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Cost & Profit Analysis of Two-Dimensional State M/M/2 Queueing Model with Multiple Vacation, Feedback, Catastrophes and Balking

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Abstract

A time-dependent solution of the two-dimensional state M/M/2 queueing system with multiple vacation, feedback, catastrophes and balking is obtained in this study. Inter-arrival and service times follow an exponential distribution with parameters λ and μ respectively. Both the servers go on vacation with probability one when there are no units in the system. All the units are ejected from the system when catastrophes occur and the system becomes temporarily unavailable. The system reactivates when new units arrive. Occurrence of catastrophes follow Poisson distribution with rate ξ. The units come and wait in the queue for service; the served units either leave the system or rejoin immediately at the early end of the queue to receive satisfactory service, known as feedback. Laplace transform approach has been used to find the time-dependent solution. The efficiency of a queuing system has been verified by evaluating some key measures along with "total expected cost" and "total expected profit". Numerical analyses have been done by using Maple software.

Key words: Time-dependent solution; Two-dimensional state model; Balking; Catastrophes; Feedback; Multiple vacation.

AMS Subject Classifications: 62K05, 05B05

1. Introduction

In the present study, the two-dimensional state model has been used to simplify the complicated transient analysis of some queueing problems. This model is used to study the queueing system more categorically for arrivals and departures. The idea of two-dimensional state for the M/M/1 queue was first given by Pegden and Rosenshine (1982). After that, two-dimensional state model has received considerable attention by many researchers to analyse various queuing systems.

Various studies have been conducted to evaluate different performance measures to verify the robustness of the system in which a server takes a break for a random period of time *i.e.* vacation. When the server returns from a vacation and finds the queue empty, it

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immediately goes on another vacation and if it finds at least one waiting unit, then it will commence service according to the prevailing service policy, *i.e.* multiple vacation. Different queueing systems with multiple vacation have been extensively investigated and effectively used in several fields including industries, computer & communication systems, telecommunication systems *etc.* Different types of vacation policies are available in literature such as single vacation, multiple vacation and working vacations. Researches on vacation models have grown tremendously in the last several years. Cooper (1970) was the first to study the vacation model and determined the mean waiting time for a unit arrive at a queue served in cyclic order. Doshi (1986) and Ke *et al.* (2010) have done outstanding researches on queueing system with vacations and released some excellent surveys. Xu and Zhang (2006) considered the Markovian multi-server queue with a single vacation (e, d)-policy. They also formulated the system as a quasi-birth-and-death process and computed the various stationary performance measures. Altman and Yechiali (2006) studied the customer's impatience in queues with server vacations. Kalidass *et al.* (2014) obtained the time-dependent solution of a single server queue with multiple vacations. Ammar (2015) analysed M/M/1 queue with impatient units and multiple vacations. Sharma and Indra (2020) investigated the dynamic aspects of a two-dimensional state single server Markovian queueing system with multiple vacations and reneging.

Also, units may be served repeatedly for many reasons, *e.g.* when a unit is unsatisfied with a service, the unit may try for a satisfactory service. For example, we visit to the online shopping store and order a full-sleeve jacket but when we receive the order it turn out to be half-sleeve jacket. Since we are unsatisfied with the service, so we go for a return policy or exchange policy provided by the shopping store and to receive satisfactory service. Many researchers have been attracted to the study of queues with feedback as large number of applications have been found in many areas including production systems, post offices, supermarkets, hospital management, financial sectors, ticket offices, grocery stores, ATMs and so forth. Takacs (1963) determined the distribution of the queue size and the first two moments of the distribution for a queue with feedback. D'Avignon and Disney (1976) studied the non-Markovian queue with a state-dependent feedback mechanism. Disney *et al.* (1980) investigated a number of random processes that occur in queues with instantaneous Bernoulli feedback. Choudhury and Paul (2005) derived two phases of heterogeneous services with Bernoulli feedback systems. Chowdhury and Indra (2020) analysed two-node tandem queue with feedback.

Queueing systems with catastrophes are getting a lot of attention nowadays and may be used to solve a wide range of real-world problems. Catastrophes may occur at any time, resulting in the loss of all the units and the deactivation of the service centre, because they are totally unpredictable in nature. Such types of queues with catastrophes play an important role in computer programs, telecommunication, ticket counter *etc.* For example, virus or hacker attacking a computer system or program causing the system fail or become idle. Chao (1995) obtained steady-state probability of the queue size and a product form solution of a queueing network system with catastrophes. Krishna Kumar *et al.* (2007) obtained time-dependent solution for M/M/1 queueing system with catastrophes. Kalidass *et al.* (2012) derived explicit closed form analytical expressions for the time-dependent probabilities of the system size. Dharmaraja and Kumar (2015) studied Markovian queueing system with heterogeneous servers and catastrophes. Chakravarthy (2017) studied delayed catastrophic model in steady state using the matrix analytic method. Suranga Sampath and

Liu (2018) studied an M/M/1 queue with reneging, catastrophes, server failures and repairs using modified Bessel function, Laplace transform and probability generating function approach. de Oliveira Souza and Rodriguez (2021) worked on fractional M/M/1 queue model with catastrophes.

Queues with balking have a wide range of practical applications in everyday life. Balking occurs if units avoid joining the queue, when they perceive the queue to be too long. Long queues at cash counters, ticket booths, banks, barber shops, grocery stores, toll plaza *etc.* Kumar *et al.* (1993) obtained time-dependent solution of an M/M/1 queue with balking. Zhang *et al.* (2005) analysed the M/M/1/N queueing system with balking, reneging, and server vacation. Sharma and Kumar (2012) used a single-server Markovian feedback queuing system with balking.

With above concepts in mind, we analyse a two-dimensional state M/M/2 queueing model with multiple vacation, feedback, catastrophes and balking.

Out of the many physical situations, one can be in the post office, where an unit arrives to receive the service and is unsatisfied by the service, then it re-joins at the early end of the queue to receive satisfactory service; may be considered as feedback unit. On arrival, if the unit finds a long queue and decides not to join; may be considered as balking unit and if the computer system fails due to virus or any other reason; may be considered as occurrence of catastrophes. After service completion, the server may take a break, when he finds an empty queue.

