribution [see, Khan and King (2013)], transmuted log-logistic distribution [see, Aryal (2013)], transmuted exponential distribution (Kumar *et al.* (2015)) and many more.

$$g(x)=\frac{1}{e\text{-}1}\, f(x)\; e^{F(x)} \tag{1}$$

The transformation (1) is known as DUS transformation and is used for generating the new distribution. The cumulative function and hazard rate consistent to the g(x) are specified in (2) and (3) respectively.

$$G(x)=\frac{1}{e\text{-}1}\,[e^{F(x)}\text{-}1] \tag{2}$$

and

$$h(x)=\frac{1}{e\text{-}e^{F(x)}}\, f(x) e^{F(x)} \tag{3}$$

2. DUS Transformation of Weibull Distribution

In this section, we have proposed a probability density function of a newly formed distribution obtained using DUS transformation technique for Weibull distribution as a baseline distribution. The distribution will be useful for lifetime modeling.

Using equation (1) the probability density function of DUS$_W$(k, λ)-distribution is given by

$$g(x) =\frac{1}{e\text{-}1}\, f(x)\; e^{F(x)} \tag{4}$$

The probability density function of the two parameter Weibull distribution is

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k\text{-}1}\; e^{-\left(\frac{x}{\lambda}\right)^k} \tag{5}$$

and

$$F(x) = 1- e^{-\left(\frac{x}{\lambda}\right)^k} \tag{6}$$

Now, putting f(x) and F(x) in equation (1), we get

$$g(x) = \left(\frac{x}{\lambda}\right)^{k\text{-}1}\; e^{-\left(\frac{x}{\lambda}\right)^k}\; e^{\left(1- e^{-\left(\frac{x}{\lambda}\right)^k}\right)}, x>0, \lambda>0, k>0 \tag{7}$$

Equation (7) represents the probability density function of DUS$_W$(k, λ)-distribution (DUS transformed Weibull distribution) with k as a shape parameter and λ as a scale parameter. The shape of newly developed distribution for various values of parameters is given.

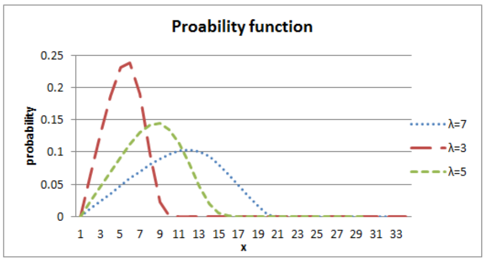

Figure 1: Probability density function of DUS$_W$(*k*, *λ*)-distribution when *k* = 2 is fixed and *λ* is varied (*λ* = 3, 5, 7)

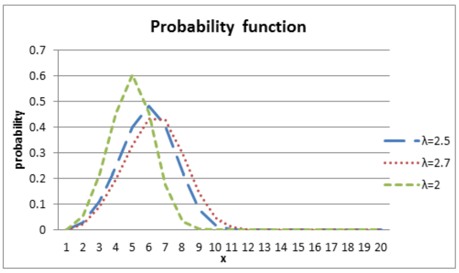

Figure 2: Probability density function of DUS$_W$(*k*, *λ*)-distribution when *k* = 3 is fixed and scale parameter *λ* is varied (*λ* = 2.5, 2.7, 2)

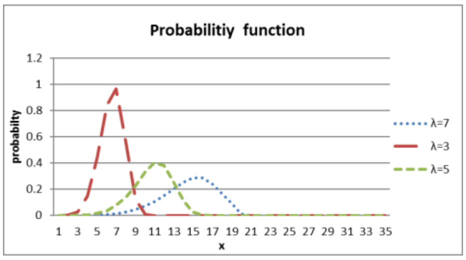

Figure 3: Probability density function of DUS$_W$(*k*, *λ*)-distribution when *k* = 5 is fixed and scale parameter *λ* is varied (*λ* = 3, 5, 7)