The paper has been structured as follows. In Section 2, the model assumptions, notations and description are given. In Section 3 the differential-difference equations to find out the time-dependent solution are given and Section 4 describes important performance measures. Section 5 investigates the total expected cost function and total expected profit function for the given queueing system. In Section 6, we present the numerical results in the form of tables and graphs to illustrate the impact of various factors on performance measures. The last Section contains discussion on the findings and suggestions for future work.

2. Model assumptions, notations and description

- Arrivals follow Poisson distribution with parameter λ.
- There are two homogeneous servers and the service times at each server follow an exponential distribution with parameter μ.
- The vacation time of the server follows an exponential distribution with parameter w.
- After completion of the service, the dissatisfied units rejoin at the early end of the queue to receive service with probability q.
- On arrival a unit either decides to join the queue with probability β or not to join the queue with probability 1-β.
- Occurrence of catastrophes follows Poisson distribution with parameter ξ.

- Various stochastic processes involved in the system are statistically independent of each other.

Initially, the system starts with zero units and the server is on vacation, *i.e.*

$$P_{0,0,V}(0) = 1 \quad , \quad P_{0,0,B}(0) = 0 \tag{1}$$

$$\delta_{i,j} = \begin{cases} 1 & ; \, for \ i = j \\ 0 & ; \, for \ i \neq j \end{cases} \tag{2}$$

$$\sum_{i}^{j} = 0 \quad for \quad j < i$$

The two-dimensional state model
$P_{i,j,V}(t)$=The probability that there are exactly i arrivals and j departures by time t and the server is on vacation.
$P_{i,j,B}(t)$=The probability that there are exactly i arrivals and j departures by time t and the server is busy in relation to the queue.
$P_{i,j}(t)$=The probability that there are exactly i arrivals and j departures by time t.

3. The differential-difference equations for the queueing model under study

$$\frac{d}{dt}P_{i,i,V}(t) = -\lambda\beta P_{i,i,V}(t) + q\mu P_{i,i-1,B}(t)(1-\delta_{i,0}) + \xi(1 - P_{i,i,V}(t)) \quad i \geq 0 \tag{3}$$

$$\frac{d}{dt}P_{i+1,i,B}(t) = -(\lambda\beta+q\mu+\xi)P_{i+1,i,B}(t)+2q\mu P_{i+1,i-1,B}(t)(1-\delta_{i,0})+wP_{i+1,i,V}(t) \quad i \geq 0 \tag{4}$$

$$\frac{d}{dt}P_{i,j,V}(t) = -(\lambda\beta + w + \xi)P_{i,j,V}(t) + \lambda\beta P_{i-1,j,V}(t) \quad i > j \geq 0 \tag{5}$$

$$\begin{aligned}\frac{d}{dt}P_{i,j,B}(t) = -(\lambda\beta + 2q\mu + \xi)P_{i,j,B}(t) + \lambda\beta P_{i-1,j,B}(1-\delta_{i-1,j})(t) + 2q\mu P_{i,j-1,B}(t)(1-\delta_{j,0}) \\ +wP_{i,j,V}(t) \quad i > j+1\end{aligned} \tag{6}$$

The preceding equations (3) to (6) are solved by taking the Laplace transforms together with initial conditions:

$$\bar{P}_{0,0,V}(s) = \frac{\xi + s}{s(s+\lambda\beta+\xi)} \tag{7}$$

$$\bar{P}_{i,0,V}(s) = \frac{(\lambda\beta)^i(\xi+s)}{s(s+\lambda\beta+\xi)(s+\lambda\beta+w+\xi)^i} \quad i > 0 \tag{8}$$

$$\bar{P}_{i,i,V}(s) = \frac{q\mu}{s+\lambda\beta+\xi}P_{j,j-1,B}(s) \quad i > 0 \tag{9}$$

$$\bar{P}_{i,0,B}(s) = \frac{w(\lambda\beta)^i(\xi+s)}{s(s+\lambda\beta+\xi)(s+\lambda\beta+w+\xi)(s+\lambda\beta+q\mu+\xi)(s+\lambda\beta+2q\mu+\xi)^{i-1}} +$$

$$+w(\lambda\beta)^i \sum_{m=1}^{i-1} \frac{1}{s(s+\lambda\beta+\xi)(s+\lambda\beta+w+\xi)^{m+1}(s+\lambda\beta+q\mu+\xi)(s+\lambda\beta+2q\mu+\xi)^{i-m}} \quad i \geq 1 \tag{10}$$

$$\bar{P}_{i+1,i,B}(s) = \frac{2q\mu}{s+\lambda\beta+q\mu+\xi} P_{i+1,i-1,B}(s)$$
$$+\frac{q\mu w\lambda\beta}{(s+\lambda\beta+\xi)(s+\lambda\beta+w+\xi)(s+\lambda\beta+q\mu+\xi)} P_{i,i-1,B}(s) \quad i > 0 \tag{11}$$

$$\bar{P}_{i,j,V}(s) = \frac{(q\mu)}{(s+\lambda\beta+w+\xi)} \frac{(\lambda\beta)^{i-j}}{(s+\lambda\beta+w+\xi)^{i-j}} P_{j,j-1,B}(s) \quad i > j \geq 1 \tag{12}$$

$$\bar{P}_{i,j,B}(s) = \frac{\lambda\beta}{s+\lambda\beta+2q\mu+\xi} P_{i-1,j,B}(s) + \frac{2q\mu}{s+\lambda\beta+2q\mu+\xi} P_{i,j-1,B}(s) + \frac{q\mu}{s+\lambda\beta+\xi}$$
$$\frac{w}{s+\lambda\beta+2q\mu+\xi} \frac{(\lambda\beta)^{i-j}}{(s+\lambda\beta+w+\xi)^{i-j}} P_{j,j-1,B}(s) \quad i > j+1, j > 0 \tag{13}$$

It is seen that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}[\bar{P}_{i,j,V}(s) + \bar{P}_{i,j,B}(s)(1-\delta_{i,j})] = \frac{1}{s} \tag{14}$$

and hence

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i}[\bar{P}_{i,j,V}(t) + \bar{P}_{i,j,B}(t)(1-\delta_{i,j})] = 1 \tag{15}$$

a verification.