The shape of DUS transformed is pretty flexible, including moderately positively skewed, approximately symmetric and moderately negatively skewed shapes for different values of parameters, the DUS$_W$(*k*,*λ*)-distribution seems to be a viable model for life-length of components and systems as well as non-negative variables. The cdf of DUS$_W$(*k*,*λ*)-distribution can be written as

$$G(x)=\frac{1}{e-1}\left(e^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)}-1\right) \tag{8}$$

whereas the survival function of the distribution is obtained as

$$S(x)=1-\frac{1}{e-1}\left(e^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)}-1\right) \tag{9}$$

Using equations (8) and (9), the hazard function is obtained

$$h(x)=\left(\frac{\left(\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)e^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)}}{\left(e-e^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^{k}}\right)}\right)}\right) \tag{10}$$

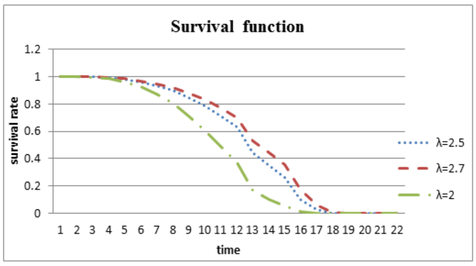

Figure 4: Survival function of DUS$_W$(k, λ)-distribution when k = 3 is fixed and λ is varied (λ = 2.5, 2.7, 2)

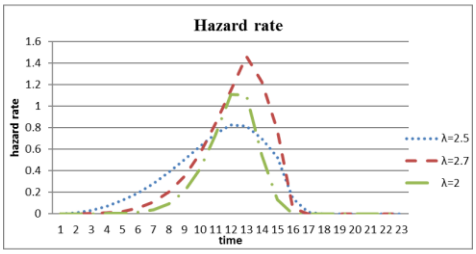

Figure 5: Hazard rate of DUS$_W$(k, λ)-distribution when k = 3 is fixed and λ is varied (λ = 2.5, 2.7, 2)

The graph of hazard rate shows that at the starting of time, hazard rate has an increasing trend whereas after completing its median time (approximately), it drastically goes down to zero.

3. Statistical Properties of DUS$_W$(k, λ) Distribution

The mean of DUS$_W$(k, λ) distribution is obtained as

$$E(x)=\int_{0}^{\infty} x\, g(x)dx$$

$$\mathrm{E(x)}=\lambda \frac{e}{e-1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma\left(i+\frac{1}{k}+1\right) \frac{1}{j^{\left(i+\frac{1}{k}+1\right)}} \tag{11}$$

whereas the median of $\mathrm{DUS}_W(k, \lambda)$ distribution is the solution of the following,

$$\mathrm{G(M)} = \frac{1}{2}$$

for M and the same is obtained as follows,

$$\text{Median} = \mathrm{m} = \lambda \left(\ln\left(\frac{1}{(1-\ln(1+(e-1)0.5))}\right)\right)^{\frac{1}{k}} \tag{12}$$

In order to obtain the mode of the distribution, differentiating equation (7) with respect to x, we get

$$\mathrm{g}'(x)=\left(\frac{k}{\lambda}\right)\left(\frac{1}{e-1}\right)\left(\frac{(k\text{-}1)}{\lambda}\left(\frac{x}{\lambda}\right)^{k\text{-}2}\left(-\frac{1}{\lambda}\right)^{k} e^{-\left(\frac{x}{\lambda}\right)^k}\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)\left(-e^{\left(1-e^{-\left(\frac{x}{\lambda}\right)^k}\right)}\right)\right) \tag{13}$$

It is easy to show that $g''(x)$ is a decreasing function hence the expression for the mode may be obtained by putting equation (13) equal to zero.

The harmonic mean of $\mathrm{DUS}_W(k, \lambda)$ distribution is obtained by solving the following expression and is obtained as

$$\frac{1}{\mathrm{H}} = \int_0^{\infty} \frac{1}{x} \mathrm{g}(x)\mathrm{d}x$$

$$\mathrm{H} = \left[\frac{1}{\lambda}\frac{e}{e\text{-}1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma\left(i-\frac{1}{k}+1\right) \frac{1}{j^{\left(i-\frac{1}{k}+1\right)}}\right]^{-1} \tag{14}$$

The variance of $\mathrm{DUS}_W(k, \lambda)$ distribution can be obtained as

$$\mathrm{Var}(x) = \lambda^2 \frac{e}{e\text{-}1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma\left(i+\frac{2}{k}+1\right) \frac{1}{j^{\left(i+\frac{2}{k}+1\right)}} - \left(\lambda \frac{e}{e\text{-}1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \Gamma\left(i+\frac{1}{k}+1\right) \frac{1}{j^{\left(i+\frac{1}{k}+1\right)}}\right)^2 \tag{15}$$

The moment generating function of $\mathrm{DUS}_W(k, \lambda)$ distribution is obtained as

$$\mathrm{M}_x(t) = \mathrm{E}(\mathrm{e}^{tx})$$

$$\mathrm{E}(\mathrm{e}^{tx}) = \frac{e}{e\text{-}1} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left[i-\frac{t\lambda}{k}+1\right]^{-1} \tag{16}$$

The characteristic function of $\mathrm{DUS}_W(k, \lambda)$-distribution for the variable X is obtained as

$$\phi_x(t) = \mathrm{E}(e^{itx})$$

$$\mathrm{E}(\mathrm{e}^{itx}) = \frac{e}{e\text{-}1}\ \sum_{j=0}^{\infty} \frac{(\text{-}1)^j}{j!}\left[\frac{j\text{-}it\lambda+1}{k}\right]^{-1} \tag{17}$$

The raw moments of DUS$_W$(k, λ) distribution are obtained as follow

$$\mu'_r = \mathrm{E}(x^r) = \lambda^r \frac{e}{e\text{-}1}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!}\Gamma\left(i+\frac{r}{k}+1\right)\ \frac{1}{j^{\left(i+\frac{r}{k}+1\right)}} \tag{18}$$

The quantile function for DUS$_W$(k, λ)-distribution is obtained as

$$x_q = \lambda\left[\ln\left[\frac{1}{1-\ln\left(1+q(e-1)\right)}\right]\right]^{\frac{1}{k}} \tag{19}$$

4. Estimation of the Parameters of DUS$_W$(k, λ) Distribution

In order to assess the real life application of the DUS$_W$(k, λ) distribution, the parameters of the distribution are estimated. We estimate the parameters 'k' and 'λ' of DUS$_W$(k,λ) distribution using the maximum likelihood estimation method. By definition

$$\frac{\partial \ln \mathrm{L}(k;x_1,x_2,\ldots,x_n)}{\partial k} = 0 \tag{20}$$

and

$$\frac{\partial \ln \mathrm{L}(\lambda;x_1,x_2,\ldots,x_n)}{\partial \lambda} = 0 \tag{21}$$

So solving the equations simultaneously, we have

$$\frac{n}{k}+\ln\left[\prod_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{k-1}\right]-\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{k}\ln\left(\frac{k}{\lambda}\right)-\sum_{i=1}^{n}\left(\mathrm{e}^{1-\left(\frac{x_i}{\lambda}\right)^k}\right)\left(\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^{k}\ln\left(\frac{k}{\lambda}\right)\right)=0 \tag{22}$$

$$-\frac{n}{\lambda^2}+\frac{1-k}{\lambda}+\frac{k}{\lambda}\sum_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k-\frac{k}{\lambda}\sum_{i=1}^{n}\left(e^{1-\left(\frac{x_i}{\lambda}\right)^k}\right)\left(\frac{x_i}{\lambda}\right)^k=0 \tag{23}$$