4. Performance measures

(a) The Laplace transform of $P_{i.}(t)$ the probability that exactly i units arrive by time t; when initially there are no units in the system is given by

$$\bar{P}_{i.}(s) = \sum_{j=0}^{i}[\bar{P}_{i,j,V}(s) + \bar{P}_{i,j,B}(s)(1-\delta_{i,j})] = \sum_{j=0}^{i}\bar{P}_{i,j}(s) = \frac{(\lambda\beta)^i}{(s+\lambda\beta)^{i+1}} \tag{16}$$

and its inverse Laplace transform is

$$P_{i.}(t) = \frac{e^{-\lambda\beta t}(\lambda\beta t)^i}{i!} \tag{17}$$

The arrivals follow a Poisson distribution as the probability of the total number of arrivals is not affected by vacation time of the server.
(b) $P_{.j}(t)$ is the probability that exactly j units have been served by time t. In terms of $P_{i,j}(t)$ we have

$$P_{.j}(t) = \sum_{i=j}^{\infty} P_{i,j}(t) \tag{18}$$

(c)The probability of exactly n units in the system at time t, denoted by $P_n(t)$, can be expressed in terms $P_{ij}(t)$ as

$$P_n(t) = \sum_{j=0}^{\infty} P_{j+n,j}(t) \tag{19}$$

(d) The Laplace transform of mean number of arrivals by time t is

$$\sum_{i=0}^{\infty} i\bar{P}_{i.}(s) = \frac{\lambda\beta}{s^2} \tag{20}$$

and inverse of the mean number of arrivals by time t is

$$\sum_{i=0}^{\infty} i\ P_{i.}(t) = \lambda t \tag{21}$$

(e) The mean number of units in the queue is calculated as follows

$$Q_L(t) = \sum_{n=0}^{\infty} nP_V(t) + \sum_{n=2}^{\infty}(n-2)P_B(t) \tag{22}$$

where $n = i - j$.

5. Cost function and profit function

For the given queueing system, the following notations have been used to represent various costs to find out the total expected cost and total expected profit per unit time
Let
C_H: Cost per unit time for unit in the queue.
C_B: Cost per unit time for a busy server.
C_μ: Cost per service per unit time.
C_V: Cost per unit time when the server is on vacation.
$C_{\mu-q}$: Cost per unit time when a unit rejoins at the early end of the queue as a feedback unit.
If I is the total expected amount of income generated by delivering a service per unit time then
a) Total expected cost per unit at time t is given by

$$TC(t) = C_H * Q_L(t) + C_B * P_B(t) + C_V * P_V(t) + \mu * (C_\mu + C_{\mu-q}) \tag{23}$$

b) Total expected income per unit at time t is given by

$$TE_I(t) = I * \mu * (1 - P_V(t)) = I * \mu * P_B(t) \tag{24}$$

c) Total expected profit per unit at time t is given by

$$TE_P(t) = TE_I(t) - TC(t) \tag{25}$$

6. Numerical results

6.1. Numerical validity check

1. For the state when the server is on vacation

$$P_V(t) = \sum_{j=0}^{i} P_{i,j,V}(t) \tag{26}$$

2. For the state when the server is busy in relation to the queue

$$P_B(t) = \sum_{j=0}^{i-1} P_{i,j,B}(t) \tag{27}$$

3. The probability $P_{i.}(t)$ that exactly i units arrive by time t is

$$P_{i.}(t) = \sum_{j=0}^{i} P_{i,j}(t) = \sum_{j=0}^{i} P_{i,j,V}(t) + \sum_{j=0}^{i-1} P_{i,j,B}(t) \tag{28}$$

4. A numerical validity check of inversion of $\bar{P}_{i,j}(s)$ is based on the relationship

$$P_r\{i \quad arrivals \quad in \quad (0,t)\} = \frac{e^{-(\lambda\beta t)} * (\lambda\beta t)^i}{i!} = \sum_{j=0}^{i} P_{i,j}(t) = P_{i.}(t) \tag{29}$$

The probabilities of this model shown in last column of Table 1 given below are consistent to the last column of "Pegden and Rosenshine (1982)"

Table 1: Numerical validity check of inversion $\bar{P}_{i,j}(s)$

λ	μ	t	i	w	q	ξ	β	$\frac{e^{-(\lambda t)}*(\lambda t)^i}{i!}$	$\sum_{j=0}^{i} P_{i,j,V}(t)$	$\sum_{j=0}^{i-1} P_{i,j,B}(t)$	$\sum_{j=0}^{i} P_{i,j}(t)$
1	2	3	1	1	1	0	1	0.149361	0.12688	0.02247	0.14936
1	2	3	3	1	1	0	1	0.224041	0.14971	0.07433	0.22404
1	2	3	5	1	1	0	1	0.100818	0.05262	0.04818	0.10081
2	2	3	1	1	1	0	1	0.014871	0.01263	0.00223	0.01487
2	2	3	3	1	1	0	1	0.089234	0.05962	0.02960	0.08923
2	2	3	5	1	1	0	1	0.160622	0.08384	0.07677	0.16062
1	2	4	1	1	1	0	1	0.073261	0.06443	0.00882	0.07326
1	2	4	3	1	1	0	1	0.195366	0.14187	0.05349	0.19536
1	2	4	5	1	1	0	1	0.156292	0.09401	0.06227	0.15629
2	2	4	1	1	1	0	1	0.002682	0.00236	0.00032	0.00268
2	2	4	3	1	1	0	1	0.028625	0.02078	0.00783	0.02862
2	2	4	5	1	1	0	1	0.091602	0.05510	0.03650	0.09160
2	4	4	5	1	1	0	1	0.091602	0.07219	0.01940	0.09160
1	2	4	4	1	1	0	1	0.195366	0.12931	0.06605	0.19536
1	2	3	6	1	1	0	1	0.050409	0.02299	0.02741	0.05040