Equations (22) and (23) were difficult to solve analytically, some numerical methods may be used to solve the equations simultaneously for k and λ respectively. In order to estimate the parameters analytically, we have estimated the parameters using method of moments. Following equations will be solved to estimate the parameters k and λ,

$$\acute{\mu}_1 = m'_1$$

and

$$\acute{\mu}_2 = m_2'$$

$$\hat{\lambda} = \frac{\bar{x}}{\frac{e}{(e\text{-}1)}\sum_{i=1}^{\infty}\frac{(\text{-}1)^i}{i!}\sum_{j=1}^{\infty}\frac{(\text{-}1)^j}{j!}\Gamma\frac{(k(i+1)+1)}{k}\frac{1}{j^{(\frac{k(i+1)+1}{k}}}} \tag{24}$$

for parameter k,

$$\frac{1}{n}\left(A\left(\frac{e\text{–}1}{e}\right)BC\right) = 0$$

where $A = n\sum x_i^2 - (\sum x_i)^2$; $B = \left(\frac{1}{\sum_{i=1}^{\infty}\frac{(\text{-}1)^i}{i!}\sum_{j=1}^{\infty}\frac{(\text{-}1)^j}{j!}\Gamma\frac{(k(i+1)+1)}{k}\frac{1}{j^{(\frac{k(i+1)+1}{k}}}}\right)^2$ and

$$C = \sum_{i=1}^{\infty}\frac{(-1)^i}{i!}\sum_{j=1}^{\infty}\frac{(-1)^j}{j!}\,\Gamma\left(\frac{(k(i+1)+2)}{k}\right)\frac{1}{j^{(\frac{k(i+1)+2}{k}}} = 0. \tag{25}$$

As the expression for k may not be solved analytically therefore numerical method may be used to estimate the parameters k and λ. A simulation study is carried out taking 1000 samples of various sizes n drawn from the $DUS_W(k, \lambda)$ distribution for different values of the parameters k and λ. For inversion theorem the relation $X = F^{-1}(u)$ is used to generate the random values for the variable X with the given distribution function. By definition

$$F(x) = u$$

$$x = F^{-1}(u)$$

$$\hat{x} = \lambda\left[\ln\left[\frac{1}{1-\ln\left(1+u(e-1)\right)}\right]\right]^{1/k} \tag{26}$$

Hence the above expression is used to generate random samples from the $DUS_W(k, \lambda)$ distribution for the given values of the parameters. A computer program is developed to obtain the mean values of the $DUS_W(k, \lambda)$-distribution using R language. For each pair of values (k, λ), various values of the mean of means are obtained. For a given data, the mean will be calculated and the parameters will be estimated for the given mean using the Tables generated for DUS transformed Weibull distribution. The values of the mean of transformed data of the DUS Weibull distribution are presented in the Tables 1 to 9 in the Appendix.

5. Real Life Application

To assess the applicability of $DUS_W(k, \lambda)$ distribution, we have considered a real data of 63 observations related to the strengths of 1.5 cm glass fibers. This set was obtained by workers at the UK National Physical Laboratory and was used by Smith and Naylor (1987) whereas the second data set was about the hole diameter (Dasgupta, 2011). The first data set is related to the strengths of 1.5 cm glass fibers, a total of 63 observations were obtained and are given as follows

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28, 1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50, 1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58, 1.59, 1.60, 1.61, 1.63, 1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68, 1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82, 1.84, 1.84, 2.00, 2.01, 2.24.

After arranging the above data, arithmetic mean of the transformed data is calculated which comes out to be 1.506. Now searching this value in table 5, we find that the value of the mentioned mean is 1.506 for $k = 3$, $\lambda = 1.5$ and $n = 63$. The $DUS_W(k, \lambda)$ distribution is fitted on the data using $\lambda = 1.5$ and $k = 3$. The chi-square goodness of fit test ($\chi^2 = 3.9168$, $p = 0.86$) revealed that the $DUS_W(k, \lambda)$ model is a good fit model on the data of strengths of glass fibers. Further, the Weibull distribution is fitted on the data for the same choice of the parameters $\lambda = 1.5$ and $k = 3$. The chi-square goodness of fit revealed that the $DUS_W(k, \lambda)$ distribution is a better fit model compared to two-parameter Weibull distribution**.**