Table 2: Probabilities of exactly n units in the system at time t

n	$t = 1$	$t = 2$	$t = 3$	t=4	$t = 5$
0	0.3064059	0.2756176	0.2194939	0.1313295	0.0614414
1	0.3512870	0.3112144	0.2199253	0.1139011	0.0462964
2	0.1935174	0.1626266	0.0998428	0.0447517	0.0158779
3	0.0917319	0.0804006	0.0421561	0.0161968	0.0049851
4	0.0368324	0.0375051	0.0167183	0.0053778	0.0014074
5	0.0123808	0.0160725	0.0061926	0.0016003	0.0003411
6	0.0033284	0.0059066	0.0020629	0.0004081	0.0000646

Table 3: Probabilities of exactly j departures by time t

j	$t=1$	$t=3$	$t=5$	t=7	$t=10$
0	0.67968674	0.099509402	0.007046060	0.000420927	0.00016181
1	0.17711300	0.081454841	0.009095394	0.000581746	0.00005037
2	0.09298400	0.117972870	0.018124103	0.001391221	0.00005942
3	0.03598100	0.140843910	0.031453844	0.003012981	0.00008380
4	0.01081200	0.140978350	0.047484726	0.005811690	0.00014277
5	0.00259600	0.119261779	0.061887004	0.009825172	0.00026424
6	0.00050430	0.084578352	0.068296881	0.014198767	0.00046744

6.2. Sensitivity analysis

This part focuses on the impact of the arrival rate (λ), service rate (μ), vacation rate (w), catastrophes rate (ξ), feedback probability (q) and balking probability (1-β) on the probability when the server is on vacation ($P_V(t)$), probability when the server is busy ($P_B(t)$), expected queue length ($Q_L(t)$), total expected cost ($TC(t)$), total expected income ($TE_I(t)$) and total expected profit ($TE_P(t)$) at time t. To calculate the numerical results for the sensitivity of the queueing system one parameter varied while keeping all the other parameters fixed.

Impact of arrival rate: We examine the behaviour of the queueing system using measures of effectiveness along with cost and profit analysis by varying λ with time t, while keeping all other parameters fixed; μ=5, w=2, β=0.5, ξ=0.0001, q=0.7, C_H=10, C_B=8, C_V=5, C_μ=4, $C_{\mu-q}$=2, I=100 and N=8. In Table 4, we observe that as the value of λ increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 4: Measures of effectiveness versus λ

t	λ	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	1.00	0.905411	0.094588	0.2173509	37.457268	47.2940	9.8367320
2		0.880912	0.119086	0.2429674	37.786922	59.5430	21.756078
3		0.878559	0.121412	0.2454781	37.818872	60.7060	22.887128
4		0.878264	0.121497	0.2455127	37.818423	60.7485	22.930077
5		0.877657	0.121202	0.2447627	37.805528	60.6010	22.795472
1	1.25	0.884191	0.115808	0.2722840	38.070259	57.9040	19.833741
2		0.855421	0.144572	0.3036798	38.470479	72.2860	33.815521
3		0.852827	0.147024	0.3064271	38.504598	73.5120	35.007402
4		0.852077	0.146782	0.3056222	38.490863	73.3910	34.900137
5		0.849575	0.145499	0.3023700	38.435567	72.7495	34.313933
1	1.50	0.863798	0.136201	0.3275511	38.684109	68.1005	29.416391
2		0.831169	0.168802	0.3646441	39.152702	84.4010	45.248298
3		0.828243	0.171206	0.3671780	39.182643	85.6030	46.420357
4		0.826167	0.170030	0.3638632	39.129707	85.0150	45.885293
5		0.818924	0.166270	0.3543160	38.967940	83.1350	44.167060

Impact of service rate: The behaviour of the queueing system using measures of

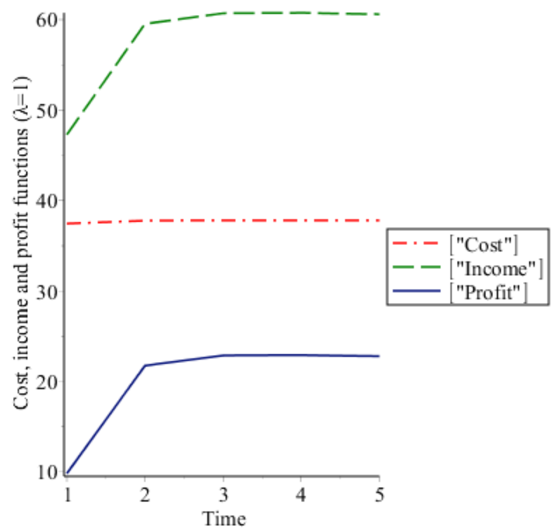

Figure 1

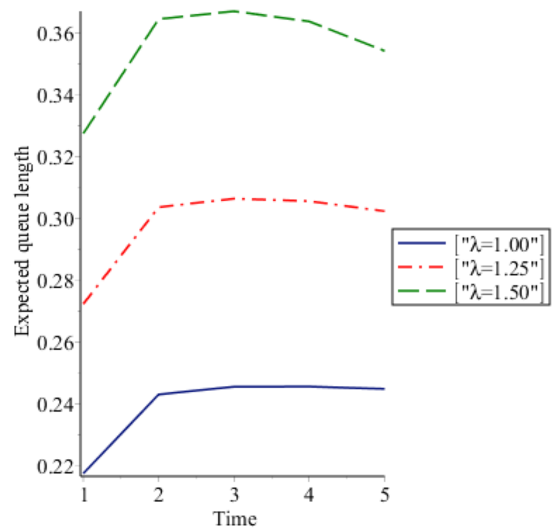

Figure 2

Figure 1: Shows the variation of cost, income and profit at an arrival rate λ=1.00 with time t while keeping the other parameters fixed (μ=5, w=2, ξ=0.0001, q=0.7, β=0.5)

Figure 2: Shows the variation of $Q_L(t)$ with time t by varying arrival rate λ(=1.00, 1.25, 1.50) while keeping the other parameters fixed (μ=5, w=2, ξ=0.0001, q=0.7, β=0.5)

effectiveness along with cost and profit analysis by varying μ with time t, while keeping all other parameters fixed; λ=1, w=2, β=0.5, ξ=0.0001, q=0.7, C_H=10, C_B=8, C_V=5, C_μ=4, $C_{\mu-q}$=2, I=100 and N=8. In Table 5, we observe that as the value of μ increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 5: Measures of effectiveness versus μ

t	μ	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	3.75	0.885251	0.114748	0.220270	30.046939	43.03050	12.983561
2		0.845732	0.154266	0.247027	30.433058	57.84975	27.416692
3		0.840988	0.158983	0.248985	30.466654	59.61862	29.151971
4		0.840598	0.159163	0.248765	30.463944	59.68612	29.222181
5		0.840100	0.158760	0.247921	30.433790	58.78500	28.351210
1	4.25	0.894093	0.105906	0.218863	33.006343	45.01005	12.003707
2		0.861897	0.138101	0.244849	33.362783	58.69292	25.330142
3		0.858481	0.141490	0.247023	33.394555	60.13325	26.738695
4		0.858160	0.141601	0.246935	33.392958	60.18042	26.787467
5		0.857609	0.141250	0.246144	33.371485	59.60625	26.234765
1	4.75	0.901872	0.098127	0.217789	35.972266	46.61032	10.638059
2		0.875156	0.124842	0.243461	36.309126	59.29995	22.990824
3		0.872529	0.127443	0.245866	36.340849	60.53542	24.194576
4		0.872229	0.127532	0.245867	36.340071	60.57770	24.237629
5		0.871638	0.127221	0.245106	36.319018	59.95497	23.635957

Impact of vacation rate: We observe that the behaviour of the queueing system

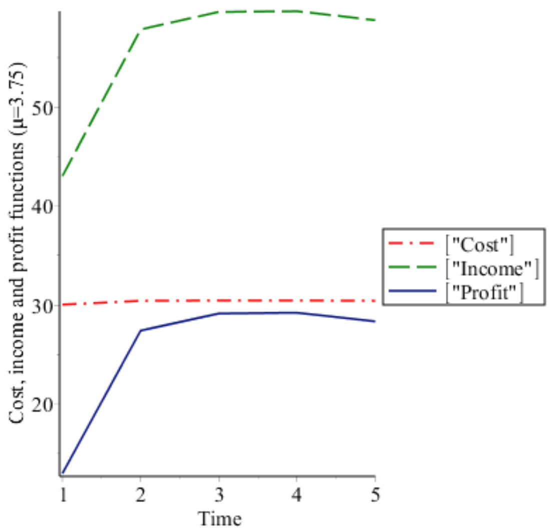

Figure 3

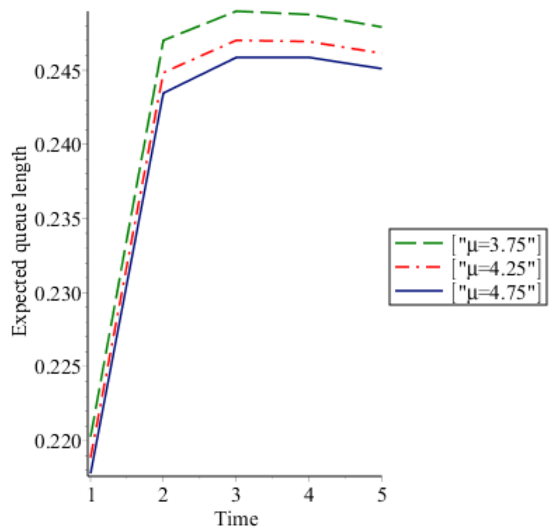

Figure 4

Figure 3: Shows the variation of cost, income and profit at a service rate μ=3.75 with time t while keeping the other parameters fixed (λ=1, w=2, ξ=0.0001, q=0.7, β=0.5)
Figure 4: Shows the variation of $Q_L(t)$ with time t by varying service rate μ(=3.75, 4.25, 4.75) while keeping the other parameters fixed (λ=1, w=2, ξ=0.0001, q=0.7, β=0.5)

using measures of effectiveness along with cost and profit analysis by varying w with time t, while keeping all other parameters fixed; λ=1, μ=5, q=0.7, β=0.5, ξ=0.0001, C_H=10, C_B=8, C_V=5, C_μ=4, $C_{\mu-q}$=2, I=100 and N=8. In Table 6 , we observe that as the value of w increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Impact of catastrophes rate: We see that the behaviour of the queueing system using measures of effectiveness, along with cost and profit analysis by varying ξ with time t, while keeping all other parameters fixed; λ=1, μ=5, w=2, q=0.7, β=0.5, C_H=10, C_B=8, C_V=5 and C_μ=4, $C_{\mu-q}$=2, I=100, N=8. In Table 7, we observe that as the value of ξ increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Impact of feedback probability: We observe that the behaviour of the queueing system using measures of effectiveness along with cost and profit analysis by varying q with time t, while keeping all other parameters fixed; λ=1, μ=5, w=2, β=0.5, ξ=0.0001, C_H=10, C_B=8, C_V=5, C_μ=4, $C_{\mu-q}$=2, I=100 and N=8. In Table 8, we observe that as the value of q increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Impact of joining probability: We observe that the behaviour of the queueing system using measures of effectiveness along with cost and profit analysis by varying β with time t, while keeping all other parameters fixed; λ=1, μ=5, w=2, q=0.7, ξ=0.0001, C_H=10, C_B=8, C_V=5, C_μ=4, $C_{\mu-q}$=2, I=100 and N=8. In Table 9, we observe that as the value of β increases with time t, $P_B(t)$, $Q_L(t)$, $TC(t)$, $TE_I(t)$ and $TE_P(t)$ increases but $P_V(t)$ decreases.