The second data set of 50 observations (in the unit of millimeter) is related to different machines under comparison for the similar operations in the same site of a factory and was used by Dasgupta (2011). The observations are given below

0.04, 0.02, 0.06, 0.12, 0.14, 0.08, 0.22, 0.12, 0.08, 0.26, 0.24, 0.04, 0.14, 0.16, 0.08, 0.26, 0.32, 0.28, 0.14, 0.16, 0.24, 0.22, 0.12, 0.18, 0.24, 0.32, 0.16, 0.14, 0.08, 0.16, 0.24, 0.16, 0.32, 0.18, 0.24, 0.22, 0.16, 0.12, 0.24, 0.06, 0.02, 0.18, 0.22, 0.14, 0.06, 0.04, 0.14, 0.26, 0.18, 0.16.

After arranging the above mentioned data, its mean is calculated as 0.1632. Now searching this value in Table 4, we have found that the value of the mentioned mean is 0.1632 for $k = 1.5$, $\lambda = 0.15$ and $n = 50$. The chi-square goodness of fit test ($\chi^2 = 24.8039$, $p = 0.81$) concluded that the proposed model is a good fit for the given data set.

6. Conclusion

From the simulation study, it is evident that the proposed DUS transformed Weibull distribution is a flexible model for application. The distribution may be used as a lifetime model and can be fitted on the life length of various components.

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APPENDIX

Table 1: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 10

k/λ	2	2.2	2.3	2.4	2.5	2.6	2.7	3	3.4
1	2.517245	2.811399	2.885155	2.986473	3.137331	3.283466	3.399174	3.787712	4.336345
1.15	2.376857	2.566277	2.698917	2.861838	2.956662	3.07399	3.183724	3.470086	3.948485
1.25	2.23171	2.469243	2.583506	2.736134	2.837344	2.939732	3.041505	3.372101	3.849027
1.5	2.155965	2.33097	2.477656	2.59744	2.696651	2.774533	2.891594	3.204253	3.608331
1.75	2.076177	2.275707	2.379886	2.484644	2.620812	2.696242	2.805161	3.126713	3.504668
2	2.023193	2.234959	2.353963	2.440517	2.529771	2.645726	2.742195	3.043873	3.435698
2.25	2.00609	2.214886	2.310757	2.399094	2.512973	2.591598	2.722587	3.031537	3.392291
2.5	2.000772	2.182101	2.290734	2.376628	2.49335	2.581597	2.691004	2.97438	3.375509
3	1.969566	2.158932	2.272332	2.362112	2.460654	2.562589	2.644976	2.961288	3.343580

Table 2: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 20

k/λ	0.2	0.4	0.45	2	2.3	2.5	3	3.3	3.5
1	0.251216	0.569388	0.569388	2.504554	2.904909	3.167895	3.755406	4.124498	4.377891
1.15	0.235446	0.530137	0.530137	2.347480	2.714062	2.929789	3.530666	3.880406	4.109790
1.25	0.225846	0.511246	0.511246	2.281141	2.614768	2.839802	3.405611	3.718937	3.980987
1.5	0.214381	0.481564	0.481564	2.149555	2.466677	2.684022	3.214789	3.561088	3.778796
1.75	0.206709	0.468092	0.468092	2.083877	2.386881	2.599988	3.102101	3.449115	3.610533
2	0.204347	0.456636	0.456636	2.028692	2.342961	2.523202	3.049356	3.34684	3.555917
2.5	0.199243	0.447806	0.447806	1.991692	2.284143	2.486346	2.985243	3.272693	3.481387
3	0.195725	0.443646	0.443646	1.970628	2.262800	2.45306	2.960755	3.261811	3.443872
3.2	0.189226	0.390668	0.390668	1.894560	2.241700	2.42293	2.46892	3.241709	3.422876