Table 6: Measures of effectiveness versus w

t	w	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	2.00	0.905411	0.094588	0.217350	37.457259	47.2940	9.8367410
2		0.880912	0.119086	0.242967	37.786918	59.5430	21.756082
3		0.878559	0.121412	0.245478	37.818871	60.7060	22.887129
4		0.878264	0.121497	0.245512	37.818416	60.7485	22.930084
5		0.877657	0.120202	0.244762	37.797521	60.1010	22.303479
1	2.25	0.900868	0.099131	0.200253	37.299918	49.5655	12.265582
2		0.878961	0.121037	0.218097	37.544071	60.5185	22.974429
3		0.877366	0.122605	0.219252	37.560190	61.3025	23.742310
4		0.877175	0.122587	0.219121	37.557781	61.2935	23.735719
5		0.876583	0.121276	0.218425	37.537373	60.6380	23.100627
1	2.50	0.897077	0.102922	0.185305	37.161811	51.4610	14.299189
2		0.877506	0.122492	0.197743	37.344896	61.2460	23.901104
3		0.876420	0.123552	0.198212	37.352636	61.7760	24.423364
4		0.876276	0.123486	0.198033	37.349598	61.7430	24.393402
5		0.875691	0.122168	0.197394	37.329739	61.0840	23.754261

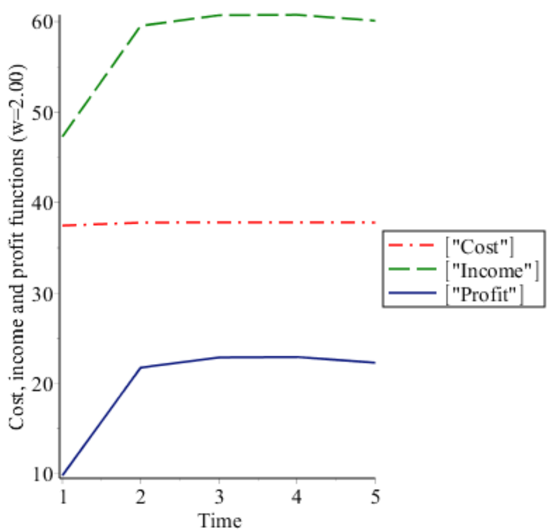

Figure 5

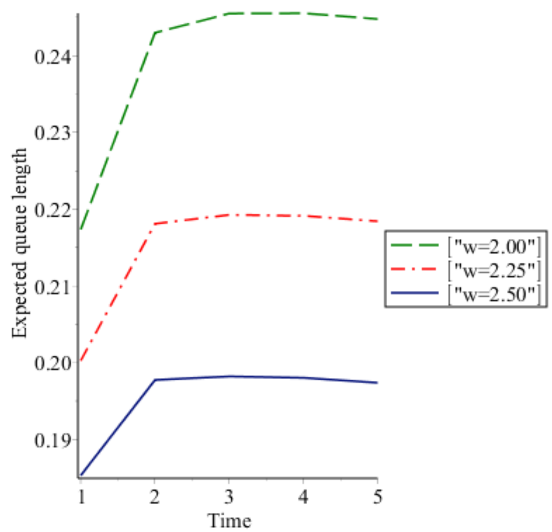

Figure 6

Figure 5: Shows the variation of cost, income and profit at a vacation rate w=2.00 with time t while keeping the other parameters fixed (λ=1, μ=5, ξ=0.0001, q=0.7, β=0.5)

Figure 6: Shows the variation of $Q_L(t)$ with time t by varying vacation rate w(=2.00, 2.25, 2.50) while keeping the other parameters fixed (λ=1, μ=5, ξ=0.0001, q=0.7, β=0.5)

7. Discussion

Figure 1 shows the variation of cost, income and profit with time t by keeping λ constant (=1.00). The value of cost, income and profit increases with increase in t upto t(=3.00, 4.00, 4.00) respectively then decreases slightly. The variation in queue length with time t is represented in figure 2 by varying the arrival rate λ(=1.00, 1.25, 1.50). Queue

Table 7: Measures of effectiveness versus ξ

t	ξ	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	0.0001	0.905411	0.094588	0.217350	37.457259	47.2940	9.8367410
2		0.880912	0.119086	0.242967	37.786918	59.5430	21.756082
3		0.878559	0.121412	0.245478	37.818871	60.7060	22.887129
4		0.878264	0.121497	0.245512	37.818416	60.7485	22.930084
5		0.877657	0.121202	0.244762	37.797521	60.1010	22.303479
1	0.0002	0.905415	0.094584	0.217343	37.457177	47.2920	9.8348230
2		0.880920	0.119078	0.242956	37.786784	59.5390	21.752216
3		0.878567	0.121404	0.245466	37.818727	60.7020	22.883273
4		0.878273	0.121489	0.245501	37.818287	60.7445	22.926213
5		0.877666	0.120194	0.244751	37.797392	60.0970	22.299608
1	0.0003	0.905420	0.094579	0.217336	37.457092	47.2895	9.8324080
2		0.880928	0.119070	0.242945	37.786650	59.5350	21.748350
3		0.878576	0.121395	0.245455	37.818590	60.6975	22.878910
4		0.878282	0.121480	0.245489	37.818140	60.7400	22.921860
5		0.877675	0.120185	0.244740	37.797255	60.0925	22.295245

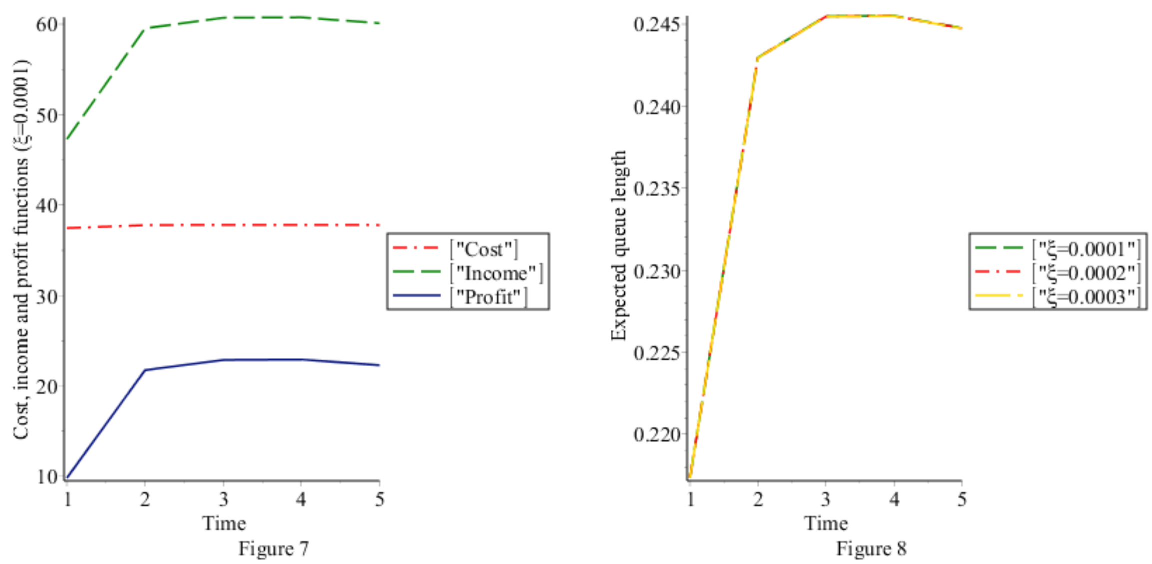

Figure 7: Shows the variation of cost, income and profit at a catastrophes rate ξ=0.0001 with time t while keeping the other parameters fixed (λ=1, μ=5, w=2, q=0.7, β=0.5)
Figure 8: Shows the variation of $Q_L(t)$ with time t by varying catastrophes rate ξ(=0.0001, 0.0002, 0.0003) while keeping the other parameters fixed (λ=1, μ=5, w=2, q=0.7, β=0.5)

length values increases with increase in time up to t(=4.00, 3.00, 3.00) also then decreases slightly. Hence we get the optimal value of t=1 when λ=1.00 and t=3 when λ=1.50 for minimum cost and maximum profit respectively.

Figure 3 shows the variation of cost, income and profit with time t by keeping μ constant (=3.75). The value of cost, income and profit increases with increase in t upto

Table 8: Measures of effectiveness versus q

t	q	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	0.55	0.888542	0.111457	0.2197238	37.531604	55.7285	18.196896
2		0.851890	0.148108	0.2461336	37.905650	74.0540	36.148350
3		0.847705	0.152266	0.2481621	37.938274	76.1330	38.194726
4		0.847350	0.152412	0.2479947	37.935993	76.2060	38.270007
5		0.846830	0.151029	0.2471729	37.914111	75.5145	37.600389
1	0.65	0.900287	0.099712	0.2179962	37.479093	49.8560	12.376907
2		0.872523	0.127475	0.2437086	37.819501	63.7375	25.917999
3		0.869757	0.130214	0.2460660	37.851157	65.1070	27.255843
4		0.869455	0.130306	0.2460511	37.850234	65.1530	27.302766
5		0.868872	0.128987	0.2452851	37.829107	64.4935	26.664393
1	0.75	0.910104	0.089895	0.2168165	37.437845	44.9475	7.5096550
2		0.888300	0.111698	0.2424285	37.759369	55.8490	18.089631
3		0.886247	0.113724	0.2450750	37.791777	56.8620	19.070223
4		0.885957	0.113804	0.2451474	37.791691	56.9020	19.110309
5		0.885330	0.112529	0.2444091	37.770973	56.2645	18.493527

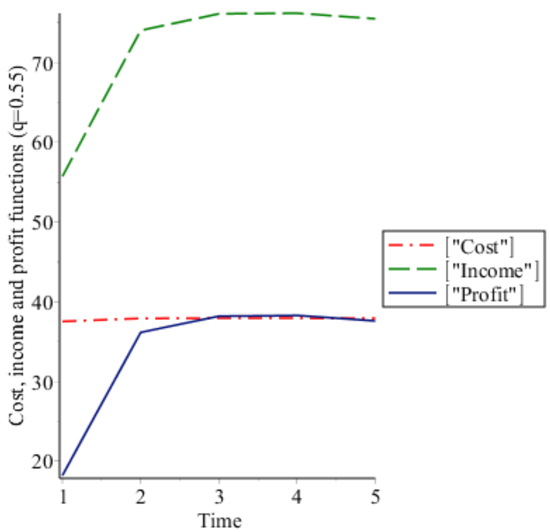

Figure 9

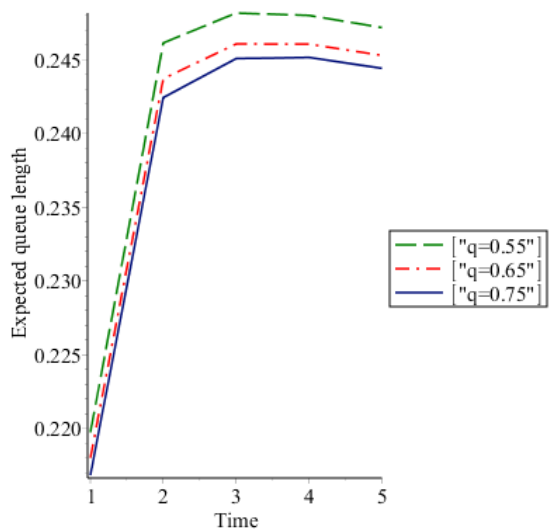

Figure 10

Figure 9: Shows the variation of cost, income and profit at a feedback probability q=0.55 with time t while keeping the other parameters fixed (λ=1, μ=5, w=2, ξ=0.0001, β=0.5)

Figure 10: Shows the variation of $Q_L(t)$ with time t by varying feedback probability q(=0.55, 0.65, 0.75) while keeping the other parameters fixed (λ=1, μ=5, w=2, ξ=0.0001, β=0.5)

t(=3.00, 4.00, 4.00) then decreases slightly. The variation in queue length with time t is represented in figure 4 by varying the service rate μ(=3.75, 4.25, 4.75). Queue length values increases with increase in time up to t(=3.00, 3.00, 4.00) also then decreases slightly. Hence we get the optimal value of t=1 when μ=3.75 and t=4 when μ=3.75 for minimum cost and maximum profit respectively.