Table 3: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 30

k/λ	2	2.3	2.5	2.7	2.8	3	3.3	3.5
1	2.504009	2.912170	3.171027	3.404401	3.499660	3.746924	4.135979	4.470881
1.15	2.347347	2.887324	2.949099	3.162511	3.252916	3.554996	3.875747	4.093062
1.25	2.273302	2.615983	2.832352	3.051734	3.186968	3.396030	3.746377	3.968723
1.5	2.138868	2.470648	2.685668	2.898625	3.009160	3.225825	3.527038	3.756814
1.75	2.071820	2.385379	2.599343	2.804341	2.895190	3.112046	3.427742	3.641127
2	2.038699	2.329662	2.554939	2.746986	2.842897	3.056338	3.365053	3.554760
2.25	2.003136	2.305109	2.500910	2.707960	2.803177	3.007277	3.308174	3.509270
2.5	1.986168	2.281053	2.486458	2.674299	2.794896	2.997515	3.268091	3.482483
3	1.971597	2.267932	2.454803	2.661883	2.754530	2.956542	3.243436	3.436502

Table 4: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 50

k/λ	0.1	0.15	0.2	0.5	1	1.5	2	2.5	3
1	0.126208	0.189445	0.252161	0.629009	1.254004	1.898039	2.531597	3.138138	3.777200
1.15	0.116912	0.176062	0.234553	0.584946	1.175404	1.755803	2.339846	2.929436	3.504400
1.25	0.113295	0.170105	0.226849	0.565130	1.134217	1.698214	2.281033	2.83516	3.390600
1.5	0.107107	0.164074	0.214397	0.534685	1.071657	1.609600	2.148722	2.666563	3.208000
1.75	0.103833	0.160124	0.206910	0.518443	1.040654	1.558920	2.074412	2.594035	3.101780
2	0.101413	0.158331	0.203733	0.507773	1.015262	1.519897	2.032245	2.540005	3.050230
2.5	0.099707	0.155434	0.198530	0.498642	0.992869	1.490626	1.993589	2.499018	2.977090
3	0.098321	0.149604	0.196730	0.491703	0.984630	1.475060	1.970222	2.462718	2.955900
3.5	0.087644	0.142777	0.192780	0.468930	0.945520	1.439970	1.935520	2.390330	2.909700

Table 5: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 63

k/λ	0.5	0.75	1	1.5	1.75	2	2.5	3	3.5
0.5	1.399557	2.136622	2.865456	4.554282	4.869946	5.709310	7.132205	8.457595	9.895400
0.75	0.784859	1.177411	1.561745	2.469169	2.763686	3.141139	3.915627	4.710288	5.481340
1	0.631977	0.943295	1.256300	1.987784	2.210679	2.528557	3.140937	3.748432	4.414500
1.5	0.536525	0.803973	1.070778	1.872589	1.890160	2.147369	2.681373	3.225480	3.755600
1.75	0.519181	0.776352	1.034615	1.685650	1.814858	2.074194	2.594843	3.118343	3.628500
2	0.507381	0.763716	1.015113	1.605890	1.773439	2.035301	2.542794	3.040332	3.558100
2.5	0.497888	0.746088	0.995065	1.591450	1.746025	1.987331	2.483504	2.989640	3.486800
2.75	0.493780	0.741295	0.989270	1.556450	1.729200	1.974721	2.471013	2.965545	3.455800
3	0.492343	0.739055	0.982879	1.510709	1.728782	1.965731	2.456757	2.955809	3.443000
3.5	0.489345	0.734759	0.979046	1.480709	1.714510	1.955997	2.44798	2.934148	3.429600