Table 9: Measures of effectiveness versus β

t	β	$P_V(t)$	$P_B(t)$	$Q_L(t)$	$TC(t)$	$TE_I(t)$	$TE_P(t)$
1	0.50	0.905411	0.094588	0.2173509	37.457268	47.2940	9.8367320
2		0.880912	0.119086	0.2429674	37.786922	59.5430	21.756078
3		0.878559	0.121412	0.2454781	37.818872	60.7060	22.887128
4		0.878264	0.121497	0.2455127	37.818423	60.7485	22.930077
5		0.877657	0.121202	0.2447627	37.805528	60.6010	22.795472
1	0.60	0.888367	0.111632	0.2612720	37.947611	55.8160	17.868389
2		0.860414	0.139580	0.2915200	38.333910	69.7900	31.456090
3		0.857871	0.142019	0.2942445	38.367952	71.0095	32.641548
4		0.857261	0.141876	0.2937015	38.358328	70.9380	32.579672
5		0.855314	0.140883	0.2911838	38.315472	70.4415	32.126028
1	0.70	0.871860	0.128139	0.3054014	38.438426	64.0695	25.631074
2		0.840733	0.159250	0.3402231	38.879896	79.6250	40.745104
3		0.837961	0.161701	0.3429228	38.912641	80.8505	41.937859
4		0.836564	0.161003	0.3408982	38.879826	80.5015	41.621674
5		0.831663	0.158467	0.3344632	38.770683	79.2335	40.462817

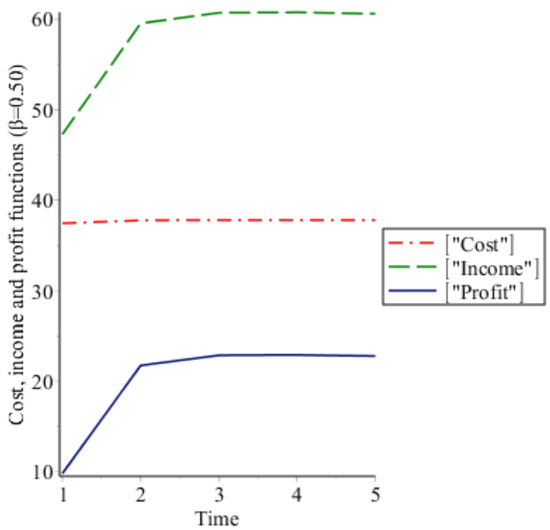

Figure 11

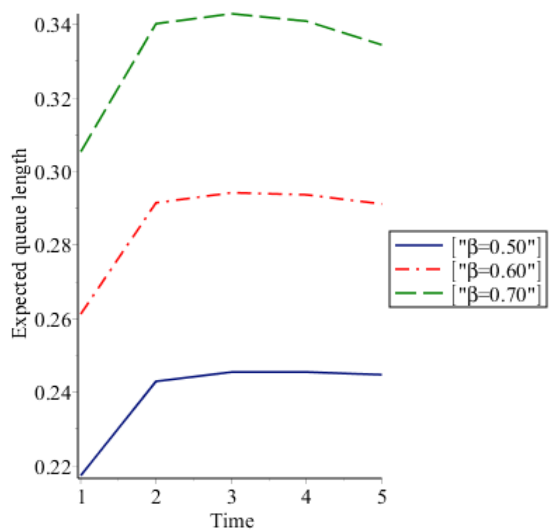

Figure 12

Figure 11: Shows the variation of cost, income and profit at a joining probability β=0.50 with time t while keeping the other parameters fixed (λ=1, μ=5, w=2, ξ=0.0001, q=0.7)
Figure 12: Shows the variation of $Q_L(t)$ with time t by varying joining probability β(=0.50, 0.60, 0.70) while keeping the other parameters fixed (λ=1, μ=5, w=2, ξ=0.0001, q=0.7)

Figure 5 shows the variation of cost, income and profit with time t by keeping w constant (=2.00). The value of cost, income and profit increases with increase in t upto t(=3.00, 4.00, 4.00) then decreases slightly. The variation in queue length with time t is represented in figure 6 by varying the vacation rate w(=2.00, 2.25, 2.50). Queue length values increases with increase in time up to t(=4.00, 3.00, 3.00) also then decreases slightly. Hence we get the optimal value of t=1 when w=2.50 and t=3 when w=2.50 for minimum cost and maximum profit respectively.

Figure 7 shows the variation of cost, income and profit with time t by keeping ξ constant (=0.0001). The value of cost, income and profit increases with increase in t upto t(=3.00, 4.00, 4.00) then decreases slightly. The variation in queue length with time t is represented in figure 8 by varying the catastrophes rate ξ(=0.0001, 0.0002, 0.0003). Queue length values increases with increase in time up to t(=4.00) then decreases slightly. Hence we get the optimal value of t=1 when ξ=0.0003 and t=4 when ξ=0.0001 for minimum cost and maximum profit respectively. Finally, the variation in rate of catastrophes shows the minor effect on cost and profit.

Figure 9 shows the variation of cost, income and profit with time t by keeping q constant (=0.55). The value of cost, income and profit increases with increase in t upto t(=3.00, 4.00, 4.00) then decreases slightly. The variation in queue length with time t is represented in figure 10 by varying the feedback probability q(=0.55, 0.65, 0.75). Queue length values increases with increase in time up to t(=3.00, 4.00, 4.00) also then decreases slightly. Hence we get the optimal value of t=1 when q=0.75 and t=4 when q=0.55 for minimum cost and maximum profit respectively.

Figure 11 shows the variation of cost, income and profit with time t by keeping β constant (=0.50). The value of cost, income and profit increases with increase in t upto t(=3.00, 4.00, 4.00) then decreases slightly. The variation in queue length with time t is represented in figure 12 by varying the joining probability β(=0.50, 0.60, 0.70). Queue length values increases with increase in time up to t(=4.00, 3.00, 3.00) also then decreases slightly. Hence we get the optimal value of t=1 when β=0.50 and t=3 when β=0.70 for minimum cost and maximum profit respectively.

8. Conclusions and future work

The time-dependent solution for the M/M/2 queueing system with multiple vacation, feedback, catastrophes and balking has been obtained using a two-dimensional state model. Based on various performance measures, total expected cost and total expected profit, the best optimal value is at t=1 when service rate=3.75 and t=3 when arrival rate=1.50 for minimum cost and maximum profit respectively. Some key measures give a greater understanding of system model behaviour. This model finds its application in post office, computer networks, supermarkets, hospital administrations, financial sector and many others.
As part of future study, this model may be examined further for Non-Markovian queues, bulk queues, tandem queues *etc.*

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