Table 6: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 100

k/λ	**2**	**2.3**	**2.5**	**2.7**	**2.8**	**3**	**3.3**	**3.5**
1	2.531597	2.894259	3.138138	3.422545	3.540568	3.777278	4.169518	4.433289
1.15	2.339846	2.696134	2.929436	3.159015	3.289008	3.504461	3.879313	4.098301
1.25	2.281033	2.603607	2.83516	3.058677	3.170763	3.390670	3.723769	3.979242
1.5	2.148722	2.475804	2.666563	2.906334	3.005207	3.208049	3.528592	3.767585
1.75	2.074412	2.377109	2.594035	2.795645	2.900345	3.101789	3.436162	3.634684
2	2.032245	2.332237	2.540005	2.735711	2.76723	3.050239	3.34302	3.556535
2.5	1.993589	2.283756	2.499018	2.687975	2.785568	2.977095	3.277005	3.479074
3	1.970222	2.267965	2.462718	2.658942	2.749937	2.955899	3.248275	3.438107

Table 7: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 300

k/λ	**2**	**2.3**	**2.5**	**2.7**	**2.8**	**3**	**3.3**	**3.5**
1	2.524565	2.898132	3.142753	3.403772	3.528445	3.774485	4.156479	4.411083
1.15	2.346215	2.699438	2.937056	3.174084	3.285149	3.517790	3.878156	4.102618
1.25	2.272043	2.605824	2.832281	3.063584	3.174093	3.407877	3.748495	3.969933
1.5	2.148019	2.466352	2.677368	2.896174	2.996944	3.222330	3.540961	3.750759
1.75	2.076095	2.386813	2.598522	2.802113	2.905126	3.114686	3.425741	3.625521
2	2.034038	2.336548	2.543534	2.743206	2.845385	3.049329	3.357493	3.556055
2.5	1.988411	2.288006	2.482313	2.683379	2.782118	2.984053	3.277599	3.482176
3	1.967270	2.264435	2.460363	2.656226	2.757550	2.949514	3.248816	3.442170

Table 8: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 500

k/λ	**2**	**2.3**	**2.5**	**2.7**	**2.8**	**3**	**3.3**	**3.5**
1	2.518699	2.898791	3.154413	3.399437	3.531351	3.777867	4.154576	4.404194
1.15	2.345587	2.699761	2.939658	3.166351	3.287146	3.523665	3.877953	4.105032
1.25	2.268411	2.609544	2.836316	3.063409	3.178322	3.406563	3.743683	3.971175
1.5	2.143511	2.463945	2.684016	2.892896	3.004482	3.218393	3.536348	3.754384
1.75	2.075971	2.388087	2.596298	2.801879	2.90648	3.110323	3.423013	3.632914
2	2.033545	2.33581	2.540464	2.742854	2.846879	3.051042	3.355247	3.556760
2.5	1.988111	2.286722	2.484683	2.684510	2.782593	2.982218	3.278046	3.480877
3	1.969891	2.263312	2.461614	2.657846	2.757536	2.950255	3.252892	3.443803

Table 9: Table of the means of ($\bar{X}_{trans\ weibull}$) when sample size *n* = 1000

k/λ	**2**	**2.3**	**2.5**	**2.7**	**2.8**	**3**	**3.3**	**3.5**
1	2.519694	2.898938	3.150607	3.398360	3.528852	3.781955	4.159814	4.407745
1.15	2.344847	2.699903	2.935890	3.168151	3.285946	3.523283	3.872619	4.106633
1.25	2.271496	2.609546	2.836777	3.063625	3.180062	3.403251	3.744170	3.971481
1.5	2.146576	2.468929	2.683170	2.896734	3.007135	3.215928	3.538355	3.753207
1.75	2.076046	2.388072	2.595777	2.801017	2.904149	3.113252	3.424946	3.629920
2	2.033407	2.340196	2.542027	2.745568	2.845711	3.049193	3.353305	3.556922
2.5	1.988675	2.285758	2.484805	2.684776	2.782991	2.983072	3.283154	3.479466
3	1.968194	2.263994	2.460396	2.656330	2.754646	2.952897	3.247433	3.446004